Coupled Systems: Theory & Examples

Lecture 3

Linear Degeneracies
and
Unusual Bifurcations

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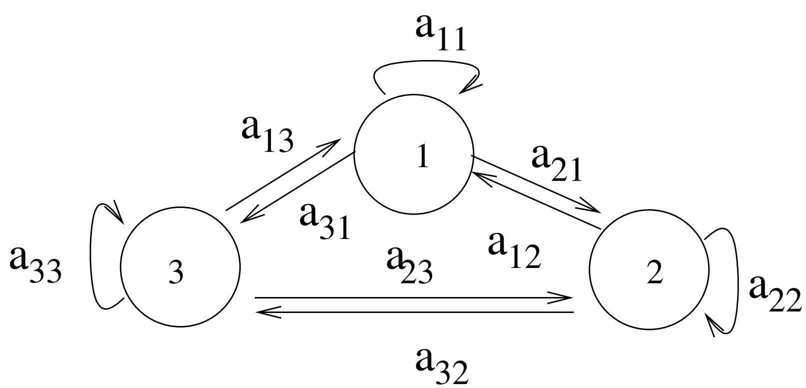
Thanks

Ian Stewart Warwick

Toby Elmhirst	*UBC*
Maria Leite	*Purdue*
Matthew Nicol	*Houston*
Marcus Pivato	*Trent*
Yunjiao Wang	*Houston*

Homogeneous Three Cell Networks



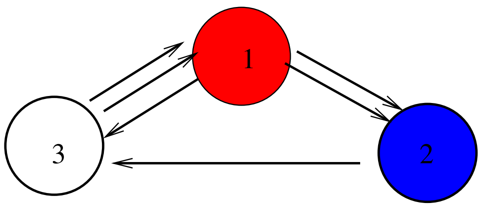
- a_{ij} = number of inputs cell i receives from cell j
- 3×3 matrix $A = (a_{ij})$ is adjacency matrix
- *Valency* $= n =$ total number of inputs per cell

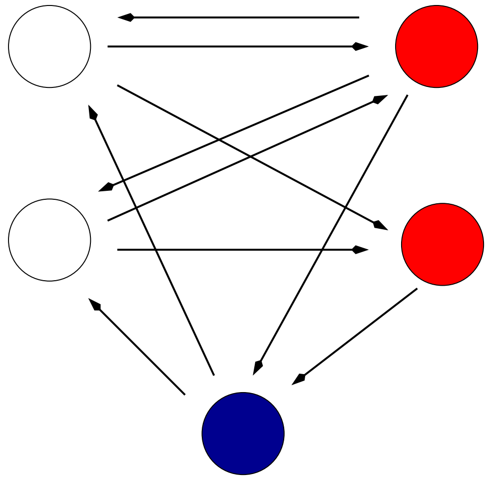
$$a_{i1} + a_{i2} + a_{i3} = n \qquad \text{for } j = 1, 2, 3$$

- 34 homogeneous three-cell valency 2 networks

Leite and G. (2005)

Network 14: Complex Eigenvalues (1)



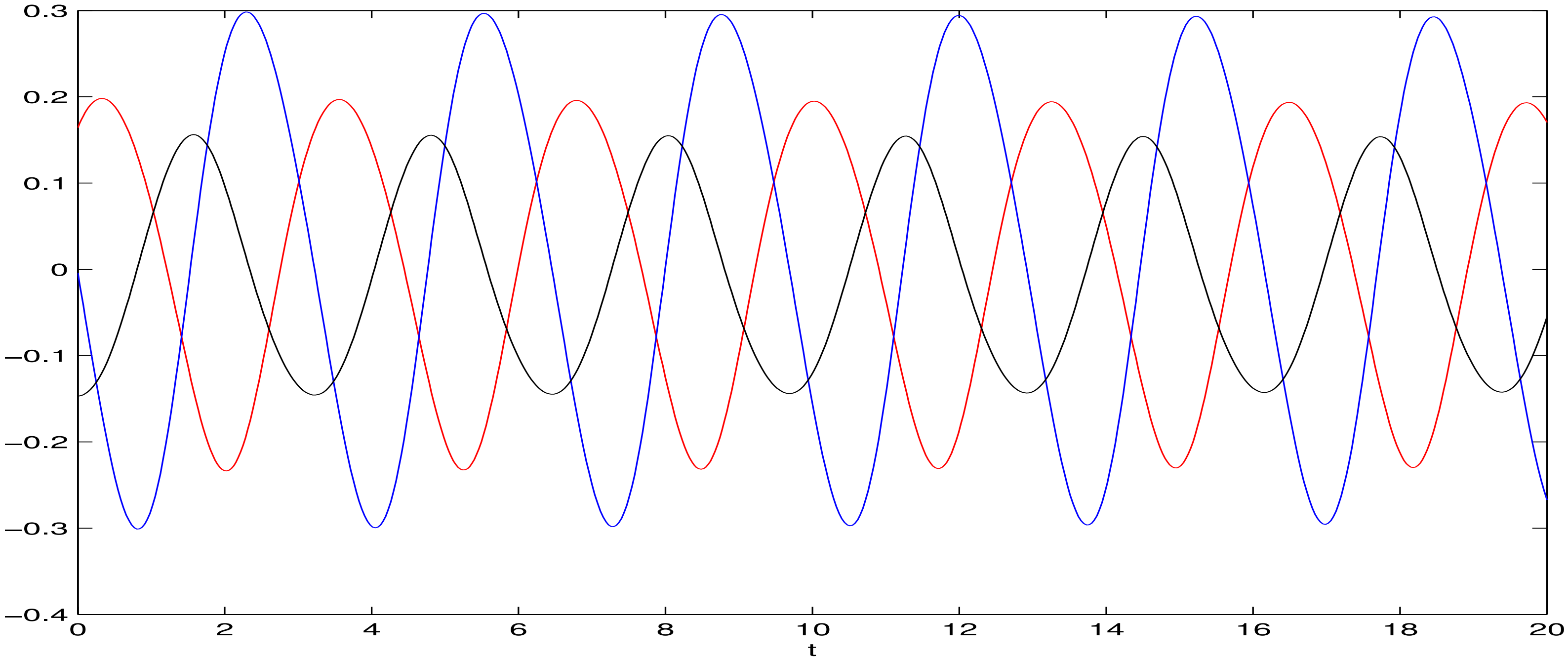


$$\begin{aligned}\dot{x}_1 &= f(x_1, \overline{x_3, x_3}) \\ \dot{x}_2 &= f(x_2, \overline{x_1, x_1}) \\ \dot{x}_3 &= f(x_3, \overline{x_1, x_2})\end{aligned} \qquad J = \begin{bmatrix} \alpha & 0 & 2\beta \\ 2\beta & \alpha & 0 \\ \beta & \beta & \alpha \end{bmatrix} = \alpha I + \beta \begin{bmatrix} 0 & 0 & 2 \\ 2 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

- Eigenvalues: $\alpha + 2\beta \quad \alpha - (1 \pm i)\beta$
- Eigenvectors: $(1, 1, 1) \quad (1, -(1 \mp i), -\frac{1 \pm i}{2})$

Network 14: Complex Eigenvalues (2)

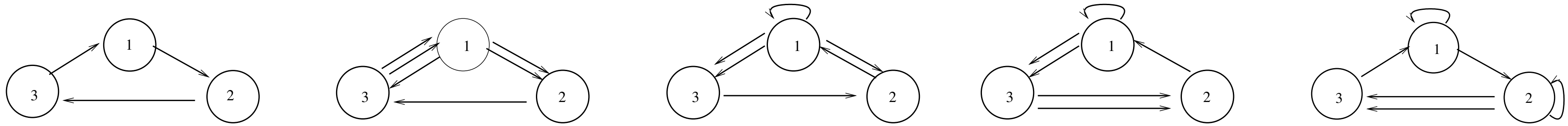
- Synchrony breaking eigenvector: $(1, -(1+i), -\frac{1+i}{2})$



- $\frac{|A_2|}{|A_1|} = \sqrt{2}$ $\quad \frac{|A_3|}{|A_1|} = \frac{1}{\sqrt{2}}$ $\quad \theta_2 - \theta_1 = \frac{5}{8}$ $\quad \theta_3 - \theta_1 = \frac{3}{8}$

Synchrony-Breaking Bifurcations

- Simple complex (no zero) eigenvalues: 2, 14, 18, 19, 24



Leite and G. (2005)

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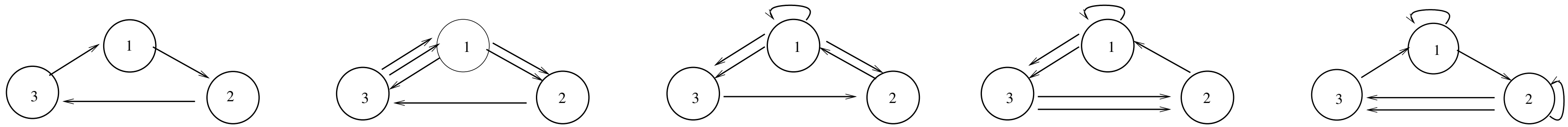
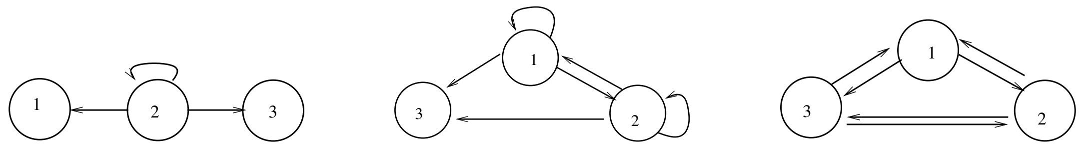
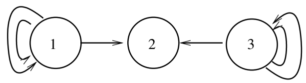


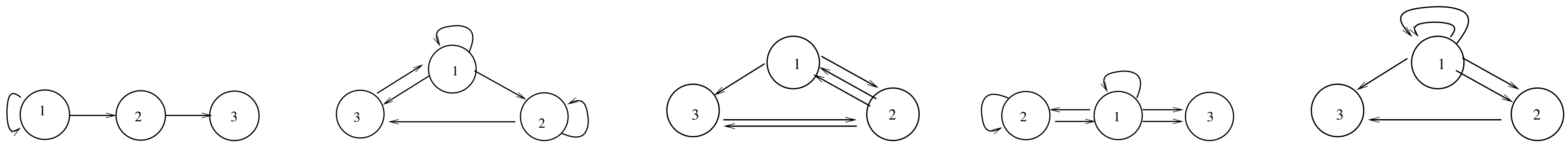
- Double with two synch-breaking eigenvectors: 4, 7, 8



- Double with synchrony preserving eigenvector: 12



- Nilpotent: 3; 6, 11, 27, 28



Leite and G. (2005)

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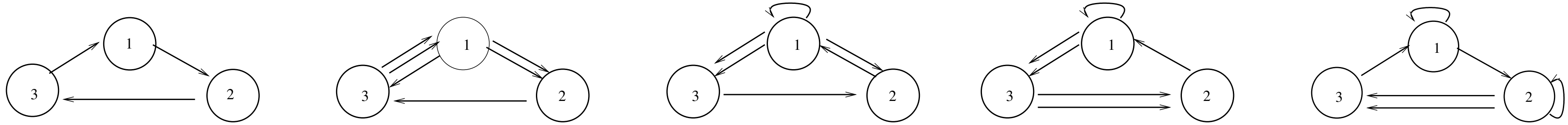

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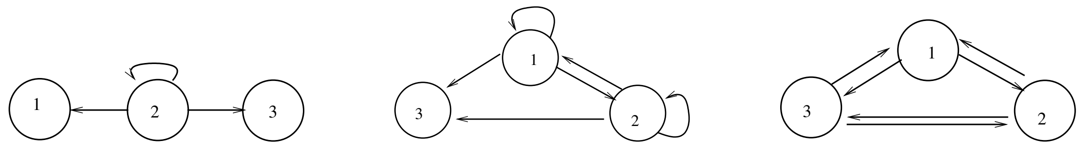

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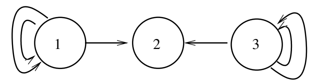

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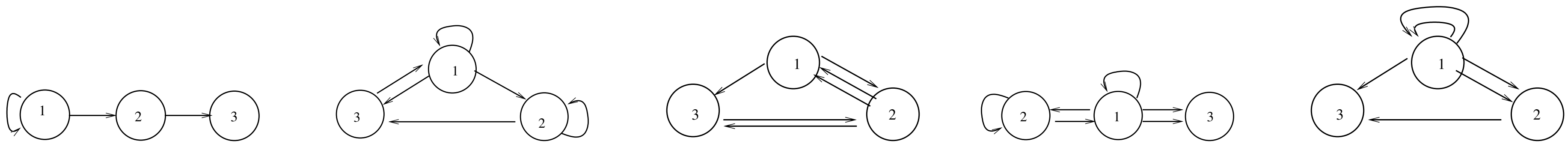
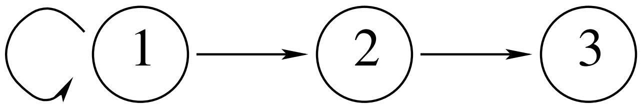


- Remaining 20 networks have simple eigenvalues

Leite and G. (2005)

Three-Cell Feed-Forward Network

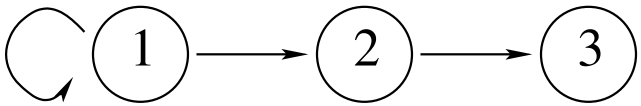


$$\begin{aligned}\dot{x}_1 &= f(x_1, x_1)\\ \dot{x}_2 &= f(x_2, x_1)\\ \dot{x}_3 &= f(x_3, x_2)\end{aligned}$$

$$J = \begin{bmatrix} \alpha+\beta & 0 & 0 \\ \beta & \alpha & 0 \\ 0 & \beta & \alpha \end{bmatrix}$$

G., Nicol, and Stewart (2004); Elmhirst and G. (2005)

Three-Cell Feed-Forward Network

- 

$$\dot{x}_1 = f(x_1, x_1)$$
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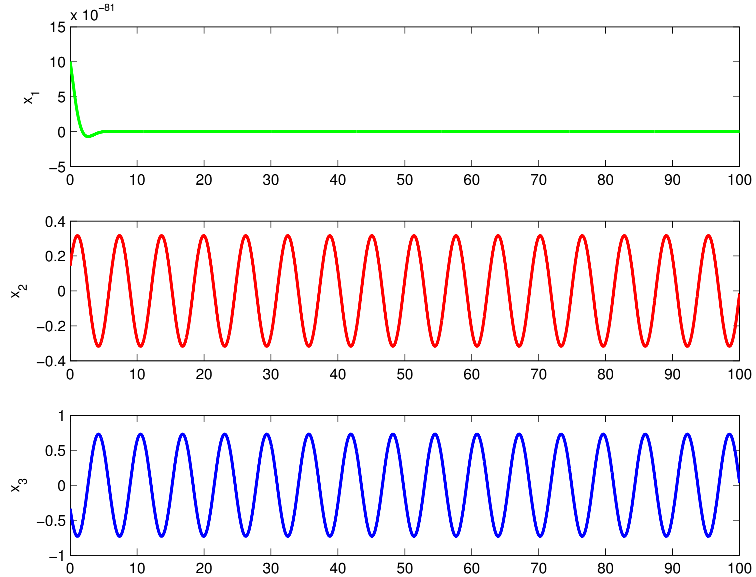
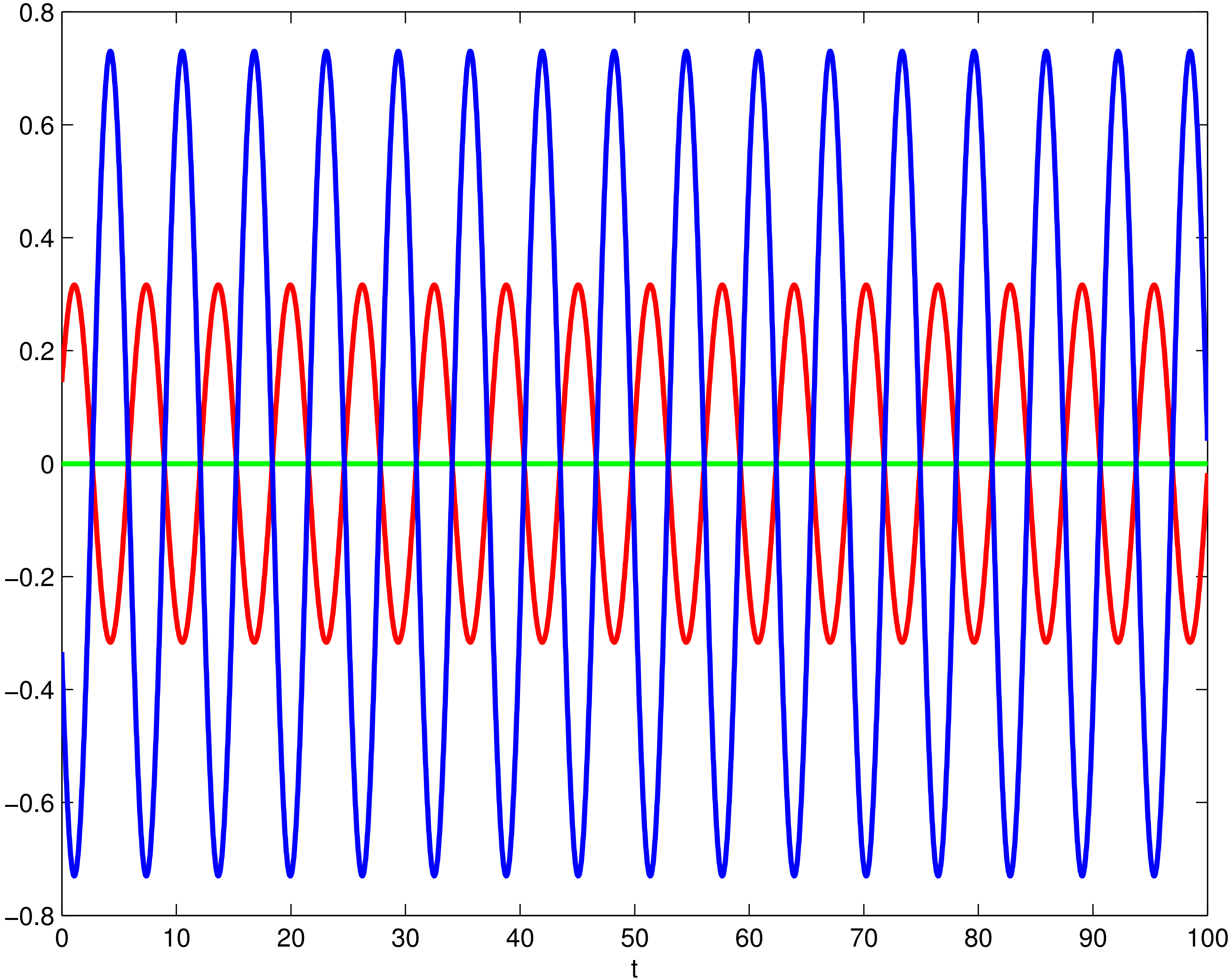
$$J = \begin{bmatrix} \alpha+\beta & 0 & 0 \\ \beta & \alpha & 0 \\ 0 & \beta & \alpha \end{bmatrix}$$

- Network supports solution by Hopf bifurcation where
$x_1(t)$ equilibrium $x_2(t), x_3(t)$ time periodic



G., Nicol, and Stewart (2004); Elmhirst and G. (2005)

$x_2(t) \approx \lambda^{1/2}$ $x_3(t) \approx \lambda^{1/6}$



Feed Forward: $\frac{1}{6}$th Power Growth Rate

$$\mathbf{f(u,v) = (\lambda + i - |u|^2)u - v; u,v \in C}$$

$$\dot{x}_1 = f(x_1, x_1) = (\lambda + i - |x_1|^2)x_1 - x_1$$

$$\dot{x}_2 = f(x_2, x_1) = (\lambda + i - |x_2|^2)x_2 - x_1$$

$$\dot{x}_3 = f(x_3, x_2) = (\lambda + i - |x_3|^2)x_3 - x_2$$

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$x_1 = 0$ is a stable equilibrium for $\lambda < 1$

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$$\dot{x}_2 = f(x_2, 0) = (\lambda + i - |x_2|^2)x_2$$

$$\dot{x}_3 = f(x_3, x_2) = (\lambda + i - |x_3|^2)x_3 - x_2$$

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$x_2(t) = \sqrt{\lambda}e^{it}$ is stable periodic solution for $0 < \lambda < 1$

$$\dot{x}_3 = f(x_3, x_2) = (\lambda + i - |x_3|^2)x_3 - x_2$$

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$$\dot{x}_3 = f(x_3, x_2) = (\lambda + i - |x_3|^2)x_3 - x_2$$

$$\dot{x}_3 = f(x_3, \sqrt{\lambda}e^{it}) = (\lambda + i - |x_3|^2)x_3 - \sqrt{\lambda}e^{it}$$

F.F. Ex. $f(u,v) = (\lambda + i - |u|^2)u - v$

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Set $\quad x_3 = ye^{it}$

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$x_3(t) = y(t)e^{it}$

$$\dot{y}e^{it} + yie^{it} = (\lambda + i - |y|^2)ye^{it} - \sqrt{\lambda}e^{it}$$

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$$\dot{y} + \boxed{yi} = (\lambda + \boxed{i} - |y|^2)y - \sqrt{\lambda}$$

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Set $y = \lambda^{\frac{1}{6}}u$

F.F. Ex. $f(u,v) = (\lambda + i - |u|^2)u - v$

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$$\dot{y} = (\lambda - |y|^2)y - \sqrt{\lambda}$$

$y(t) = \lambda^{\frac{1}{6}}u(t)$

$$\begin{aligned} \lambda^{\frac{1}{6}}\dot{u} &= (\lambda^{\frac{6}{6}} - \lambda^{\frac{2}{6}}|u|^2)\lambda^{\frac{1}{6}}u - \lambda^{\frac{3}{6}} \\ &= \lambda^{\frac{3}{6}}\left\{\lambda^{\frac{4}{6}} - (|u|^2u + 1)\right\} \end{aligned}$$

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$$\dot{u} = -\lambda^{\frac{1}{3}}\left\{(|u|^2u + 1) + \lambda^{\frac{2}{3}}u\right\}$$

F.F. Ex. $f(u, v) = (\lambda + i - |u|^2)u - v$

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$$\dot{u} = -\lambda^{\frac{1}{3}} \left\{ (|u|^2 u + 1) + \lambda^{\frac{2}{3}} u \right\}$$

Solve $\dot{u} = 0$ for equilibria

$$-(|u|^2 u + 1) + \lambda^{\frac{2}{3}} u = 0$$

F.F. Ex. $f(u,v) = (\lambda + i - |u|^2)u - v$

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Solve $\dot{u} = 0$ for equilibria

$$-(|u|^2u + 1) + \lambda^{\frac{2}{3}}u = 0$$

Use IFT to obtain branch of (stable) equilibria

$$u_0(\lambda) = -1 + O(\lambda^{\frac{2}{3}})$$

F.F. Ex. $f(u,v) = (\lambda + i - |u|^2)u - v$

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$$\dot{u} = -\lambda^{\frac{1}{3}}\left\{(|u|^2 u + 1) + \lambda^{\frac{2}{3}}u\right\}$$

$$u_0(\lambda) = -1 + O(\lambda^{\frac{2}{3}})$$

Thus $x_3(t)$ is periodic with same period as $x_2(t)$

$$x_3(t) = y(t)e^{it} = \lambda^{\frac{1}{6}}u(t)e^{it} \to \lambda^{\frac{1}{6}}u_0(\lambda)e^{it} = -\lambda^{\frac{1}{6}}e^{it} + O(\lambda^{\frac{5}{6}})$$

Feed-Forward Suggests ...

- Tune λ so that the feed-forward network is slightly subcritical; that is, the origin is stable
- Periodically force the first equation through the coupling

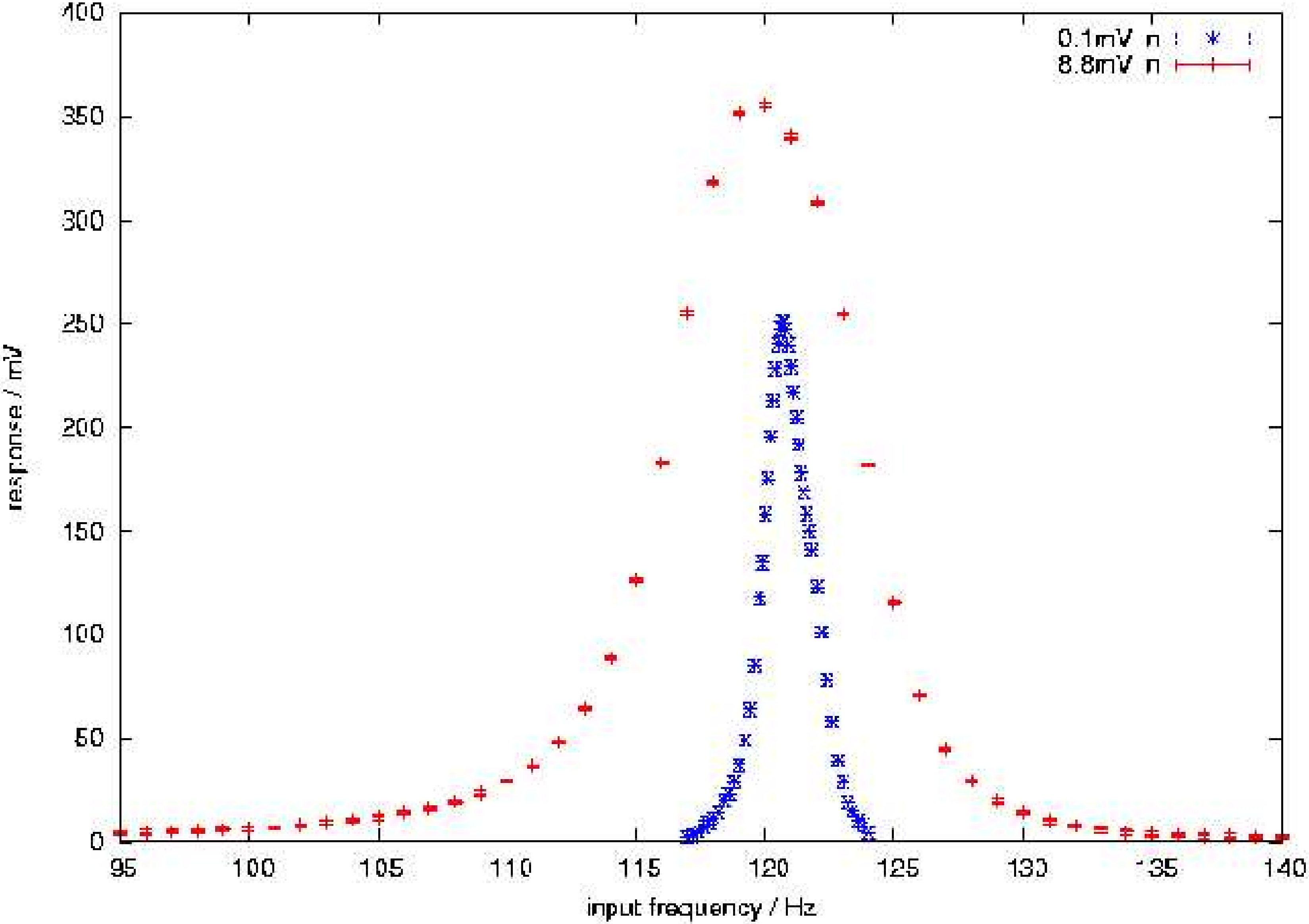
$$\dot{x}_1 = f(x_1, \varepsilon g(\omega_F t), \lambda)$$

- How does amplitude of cell 3 grow with ε?
- Guess:

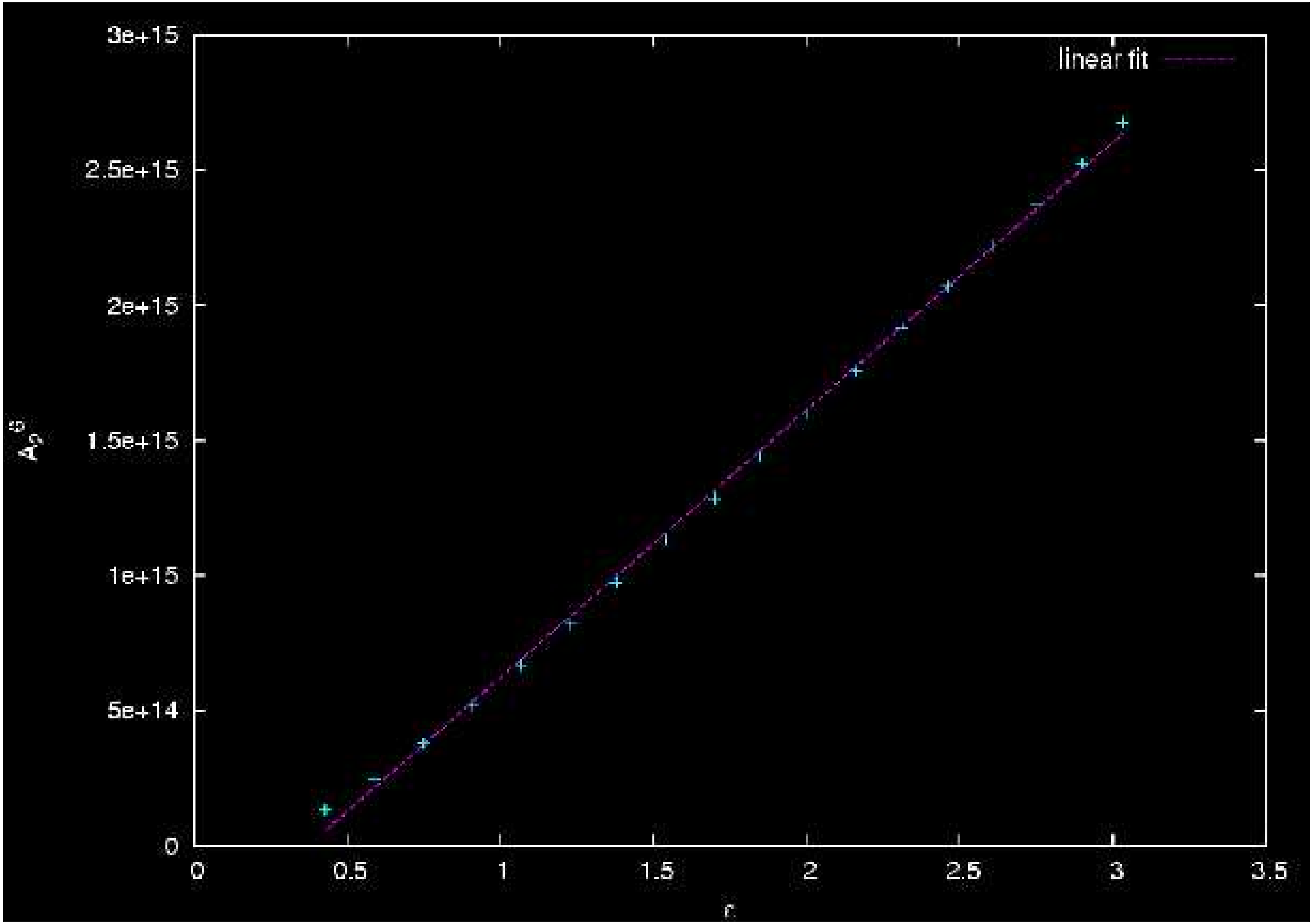
$\varepsilon^{\frac{1}{6}}$ when frequency ω_F is near ω_H

ε when frequency ω_F is far from ω_H

McCullen-Mullin Experiment (1)

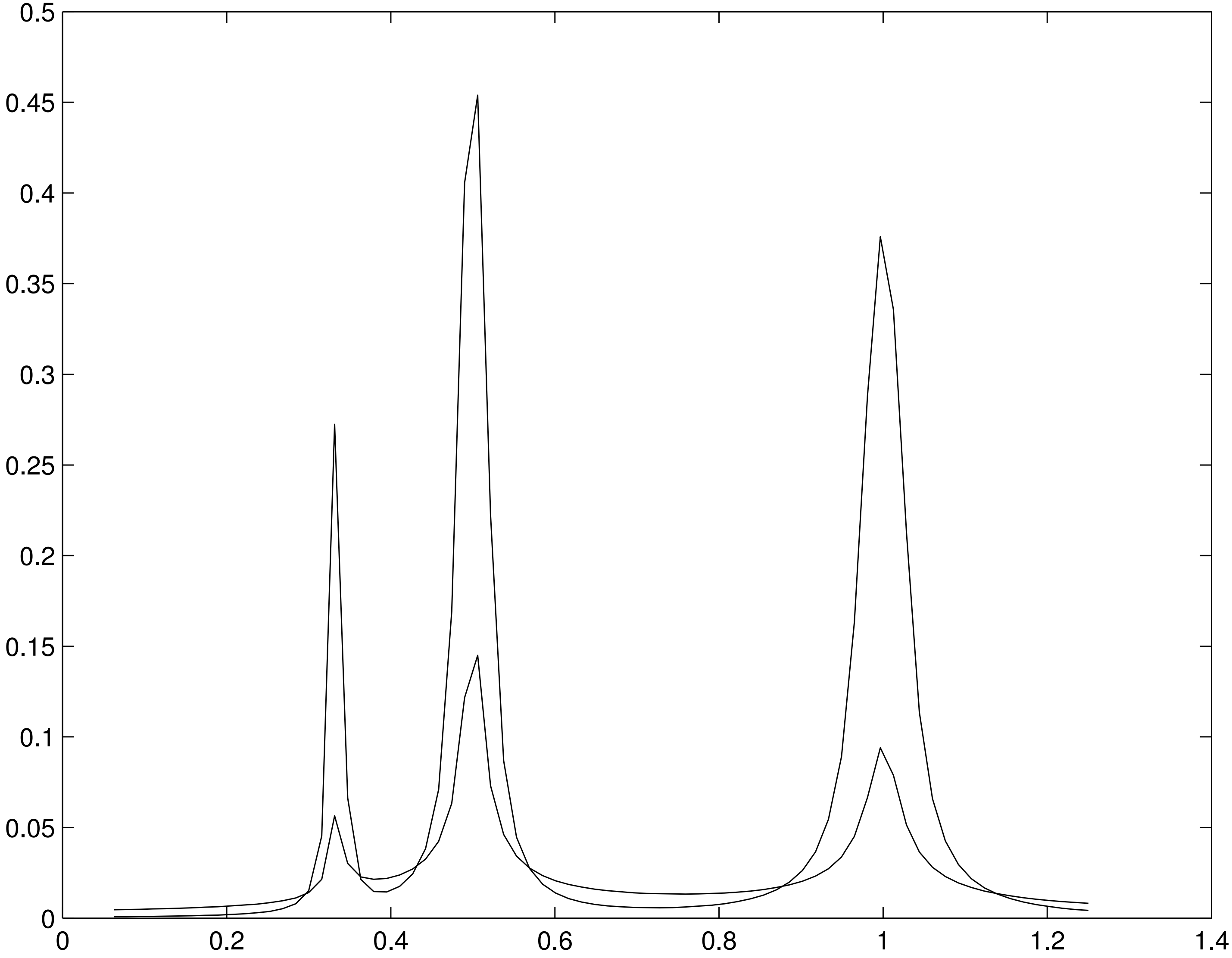


McCullen-Mullin Experiment (2)

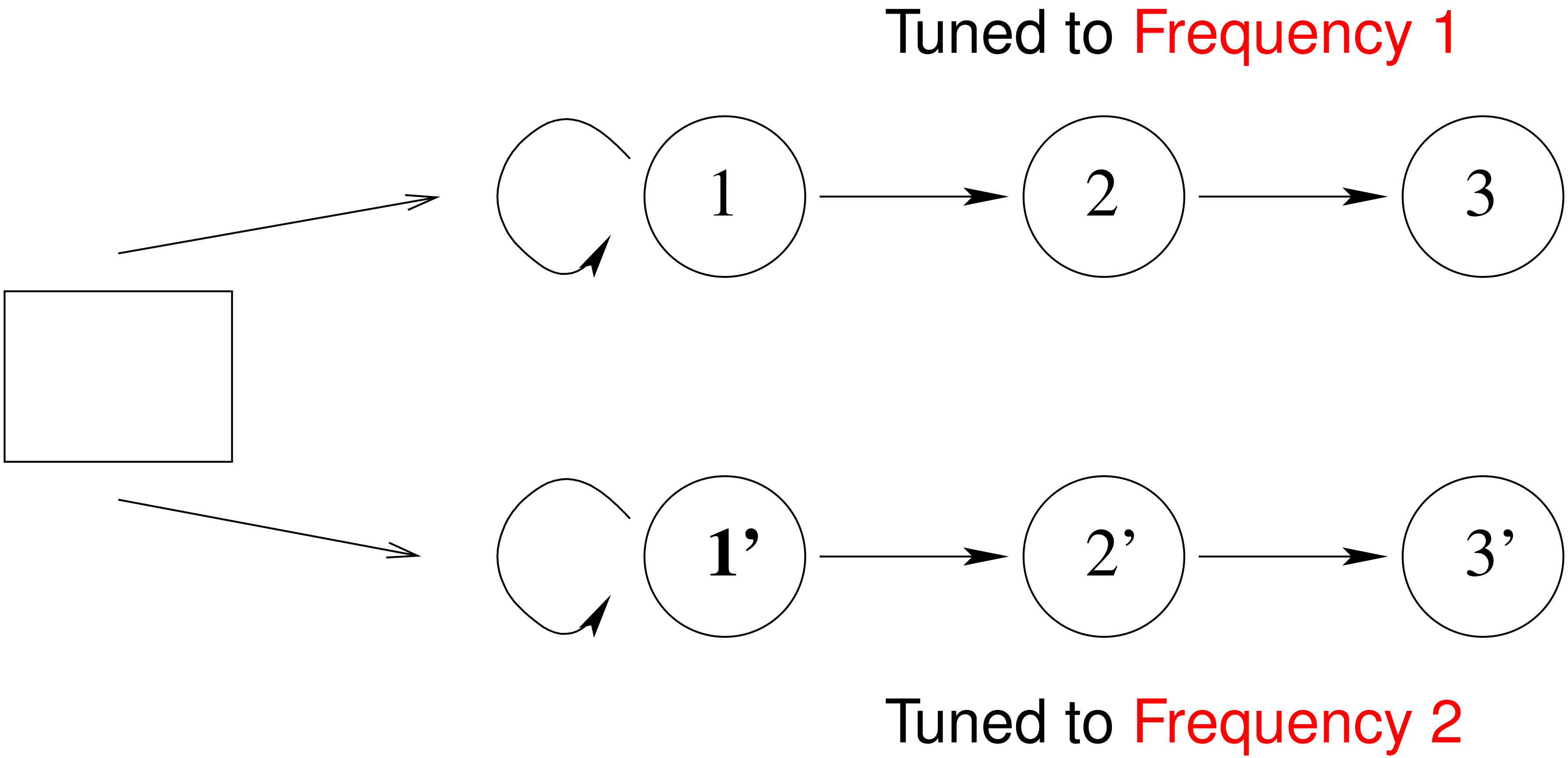


Numerics with Aronson

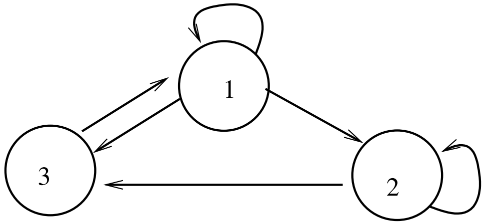
$$g(t) = \varepsilon(e^{i\omega_F t} + 2e^{2i\omega_F t} - 0.5e^{3i\omega_F t}) \quad \lambda = -0.1 \quad \varepsilon = 0.01$$



FM Switch — Broomhead



Example: Network 6



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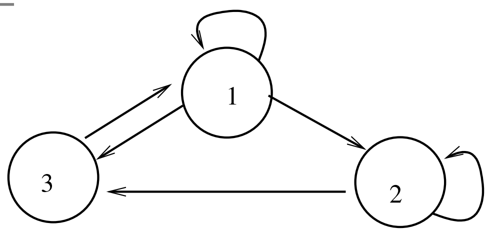
- E'values of $\alpha + 2\beta$ ($[x, x, x]^t$), α ($[x, -x, -x]^t$), α

Example: Network 6



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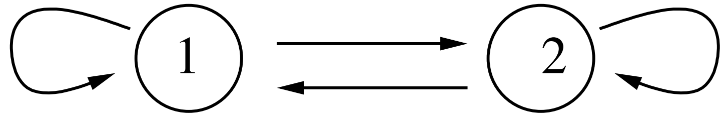
- E'values of $\alpha + 2\beta$ ($[x, x, x]^t$), α ($[x, -x, -x]^t$), α
- $x_2 = x_3$ is synchrony subspace



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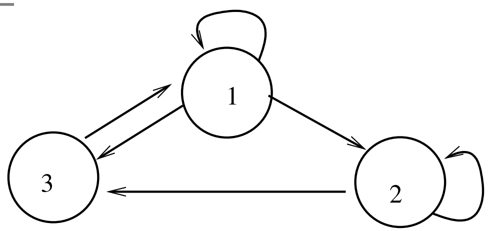

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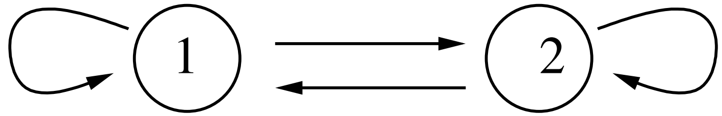

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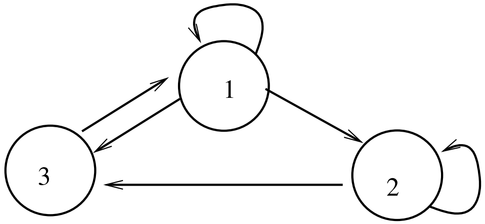
Example: Network 6



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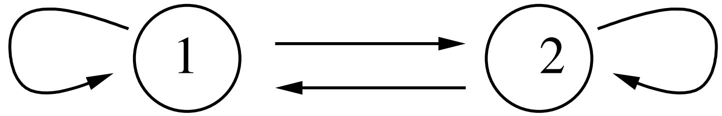

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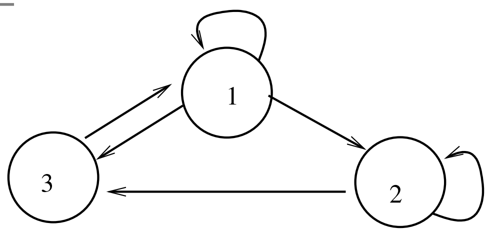

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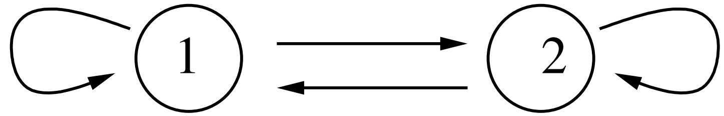

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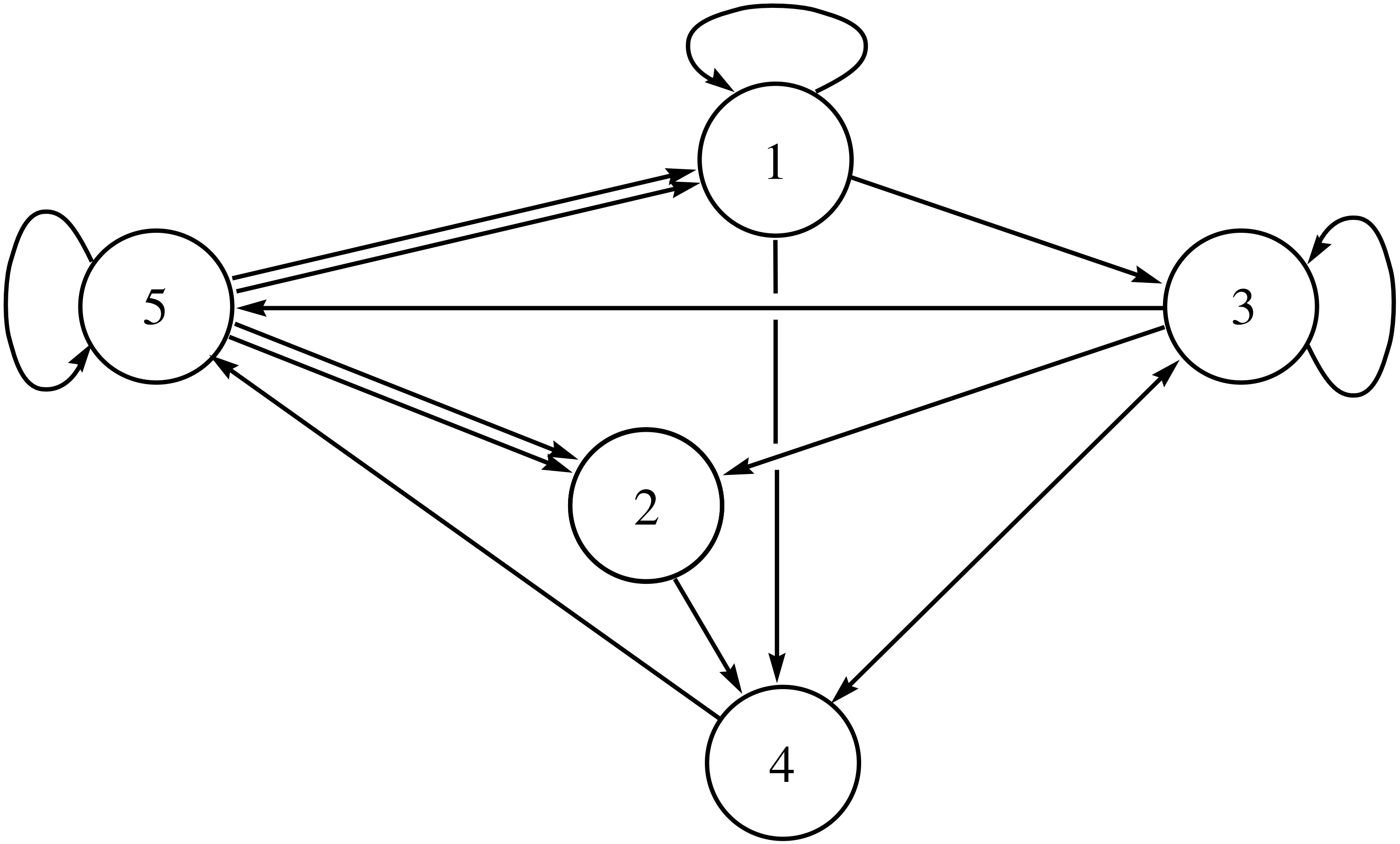
- Hopf bifurcation in $x_2 = x_3$ plane: $x_3(t) = x_2(t + \frac{1}{2})$
- 2 or 4 branches of periodic solutions
  depends on terms up to degree 5 in f
  all branches have amplitude growth rate $\lambda^{\frac{1}{2}}$

Nilpotent Hopf

Five-cell network: nilpotent Hopf generically leads to two periodic solutions with amplitude growth of λ



Nilpotent Hopf Proof

- Look for small amplitude near 2π-periodic solutions to

$$\dot{x} = F(x, \lambda),$$

where $F(0, \lambda) = 0$ and $(dF)_{0,0}$ has e'values $\pm i$

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- Define $\Phi : \mathcal{C}^1_{2\pi} \times \mathbf{R} \times \mathbf{R} \to \mathcal{C}_{2\pi}$ by

$$\Phi(x, \lambda, \tau) = (1 + \tau)\frac{dx}{ds} - F(x, \lambda); \qquad \Phi(0, \lambda, \tau) = 0$$

Nilpotent Hopf Proof (2)

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- L-S reduction to $\varphi : \mathbf{C} \times \mathbf{R} \times \mathbf{R} \to \mathbf{C}$ where

$$\varphi(e^{i\theta} z, \lambda, \tau) = e^{i\theta} \varphi(z, \lambda, \tau)$$

Let $u = |z|^2$. Then

$$\varphi(z, \lambda, \tau) = p(u, \lambda, \tau) z + q(u, \lambda, \tau) iz$$

where $p(0) = q(0) = 0$. $q_\tau(0) = -1$ in generic Hopf

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- $p(u,0,0) = u^m + \cdots$

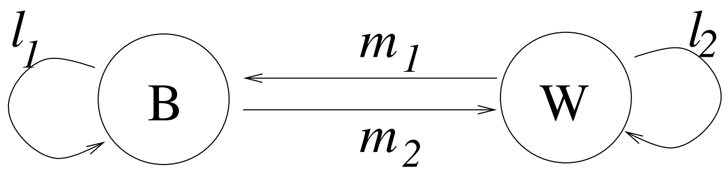
m	# solutions	growth rate
1	2	λ
2	2 or 4	$\lambda^{\frac{1}{2}}$
4	2	$\lambda^{\frac{1}{2}} \quad \lambda^{\frac{1}{6}}$

General Bifurcation Theorems

- Bifurcation to two-color solutions
- Interior symmetries

Bifurcation to Two-Color Solutions

- For each balanced two-coloring in a homogeneous network there is a codimension one bifurcation from a synchronous zero to a branch of two-colored zeros
- Quotient network for balanced two-coloring is



Homogeneous implies $k \equiv l_1 + m_1 = l_2 + m_2$

- Quotient system has form

$$\begin{aligned} \dot{x}_1 &= f(x_1, \underbrace{\overline{x_1, \cdots, x_1}}_{l_1 \text{ times}}, \underbrace{\overline{x_2, \cdots, x_2}}_{m_1 \text{ times}}, \lambda) \\ \dot{x}_2 &= f(x_2, \underbrace{\overline{x_2, \cdots, x_2}}_{l_2 \text{ times}}, \underbrace{\overline{x_1, \cdots, x_1}}_{m_2 \text{ times}}, \lambda) \end{aligned} \qquad J = \begin{pmatrix} \alpha + l_1\beta & m_1\beta \\ m_2\beta & \alpha + l_2\beta \end{pmatrix}$$

Bifurcation to Two-Color Solutions (2)

$$J = \begin{pmatrix} \alpha + l_1\beta & m_1\beta \\ m_2\beta & \alpha + l_2\beta \end{pmatrix}$$

- Eigenvalues of J are eigenvalues of matrices

  $$\alpha + k\beta \quad \text{and} \quad \alpha + (l_1 + l_2 - k)\beta$$

  Let v be eigenvector of $\alpha + (l_1 + l_2 - k)\beta$;
  then $(m_1 v, -m_2 v)^t$ is eigenvector of J.

- $\alpha + (l_1 + l_2 - k)\beta$ can have simple real eigenvalue crossing zero with nonzero speed.

- Then there exists unique branch of nontrivial solutions to $f = 0$ satisfying $x_1 \neq x_2$

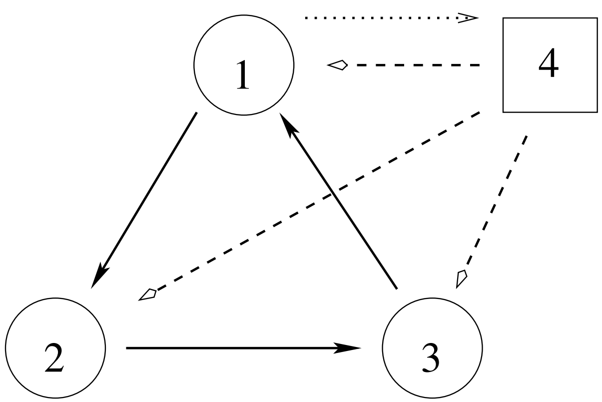
Interior Symmetries

- Let σ permute cells. σ is a symmetry iff $\forall j$

$$\sigma|_{I(j)} : I(j) \rightarrow I(\sigma(j)) \quad \text{is an input isomorphism} \qquad (1)$$

- $\mathcal{S} \subseteq \mathcal{C}$ is subset of cells. σ is an interior symmetry on $\mathcal{S}$ if
  - is the identity on $\mathcal{C} \setminus \mathcal{S}$, and
  - satisfies (1) for every $j \in \mathcal{S}$

- $\Sigma_{\mathcal{S}}$ is the group of interior symmetries on $\mathcal{S}$



$$\mathcal{S} = \{1, 2, 3\}$$
$$\Sigma_{\mathcal{S}} = \mathbf{Z}_3$$

Synchrony Breaking: Linear

- $\mathrm{Fix}(\Sigma_{\mathcal{S}}) = \mathrm{Fix}_{\mathcal{S}}(\Sigma_{\mathcal{S}}) \cup$ "$\mathcal{C} \setminus \mathcal{S}$" is flow-invariant
- Let W consist of vectors that are 0 on $\mathcal{C} \setminus \mathcal{S}$ and whose coordinates sum to 0 on orbits of $\mathcal{S}$ acting on $\mathcal{C}$.

  In example $W = \{(x_1, x_2, x_3, 0) : x_1 + x_2 + x_3 = 0\}$
- W is $\Sigma_{\mathcal{S}}$-invariant and phase space is $W \oplus \mathrm{Fix}(\Sigma_{\mathcal{S}})$

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  In example $W = \{(x_1, x_2, x_3, 0) : x_1 + x_2 + x_3 = 0\}$
- W is $\Sigma_{\mathcal{S}}$-invariant and phase space is $W \oplus \mathrm{Fix}(\Sigma_{\mathcal{S}})$
- Let X_0 be an $\Sigma_{\mathcal{S}}$-invariant equilibrium
- Synchrony space $\mathrm{Fix}(\Sigma_{\mathcal{S}})$ is invariant under $(DF)_{X_0}$; so

$$(DF)_{X_0} = \begin{bmatrix} A & 0 \\ C & B \end{bmatrix}$$

  where $A : W \to W$ commutes with $\Sigma_{\mathcal{S}}$
- Critical e'value of $A \Longrightarrow$ synchrony-breaking bifurcation

Synchrony Breaking: Nonlinear

- Assume synchrony-breaking Hopf bifurcation occurs. Let

$$\Delta \subset \Sigma_{\mathcal{S}} \times \mathbf{S}^1 \quad \text{and} \quad K = \Delta \cap \Sigma_{\mathcal{S}}$$

- Suppose E is center subspace of $A = (DF)_{X_0}|W$, and

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- Then generically there exist periodic solutions of form

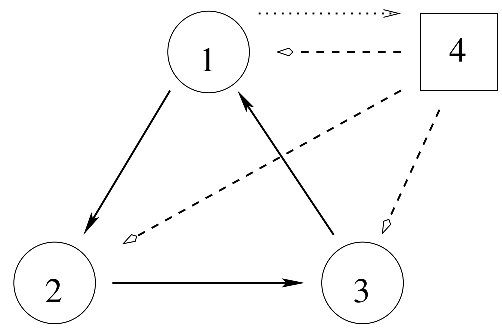
$$u(t) = w(t) + v(t) \ \in \ W \oplus \mathrm{Fix}(\Sigma_{\mathcal{S}})$$

  where

  - $v(t)$ is synchronous on K orbits of cells in $\mathcal{S}$, and
  - $w(t)$ has Δ spatio-temporal symmetry on cells in $\mathcal{S}$

Four-Cells with $\mathbf{Z}_3$-Interior Symmetry

- Synchrony-breaking Hopf in



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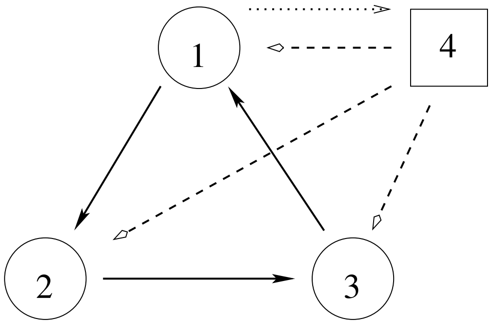
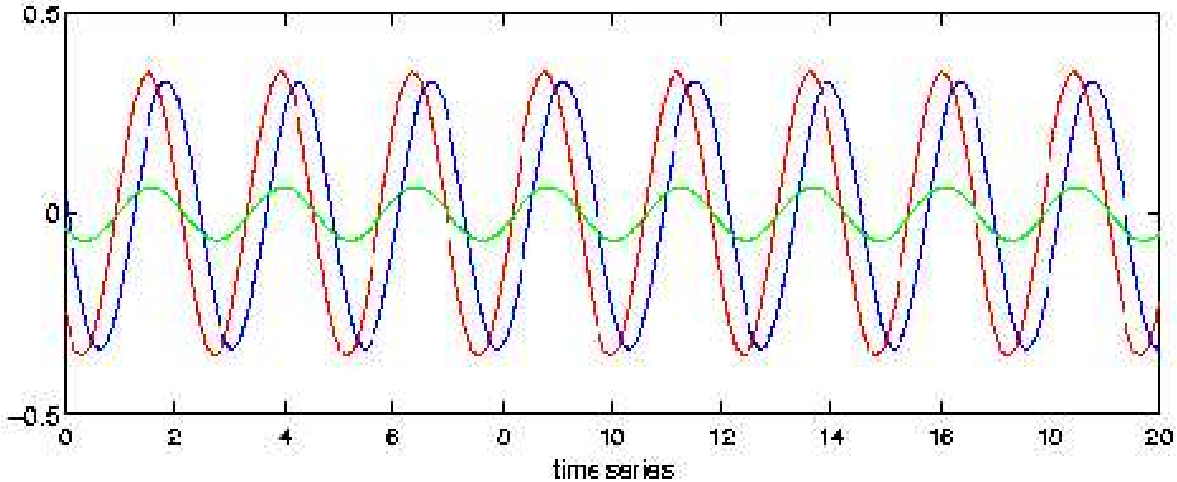


- 

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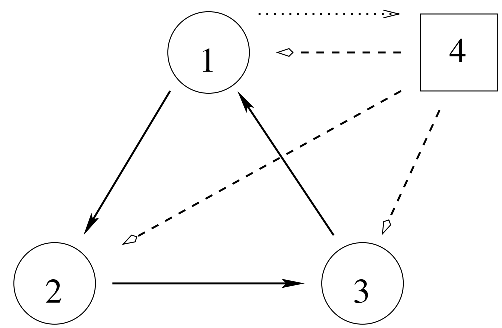
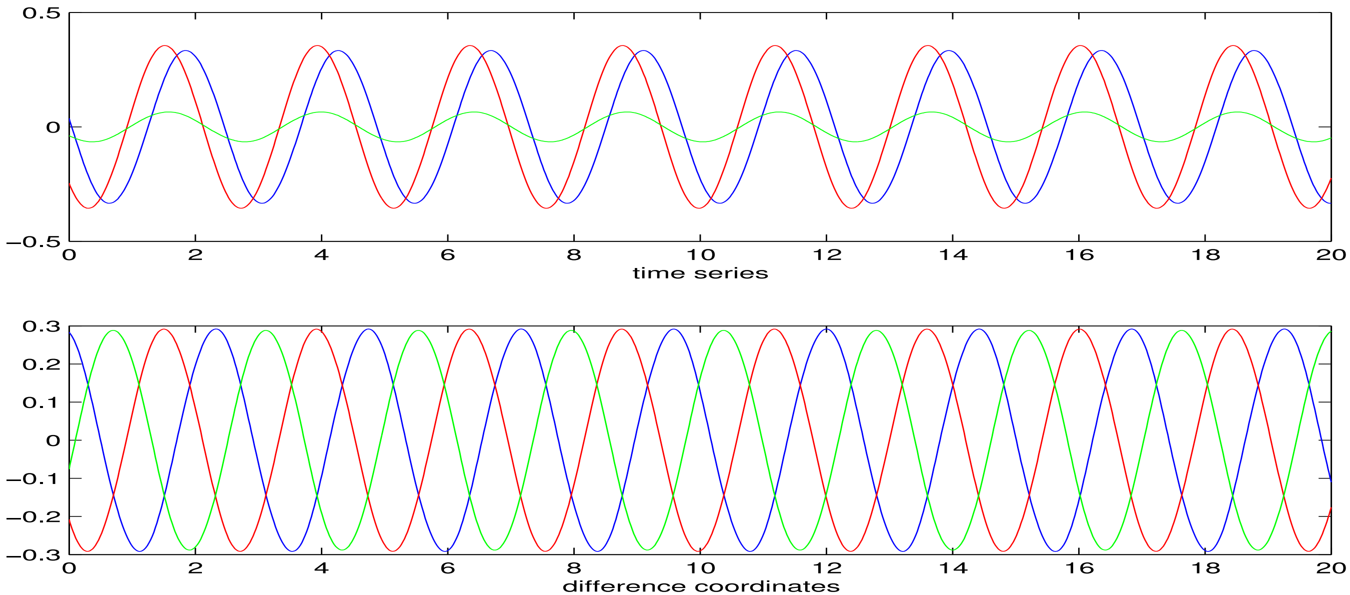


- leads to



- Lower panel shows hidden rotating wave in time series

$$x_1 - x_2 \qquad x_2 - x_3 \qquad x_3 - x_1$$

Bifurcation Summary

- Synchrony breaking bifurcations: differences due to
  - partial synchrony (quotient networks)
  - interior symmetry
  - network architecture
- Patterns of oscillations at bifurcation due to network architecture
- Two-color patterns