Attractor Reconstruction
I: Fundamentals

Tim Sauer

July 2006

Time series from dynamical system

Finite dimensional system $f : X \to X$

f may represent difference equations or time-T map of autonomous differential equations

Observation function $h : X \to R^m$

$m = 1$ univariate time series

$m > 1$ multivariate time series

Goal

Find invariant set and dynamics

Attractor reconstruction

Use observations to construct embedding, or at least 1-1 function, from attractor's phase space to some R^m.

Example

Periodic orbit in R^3

$E(x, y, z) = (x, y)$ to R^2 may or may not unfold periodic orbit

To guarantee embedding of a periodic orbit, 3 independent observations are need, generically.

Whitney Embedding Theorem 1936

Smooth manifold A of dimension d

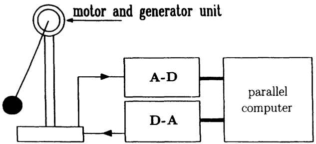
Whitney

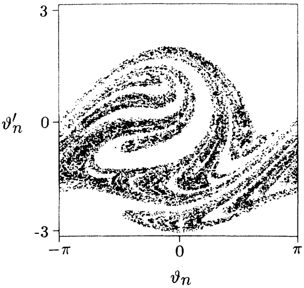
There is a C^1 - open and dense set of maps into R^{2d+1} which embed A.

Consider $m = 2d + 1$ independent measurements as a map

Embedding means individual states are distinguished by the observations

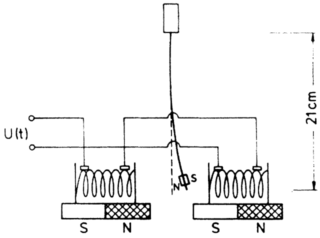
Not all attractors are manifolds!

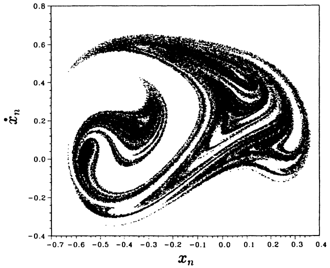




U. Dressler et al., Daimler-Benz

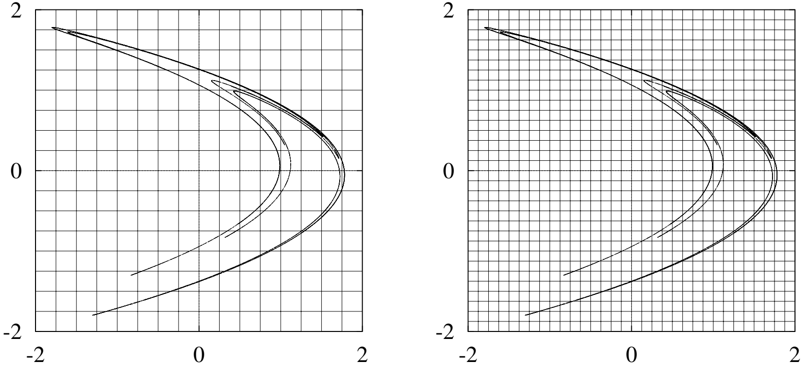
Fractal attractor



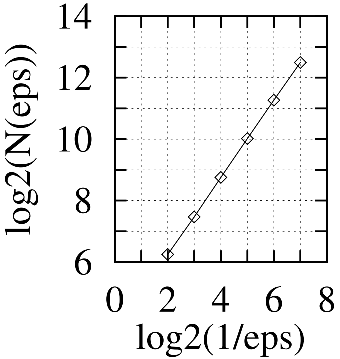


Fractal attractors lead to fractal dimensions

Box counting dimension



Box counting dimension



$N(\epsilon) = \text{number of boxes to cover} \propto \epsilon^{-d}$

$d_{\text{box}} approx 1.4$

Middle-third Cantor set

At step i, 2^i intervals of length 3^{-i} remain.

Cover set with $N(\epsilon)$ boxes of diameter ϵ

$$N(\epsilon) \sim \epsilon^{-d}$$

$$d_{\text{box}} = \lim_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log 1/\epsilon}$$

Middle-third Cantor set requires 2^i one-dim boxes of diameter $(1/3)^i$, so

$$d_{\text{box}} = \lim \frac{\log 2^i}{\log 3^i} = \frac{\log 2}{\log 3} \approx 0.63$$

Middle-3/5 Cantor set

At step i, 2^i intervals of length 5^{-i} remain.

$d_{\text{box}} = \log 2 / \log 5 \approx 0.43$

Fractal Whitney Embedding Theorem

Theorem

Let A be a compact subset of R^k of box-counting dimension d. Let n be an integer, $n > 2d$. Then for a dense set of smooth maps $F : R^k \longrightarrow R^n$,

1. F is one-to-one on A
2. F is an immersion on each compact subset C of a smooth manifold contained in A

Dense can be replaced by "prevalent".

Fractal Whitney Embedding Theorem

Idea of proof

Assume A (compact) $\subset R^k, d_{\text{box}}(A) = d, n > 2d$.
Let $F : R^k \longrightarrow R^n$.

Let $L = L(R^k, R^n)$ denote kn-dimensional cube of linear maps.

For each pair of ϵ-boxes $B_1, B_2 \subset A$, perturbations F' of F by functions from L cause $B_1 \cap B_2$ with probability $\propto \epsilon^n$.

A can be covered by ϵ^{-d} boxes of size ϵ, so there are ϵ^{-2d} pairs to consider.

The probability that two ϵ-boxes intersect in the image of F' is approximately ϵ^{n-2d}. If $n > 2d$, the probability goes to 0 with ϵ.

Intersection theory in R^1

Example: Middle-third Cantor set

Cantor set $C = \{.b_1 b_2 b_3 \ldots \text{ in base 3} : b_i = 0 \text{ or } 2\}$

$d_{\text{box}}(C) \approx 0.63$

Middle third Cantor sets always overlap

$C =$ middle third Cantor set

Let $v \in [0,1]$. Do C and $C+v$ intersect?

Then $\frac{v+1}{2} \in [0,1]$

$$\begin{aligned} \frac{v+1}{2} &= .02112012\ldots \text{ for example} \\ &= .01111011\ldots + .01001001\ldots \end{aligned}$$

$$\begin{aligned} v+1 &= .02222022\ldots + .02002002\ldots \\ &= c_1 + c_2 \end{aligned}$$

Therefore $v + 1 - c_1 = c_2$.

Middle 3/5 Cantor sets almost never overlap

C = middle 3/5 Cantor set
$C = \{.b_1 b_2 b_3 \ldots \text{ in base } 5 : \; b_i = 0 \text{ or } 4\}$

$d_{\text{box}} = \log 2 / \log 5 \approx 0.43$

Note that $\ldots 22 \ldots$ in v implies translates don't intersect.

Lebesgue almost every $v \in [0, 1]$ contains a 22

Hausdorff dimension

There is no corresponding Whitney result for Hausdorff dimension.

Example

There exists a set of Hausdorff dimension 0 in R^m such that all linear projections to R^k, $k < m$, fail to be one-to-one.

(Ittai Kan)

Univariate time series

Fractal Whitney Embedding Theorem assumes multivariate observations.

What can be done with a single observation function $h(x)$?

Idea

Replace independent observations with time delays

Define $H : R^k \rightarrow R^m$ by

$$x \longrightarrow [h(x), h(f_{-\tau}(x), h(f_{-2\tau}(x), \ldots, h(f_{-(m-1)\tau}(x)]$$

Univariate time series

What can be done with a single observation function $h(x)$?

Are

$$h(x_t), h(x_{t-\tau}), h(x_{t-2\tau}), \ldots$$

independent coordinates?

Mathematical translation

Is

$$H(x) = [h(x), h(f_{-\tau}(x)), \ldots, h(f_{-(m-1)\tau}(x))]$$

one-to-one on A for generic $h : R^k \longrightarrow R$?

Univariate time series

Main question

Is

$$H(x) = [h(x), h(f_{-\tau}(x)), \ldots, h(f_{-(m-1)\tau}(x))]$$

one-to-one on A for generic $h : R^k \longrightarrow R$?

Short answer: No.

Example

Periodic orbit, period τ. For each point on the orbit,

$$x_t = x_{t-\tau} = x_{t-2\tau} = \ldots$$

and the orbit is projected to a line segment. H cannot be 1-1.

Univariate time series

Main question

Is

$$H(x) = [h(x), h(f_{-\tau}(x)), \ldots, h(f_{-(m-1)\tau}(x))]$$

one-to-one on A for generic $h : R^k \longrightarrow R$?

Example

Periodic orbit, period 2τ. The function

$$h(x) - h(f_{-\tau}(x))$$

has at least one zero crossing x_0 on A. Then

$$h(x_0) = h(f_{-\tau}(x_0)) = h(f_{-2\tau}(x_0)) = h(f_{-3\tau}(x_0)) = \ldots$$

so x_0 and $f_{-\tau}(x_0)$ are mapped together. F cannot be 1-1.

Fractal Takens Embedding Theorem

Theorem.
Let A be a compact subset of R^k of box-counting dimension d, invariant under diffeomorphism f. Let n be an integer, $n > 2d$. Assume:

1. For every $p \leq n$, the set A_p of periodic points of period p satisfies $d_{\text{box}}(A_p) < p/2$
2. Df^p has distinct eigenvalues for each of these orbits

Then for a dense set of smooth maps $h : R^k \longrightarrow R$,

1. the corresponding delay map H is one-to-one on A
2. H is an immersion on each compact subset C of a smooth manifold contained in A

Distinct eigenvalues necessary for immersion

Let x be a fixed point of f such that $Df(x)$ has 2 linearly independent vectors v_0, v_1 with same eigenvalue λ.

Set $u = (Dh(x)v_1)v_0 - (Dh(x)v_0)v_1$. Then u is an eigenvalue of $Df(x)$ with eigenvalue λ and $Dh(x)u = 0$.

For each i,

$$D(h(f^i(x)))u = Dh(f^i(x))Df(x)\cdots Df(x)u = Dh(x)\lambda^i u = 0$$

Therefore $H(x) = [h(x), hf(x), \ldots, hf^i(x)]$ is not an immersion at x.

Takens Embedding Theorem

Packard, Crutchfield, Farmer, Shaw (PRL, 1980)

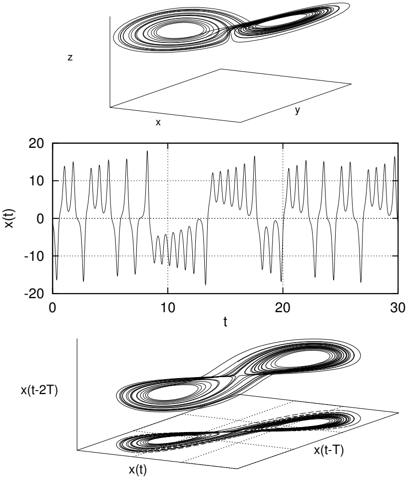
Takens (1981)

Roux, Swinney (1981)

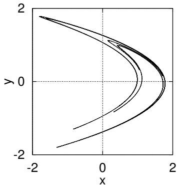
Aeyels (1981)

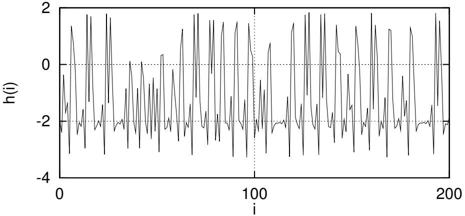
Sauer, Takens, Casdagli (1991 fractal version)

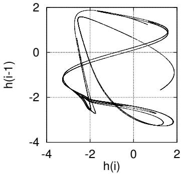
Time series from Lorenz system



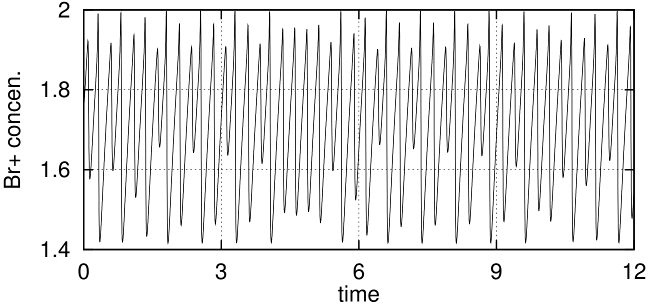
Hénon map



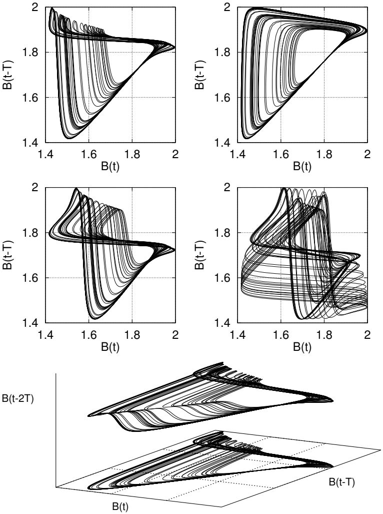




Belousov-Zhabotinskii reaction



Time series from dynamical system



Applications - Analysis

Calculating system invariants from data stream

- dimension
  - often focused on correlation dimension
  - + surrogate data as sanity check
- Lyapunov exponents
  - Eckmann, Ruelle. Rev. Mod. Phys. 1985
- determinism tests
  - Kaplan, Glass. Phys. Rev. Lett. 1992
- critical exponents, scaling laws (bifurcations, crises, etc.)
  - E.g. Sommerer et al. Phys. Lett A 1991
- unstable periodic orbits (symbolic dynamics)
  - Gilmore, Mindlin, Glorieux, etc.

Applications

- Time series prediction
- Noise reduction
- Control of chaos
- Tracking, targeting and goal dynamics

Coping with Chaos (Ott, Sauer, Yorke) Wiley, 1994.

Nonlinear Time Series (Kantz, Schreiber) Cambridge, 1997, 2003.

TISEAN package (Hegger, Kantz, Schreiber, 1999)

Time series prediction

Typical methodology:

Fit local linear AR model in embedding space of dynamics, using evolution of near neighbors over short time interval. Use local model to predict.
(Long history of nearest-neighbor prediction in statistical literature.)

Ingredients:

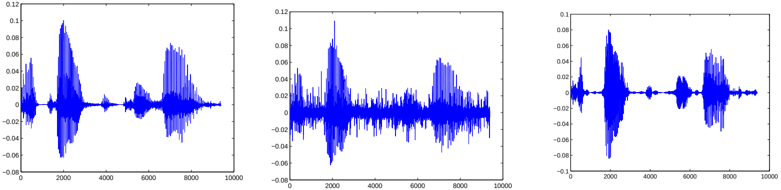
- weighted linear regression (Tukey's tricubic)
- Fourier interpolation to "fatten" attractor
- Use of principal component analysis to project out noise

Time Series Prediction (Weigend, Gershenfeld) Addison-Wesley 1994.

Noise reduction

Sample technique:

Embedding threshold estimator used in Fourier frame



Delay coordinate embedding as a tool for denoising speech signals

D. Napoletani, C. Berenstein, T. Sauer, D. Struppa, D. Walnut (2005)

Control from time series reconstruction

BZ reaction in continuous-flow stirred-tank reactor (Showalter et al., 1992)

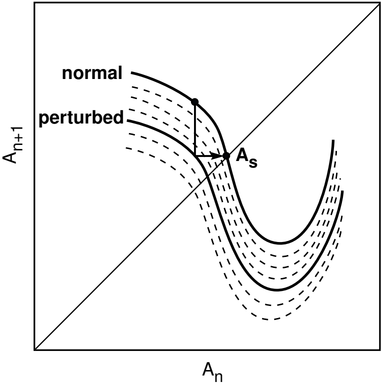
Single measurement: Bromide electrode potential vs. time

Delay time = 13 sec.

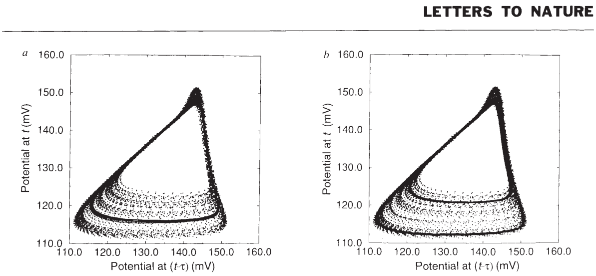
Control parameter: reactant inflow rate (cerium/bromate solution)

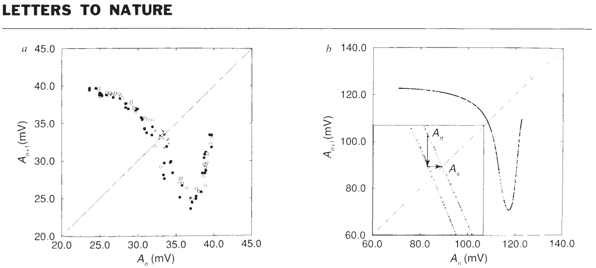
RESULT: Stabilized period 1 and 2 limit cycles

Control from time series reconstruction



Control from time series reconstruction





Open problems in applications

Determination of directionality in coupled time series

Estimation of delay

Characterization of on-off intermittency from time series

Determinism tests

Measuring noise, observational noise, dynamical noise

Parameter and unobserved component estimation from time series

Attractor Reconstruction
II: Extensions

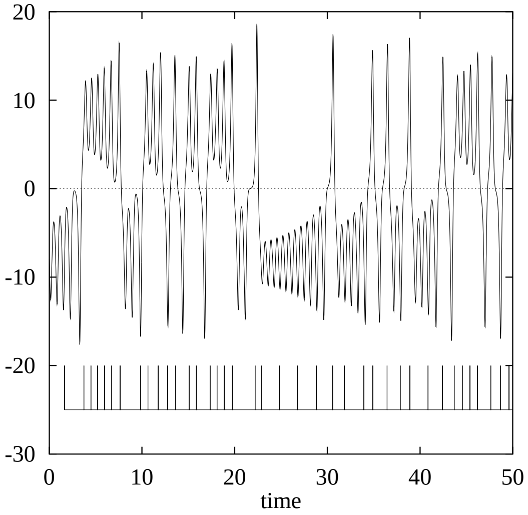
Tim Sauer

July 2006

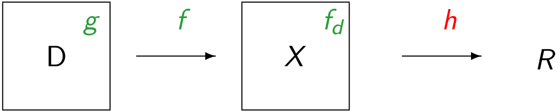
Extensions

1. Reconstruction from spike trains
2. Nonautonomous Takens Theorem
3. Stochastic Takens Theorem
4. Driver reconstruction

1. Reconstruction from spike trains



2. Nonautonomous Takens

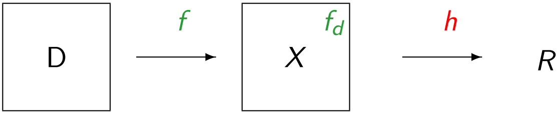


$$d_{i+1} = g(d_i) \text{ and } x_{i+1} = f(x_i, d_i)$$

Goal

Reconstruct $D \times X$, recording only from X.

3. Stochastic Takens

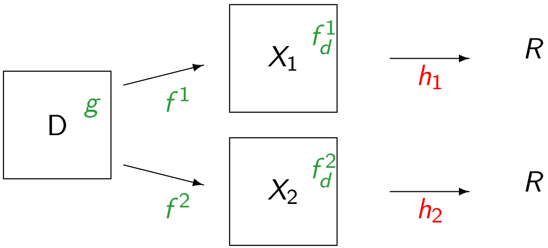


$\Omega = D^\infty$
$\omega = (\ldots, d_{-1}, d_0, d_1, \ldots)$
σ is shift map and $x_{i+1} = f(x_i, \omega)$

Goal

Reconstruct fibers over ω.

4. Driver reconstruction

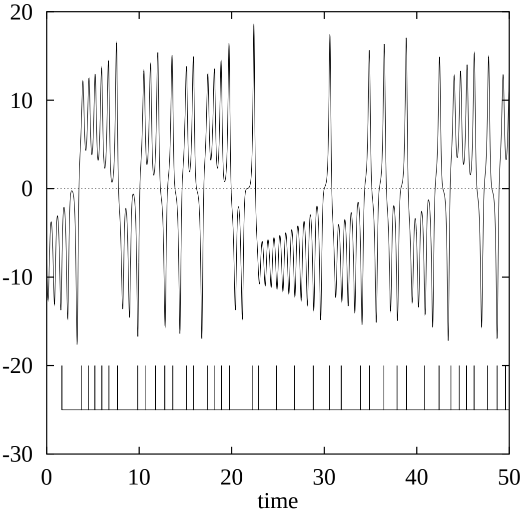


$$
\begin{aligned}
d_{i+1} &= g(d_i) \\
x_{i+1}^1 &= f^1(x_i^1, d_i) \\
x_{i+1}^2 &= f^2(x_i^2, d_i)
\end{aligned}
$$

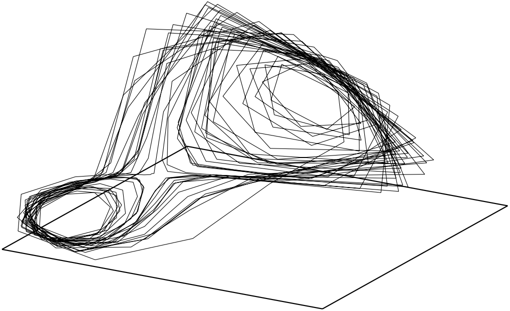
Goal

Reconstruct D, recording from X_1 and X_2.

Reconstruction from spike trains



Reconstruction from spike trains



Reconstruction coordinates are interspike intervals

$$[T_{i+1} - T_i, T_{i+2} - T_{i+1}, T_{i+3} - T_{i+2}].$$

Reconstruction of spike trains

Integrate and fire hypothesis

Let $S(t) > 0$ be a signal, $\Theta > 0$ threshold.

Define "firing times" $T_1 < T_2 < T_3 \ldots$ by $\displaystyle\int_{T_i}^{T_{i+1}} S(t)dt = \Theta$.
Interspike intervals are $T_{i+1} - T_i$.

Example. Lorenz equations

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= \rho x - y - xz \\ \dot{z} &= -\beta x + xy \end{aligned}$$

where $\sigma = 10, \rho = 28, \beta = 8/3, S(t) = (x+2)^2, \Theta = 60$.

Reconstruction from spike trains

Let $S(t) > 0$ be a signal, $\Theta > 0$ threshold.

Define "firing times" $T_1 < T_2 < T_3 \ldots$ by $\displaystyle\int_{T_i}^{T_{i+1}} S(t)dt = \Theta.$

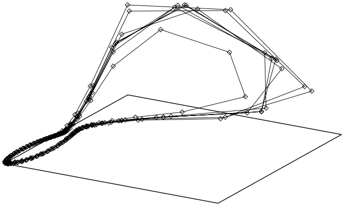
Theorem. Let $\dot{x} = f(x)$ be an autonomous system of differential equations on R^k with compact invariant set A. Assume that A contains at most a finite number of equilibrium points and $m > 2d_{\text{box}}(A)$. Then there is a residual set of positive-valued output functions h for which the interspike intervals

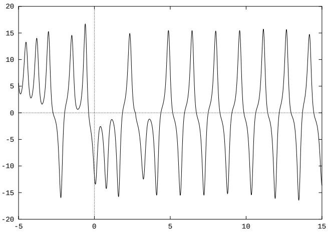
$$[T_{i+1} - T_i, T_{i+2} - T_{i+1}, \ldots, T_{m+1} - T_m]$$

created from the integrate-and-fire hypothesis uniquely define states of A.

T. Sauer, "Reconstruction of integrate-and-fire dynamics", in Nonlinear dynamics and time series: Building a bridge between the natural and statistical sciences, Eds. C. Cutler, D. Kaplan, AMS (1997)

Subthreshold control of spike trains



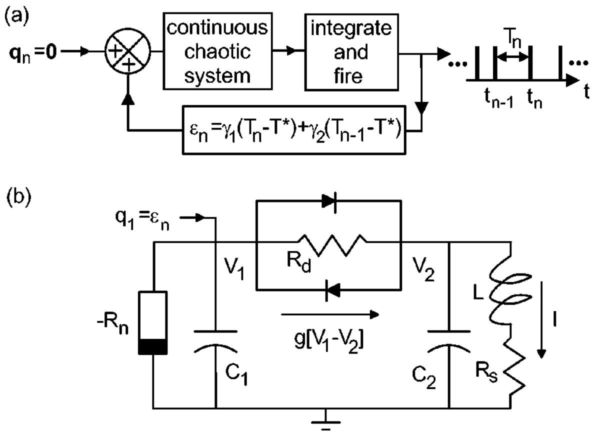


Small perturbations based on spike train observations are used to control Lorenz attractor input to integrate-and-fire generator.

Experimental control

Experimental control of a chaotic point process using interspike intervals

G. M. Hall, S. Bahar, D. Gauthier at Duke University



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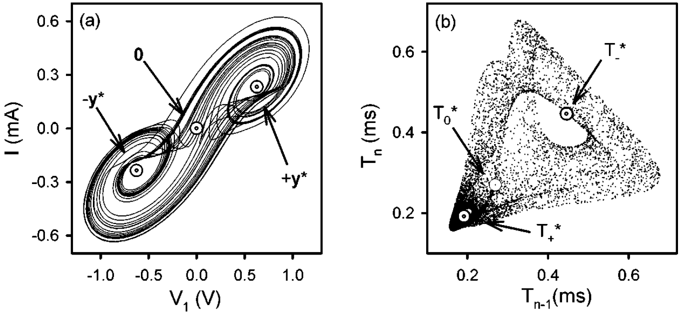
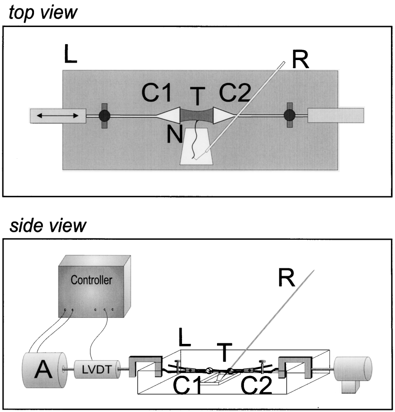


Experimental spike train reconstruction

Encoding chaos in neural spike trains

K. Richardson, T. Imhoff, P. Grigg, J. Collins at Boston Univ.



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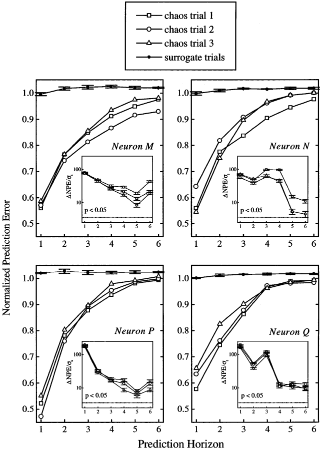
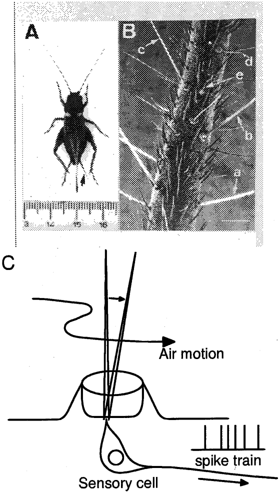


Experimental spike train reconstruction

Analysis of neural spike trains with ISI reconstruction

H. Suzuki, K. Aihara, J. Murakami, T. Shimozawa at Univ. Tokyo



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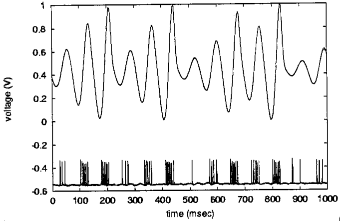
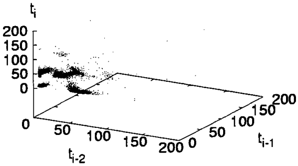
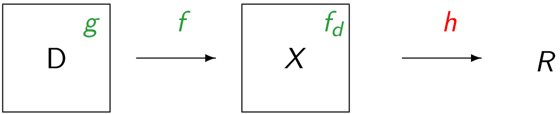


(a)



(b)

Nonautonomous Takens



$$d_{i+1} = g(d_i) \text{ and } x_{i+1} = f(x_i, d_i)$$

Goal

Reconstruct $D \times X$, recording only from X.

Nonautonomous Takens

Theorem 1. Let D and X be compact manifolds, $\dim(D) = d$, $\dim(X) = k \geq 1$. Let $m \geq 2d + 2k + 1$, and assume the periodic orbits of period $< 2m$ of $g : D \to D$ are isolated and have distinct eigenvalues. Then there exists and open, dense set of C^1 functions $f : D \times X \to X$ and $h : X \to R$ for which the m-dimensional delay map is an embedding.

J. Stark (1999)

Nonautonomous Takens

Theorem 2. (Fiber version) Let D and X be compact manifolds, $\dim(D) = d$, $\dim(X) = k \geq 1$. Let $m \geq 2k + 1$, and assume the periodic orbits of period $< m$ of $g : D \to D$ are isolated and have distinct eigenvalues. Then there exists a residual set of C^1 functions $f : D \times X \to X$ and $h : X \to R$ and for any such f, h an open dense subset of $d \in D$ for which the m-dimensional delay map is an embedding of the fiber over d.

J. Stark (1999)

Stochastic Takens

$$\boxed{\mathrm{D}} \xrightarrow{\;f\;} \boxed{X \quad f_d} \xrightarrow{\;h\;} R$$

$\Omega = D^\infty$
$\omega = (\ldots, d_{-1}, d_0, d_1, \ldots)$
σ is shift map and $x_{i+1} = f(x_i, \omega)$

Goal

Reconstruct fibers over ω.

Stochastic Takens

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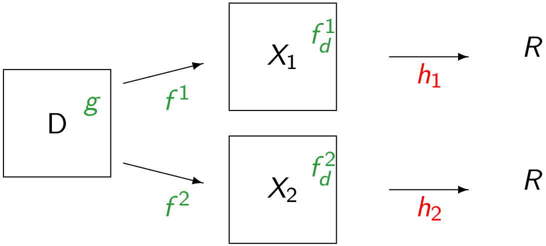
J. Stark, D. Broomhead, M. Davies, J. Huke (2003)

Stochastic Takens

Theorem 2. (Measure theoretic fiber version) Let D and X be compact manifolds, $\dim(D) = d$, $\dim(X) = k \geq 1$, and let μ be a probability measure on D which is absolutely continuous w.r.t. Lebesgue. Let $m \geq 2k + 1$. Then there exists a residual set of C^1 functions $f : D \times X \to X$ and $h : X \to R$ and for any such f, h the subset of $\omega \in \Omega$ for which the m-dimensional delay map is an embedding of the fiber over ω is full measure.

J. Stark, D. Broomhead, M. Davies, J. Huke (2003)

Driver reconstruction



$$
\begin{aligned}
d_{i+1} &= g(d_i) \\
x_{i+1}^1 &= f^1(x_i^1, d_i) \\
x_{i+1}^2 &= f^2(x_i^2, d_i)
\end{aligned}
$$

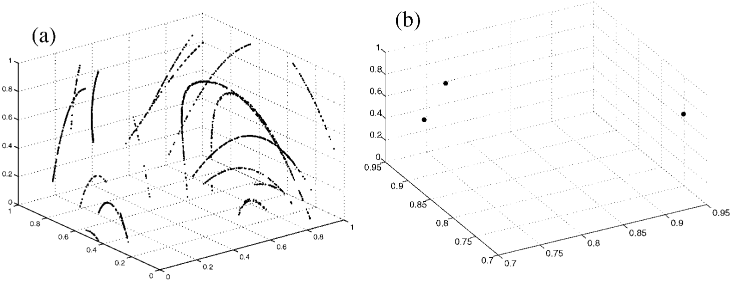
Goal

Reconstruct D, recording from X_1 and X_2.

Driver reconstruction

Algorithm based on Nonautonomous Takens Theorem uses observed time series to identify states of driver as equivalence classes.

Equivalence classes give semi-conjugacy with driver dynamics g.



Sauer (2004)

Future directions

1. Under what conditions can measurements from subsystems be used to (generically) reconstruct system dynamics?
2. Network dynamics: use multiple measurements from network to reconstruct dynamics of network components
3. Fractal versions of nonautonomous and stochastic Takens
4. Reconstruction of leaky integrate-and-fire spike trains.
5. System identification and signal processing using multiple spike trains.