Coupled Systems: Theory & Examples

Lecture 2

Synchrony and Balanced Coloring

Reference: Golubitsky and Stewart. Nonlinear dynamics of networks: the groupoid formalism. *Bull. Amer. Math. Soc.* **43** No. 3 (2006) 305–364

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Thanks

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Synchrony Subspaces

- A polydiagonal is a subspace

$$\Delta = \{x : x_c = x_d \quad \text{for some subset of cells}\}$$

- A **synchrony subspace** is a flow-invariant polydiagonal

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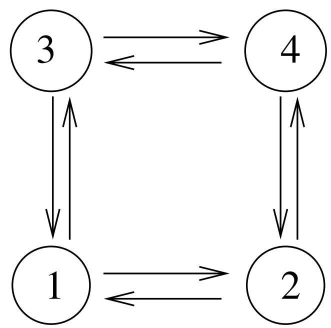
- A **synchrony subspace** is a flow-invariant polydiagonal

- Let $\sigma =$ be a permutation. Then $\mathrm{Fix}(\sigma)$ is a polydiagonal



- $\mathrm{Fix}((2\ 3)(1\ 4)) = \{(x_1, x_2, x_3, x_4) : x_2 = x_3; x_1 = x_4\}$

- Let Σ be a subgroup of network permutation symmetries. Then $\mathrm{Fix}(\Sigma)$ is a synchrony subspace

Coupled Cell Overview

Coupled cell system: discrete space, continuous time system

Has information that cannot be understood by phase space theory alone

- symmetry — synchrony and phase shifts
- network architecture — input sets, balanced colorings, quotient networks

Stewart, G., and Pivato (2003); G., Stewart, and Török (2005)

Coupled Cell Overview

Coupled cell system: discrete space, continuous time system

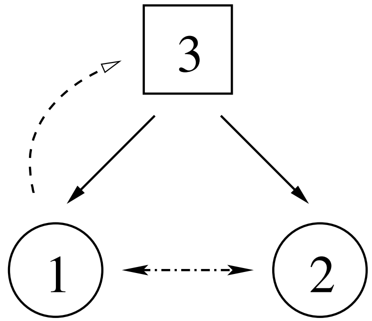
Has information that cannot be understood by phase space theory alone

- symmetry — synchrony and phase shifts
- network architecture — input sets, balanced colorings, quotient networks
- Primary Question — Which aspects of coupled cell dynamics are due to network architecture?
- Beginner Question: — Are all synchrony spaces fixed-point spaces?

  Answer: No

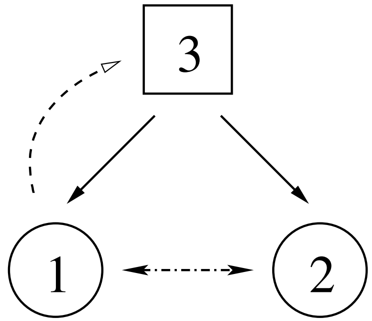
Stewart, G., and Pivato (2003); G., Stewart, and Török (2005)

Asymmetric Three-Cell Network



$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2, x_3) & x_1 &\in \mathbf{R}^k \\ \dot{x}_2 &= f(x_2, x_1, x_3) & x_2 &\in \mathbf{R}^k \\ \dot{x}_3 &= g(x_3, x_1) & x_3 &\in \mathbf{R}^\ell \end{aligned}$$

Asymmetric Three-Cell Network



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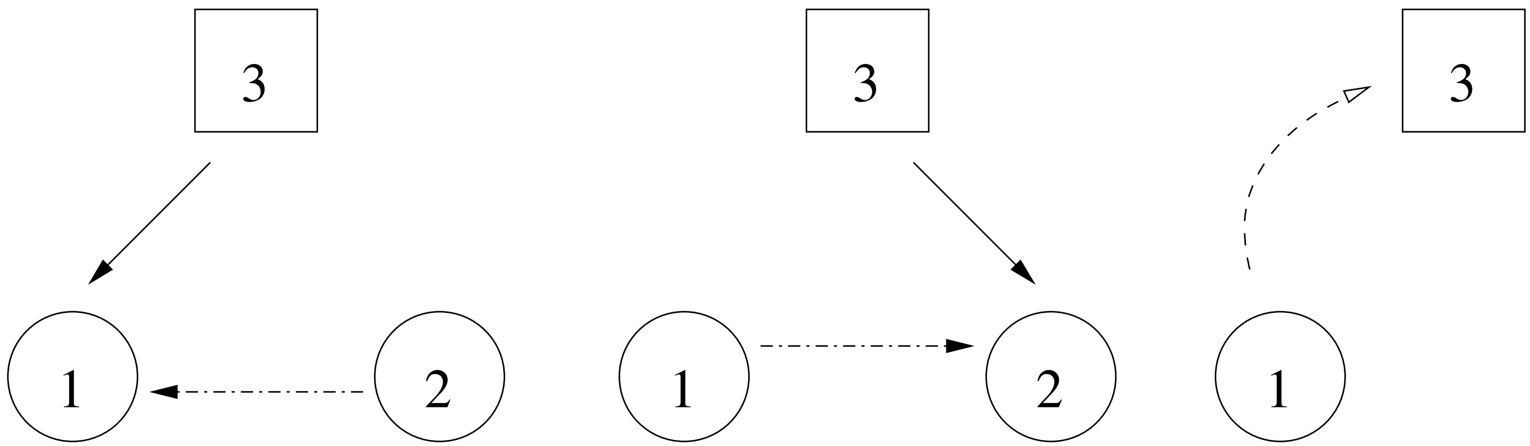
- $Y = \{x : x_1 = x_2\}$ is flow-invariant

  Restrict equations $\dot{x}_1, \dot{x}_2$ to Y:
  $$\begin{aligned} \dot{x}_1 &= f(x_1, x_1, x_3) \\ \dot{x}_2 &= f(x_1, x_1, x_3) \end{aligned}$$

- Robust synchrony exists in networks without symmetry
- Cells 1 and 2 are **identical within the network**

Input Sets

- Input set of cell j: the arrows that connect to cell j
- Key idea: cells 1, 2 have isomorphic input sets



Coupled Cell Network Definition

- A set of *cells* $\mathcal{C} = \{1, \ldots, N\}$

  Each cell has its own phase space

- An equivalence relation on cells

  Equivalent cells have the same phase space

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- Equivalent arrows have equivalent tail and head cells

Local Network Symmetry

- coupled cell networks represented by directed graphs

  For each class of cells choose node symbol $\bigcirc, \Box, \triangle$

  For each class of arrows choose arrow symbol $\longrightarrow, \Rightarrow, \rightsquigarrow$

Local Network Symmetry

- coupled cell networks represented by directed graphs

  For each class of cells choose node symbol $\bigcirc, \square, \triangle$

  For each class of arrows choose arrow symbol $\rightarrow, \Rightarrow, \rightsquigarrow$

- **Input isomorphism** is arrow type preserving bijection

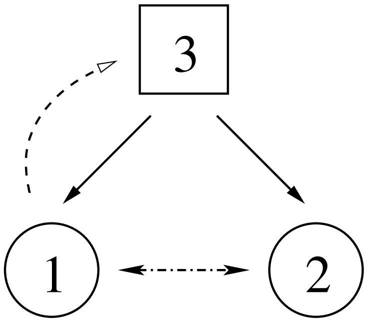
$$\beta : I(c) \rightarrow I(d)$$

  Input isomorphic cells have same equations

- $\mathcal{B}_G =$ groupoid of all input isomorphisms

- Coupled cell systems: ODEs that commute with $\mathcal{B}_G$

Asymmetric Three-Cell Network (2)



$$
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\dot{x}_3 &= g(x_3, x_1) & x_3 &\in \mathbf{R}^\ell
\end{aligned}
$$

- Two cell types: $\bigcirc \quad \Box$

  Three arrow types: $\longrightarrow \quad \dashrightarrow \quad \dashrightarrow$

- Equivalent cells 1 and 2 have same phase space $\mathbf{R}^k$

- Cells 1 and 2 are input isomorphic

  Have same systems of differential equations f

Balanced Coloring

- Let Δ be a polydiagonal
- Color equivalent cells the same color
  if cell coord's in Δ are equal
- Coloring is **balanced** if all cells with same color receive equal number of inputs from cells of a given color and a given arrow type

Stewart, G., and Pivato (2003); G., Stewart, and Török (2005)

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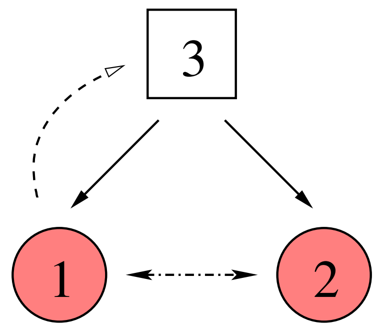

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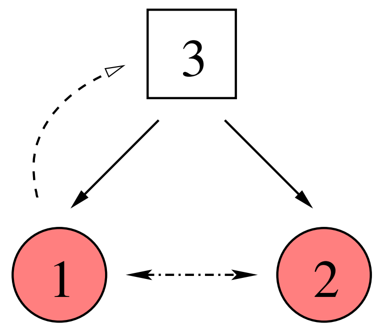
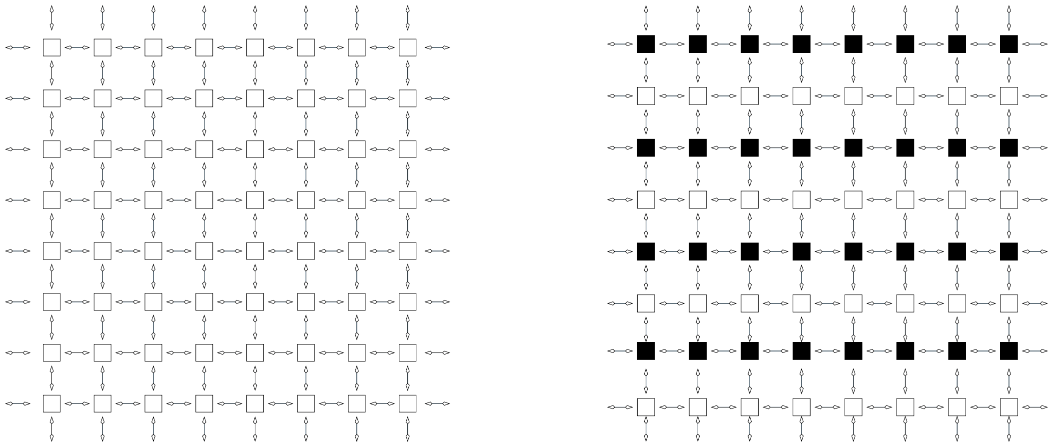


- **Theorem**: **synchrony subspace $\Longleftrightarrow$ balanced**

Stewart, G., and Pivato (2003); G., Stewart, and Török (2005)

2D-Lattice Dynamical Systems

- Consider square lattice with nearest neighbor coupling
- Form a two-color balanced relation

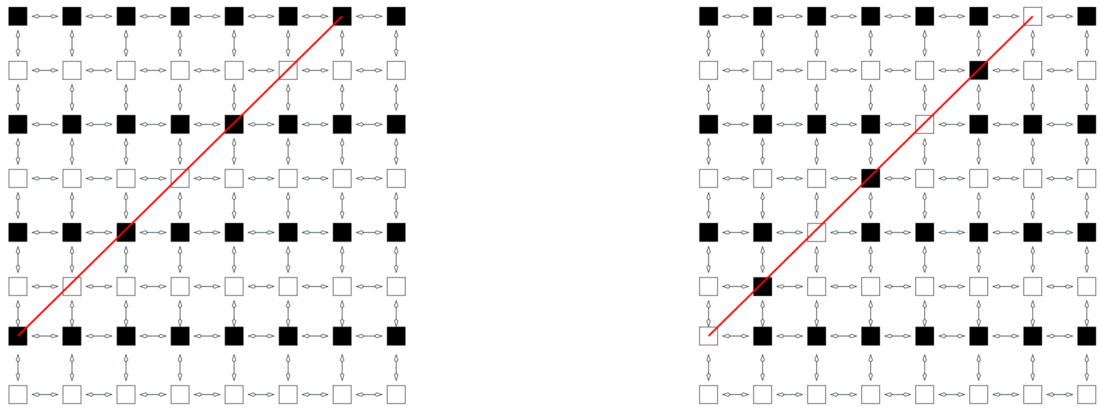


- Each black cell connected to two black and two white
  Each white cell connected to two black and two white

Stewart, G. and Nicol (2004)

Lattice Dynamical Systems (1)

- On Black/White diagonal interchange black and white



Result is balanced

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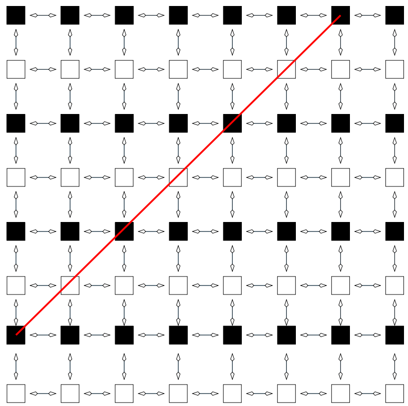
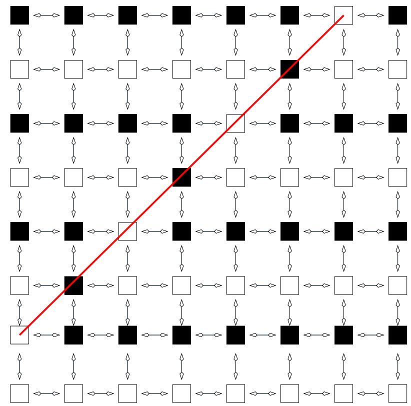
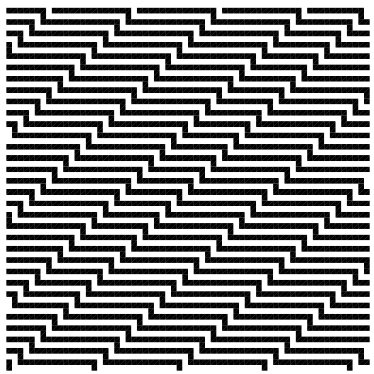
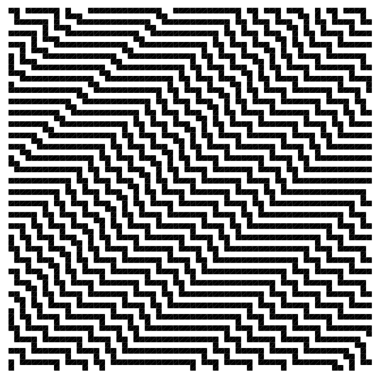




  Result is balanced

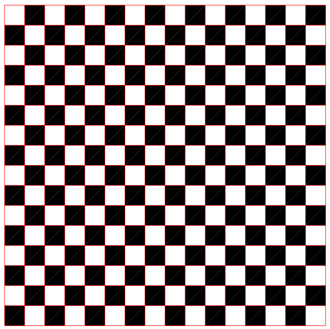
- Continuum of different synchrony subspaces



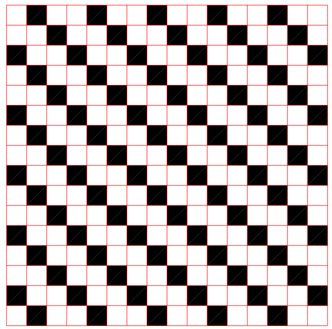


Lattice Dynamical Systems (2)

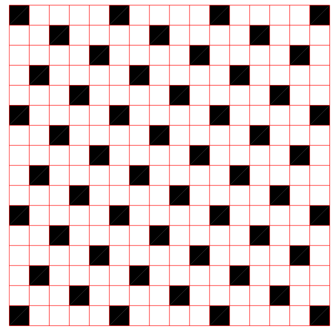
There are eight isolated balanced two-colorings on square lattice with nearest neighbor coupling



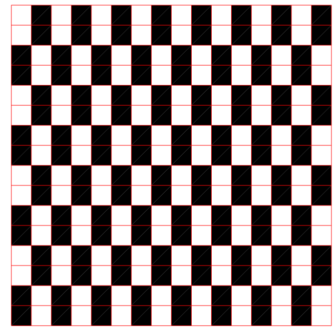
$4B \to W; 4W \to B$



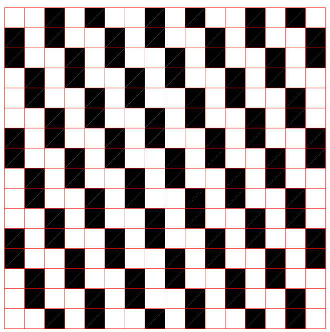
$2B \to W; 4W \to B$



$1B \to W; 4W \to B$



$3B \to W; 3W \to B$



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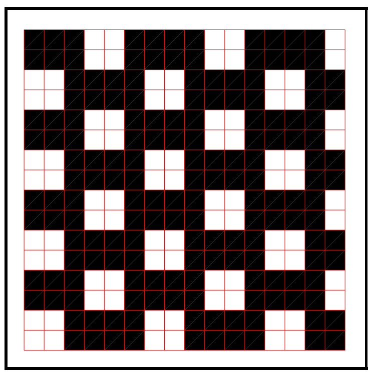
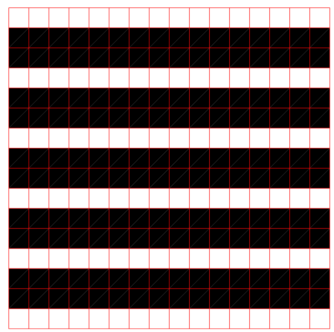


$2B \to W; 1W \to B$



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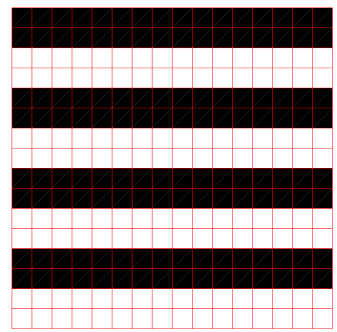


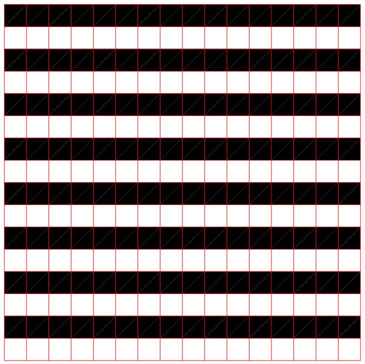
$1B \to W; 1W \to B$

Wang and G. (2004)

□ indicates nonsymmetric solution

Lattice Dynamical Systems (3)

- There are two infinite families of balanced two-colorings



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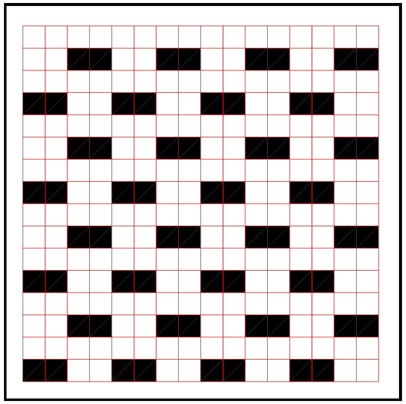

$1B \to W; 3W \to B$

- Up to symmetry these are all balanced two-colorings

Lattice Dynamical Systems (4)

- Architecture is really important

Antoneli, Dias, G., and Wang (2004)

Lattice Dynamical Systems (4)

- Architecture is really important
- For square lattice with nearest and next nearest neighbor coupling
  - No infinite families
  - For each k a finite number of balanced k colorings
  - All balanced colorings are doubly-periodic

Antoneli, Dias, G., and Wang (2004)

Windows 1

$$W_0 = \{0\} \quad \text{and} \quad W_{i+1} = I(W_i)$$

- Input set of $U = I(U) = \{c \in \mathcal{C} : c$ connects to cell in $U\}$
- $\mathcal{L} = W_0 \cup W_1 \cup \cdots$
- W_{k-1} contains all k colors of a balanced k-coloring

Windows 2

- $\mathrm{bd}(U) = I(U) \setminus U$

  $c \in \mathrm{bd}(U)$ is 1-determined if color of c is determined by colors of cells in U and fact that coloring is balanced

- Define p-determined inductively

Windows 2

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- Define p-determined inductively

- All NN boundary cells are not 1-determined

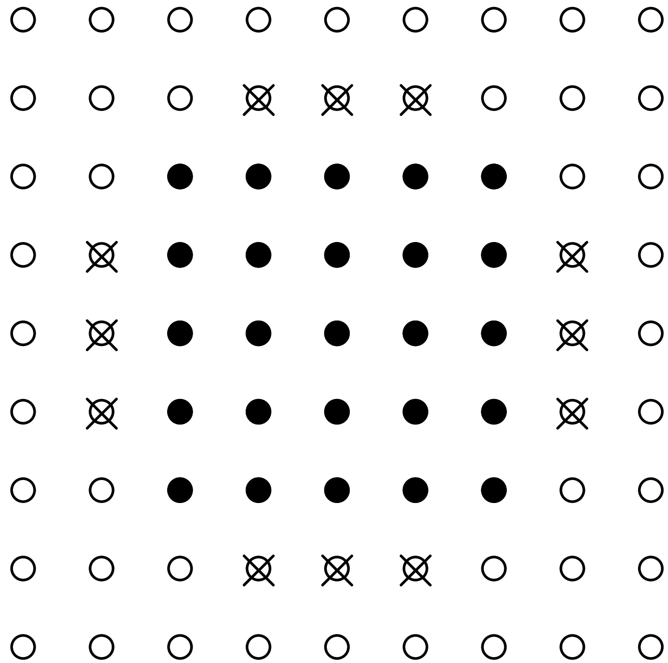
  NNN boundary cells are 1- or 2-determined

Windows 3: Square Lattice

Nearest and next nearest neighbor coupling

Black $\bullet$ indicates cells whose colors are known

$\times$ indicates 1-determined cells of W_2



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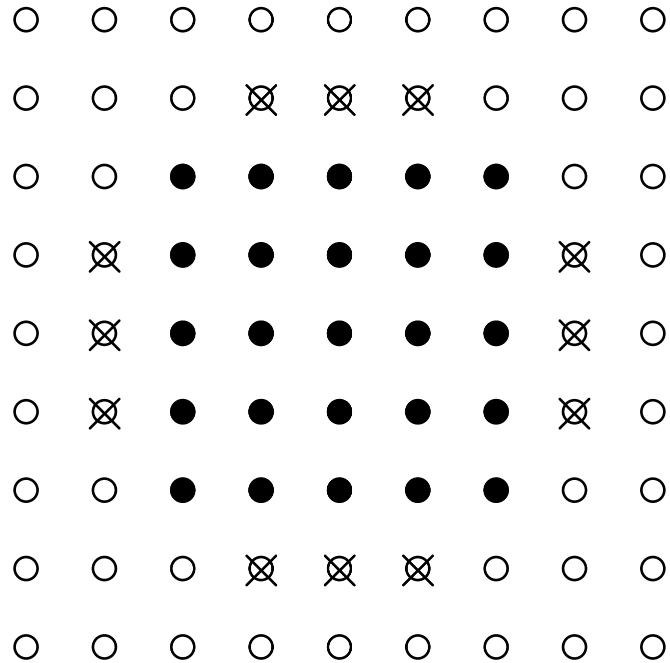

- Three cells in corners of square are 2-determined

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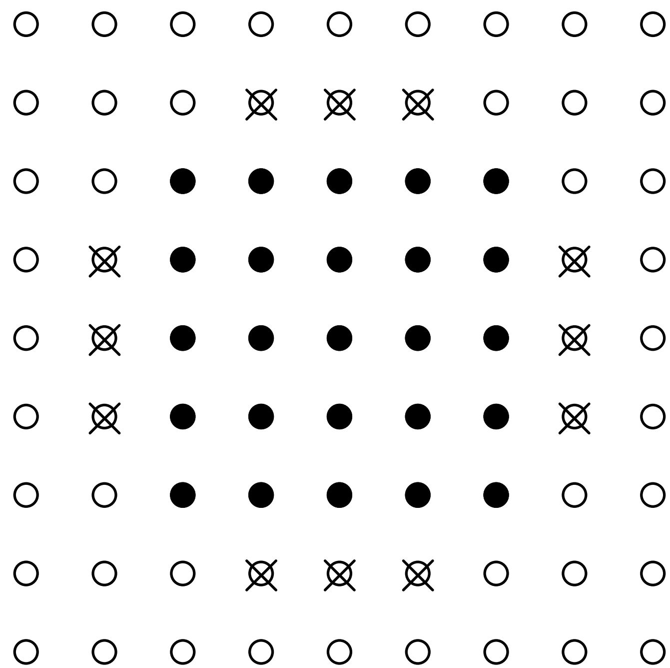
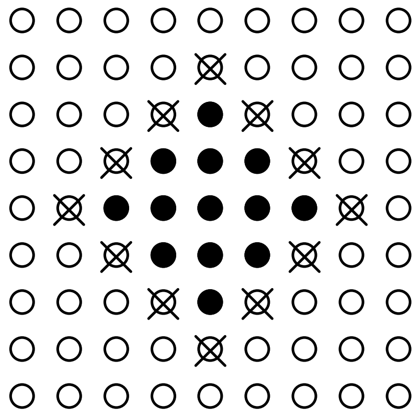


- Three cells in corners of square are 2-determined
- U determines its boundary if all $c \in \mathrm{bd}(U)$ are p-determined for some p
- W_i determines its boundary for all $i \geq 2$

Windows 4

Square lattice with Nearest neighbor coupling

W_2 is not 1-determined



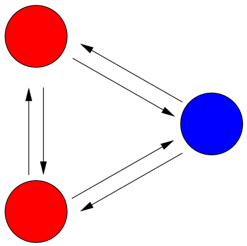
Windows 5

- W_{i_0} is a window if W_i determines its boundary $\forall\ i \geqslant i_0$
- Suppose a balanced k-coloring restricted to $\mathrm{int}(W_i)$ for some $i \geqslant i_0$ contains all k colors. Then
  - k-coloring is uniquely determined on whole lattice by its restriction to W_i
- Thm: Suppose lattice network has window. Fix k. Then
  - Finite number of balanced k-colorings on $\mathcal{L}$
  - Each balanced k-coloring is multiply-periodic

Antoneli, Dias, G., and Wang (2004)

Quotients: Self-Coupling & Multiarrows

- Balanced two-coloring of bidirectional ring

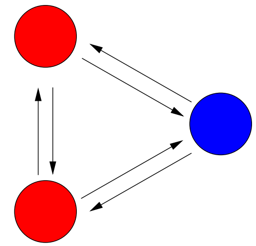


$$\begin{aligned}\dot{x}_1 &= f(x_1, x_2, x_3)\\ \dot{x}_2 &= f(x_2, x_3, x_1)\\ \dot{x}_3 &= f(x_3, x_1, x_2)\end{aligned}$$

where $f(x, y, z) = f(x, z, y)$

Quotients: Self-Coupling & Multiarrows

- Balanced two-coloring of bidirectional ring



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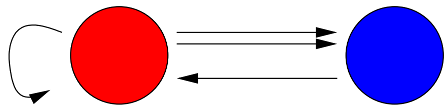
- Quotient network:



$$\begin{aligned} \color{red}{\dot{x}_1} &= f(\color{red}{x_1}, \color{red}{x_1}, \color{blue}{x_3}) \\ \color{blue}{\dot{x}_3} &= f(\color{blue}{x_3}, \color{red}{x_1}, \color{red}{x_1}) \end{aligned} \qquad \text{where } f(x,y,z) = f(x,z,y)$$

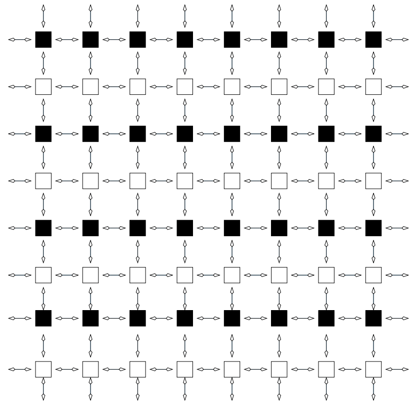
Quotient Networks

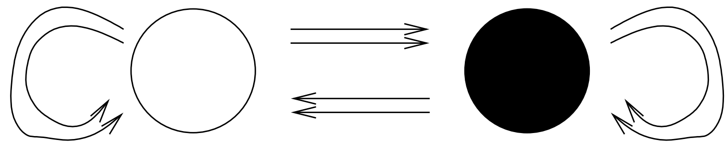
- Given cell network $\mathcal{C}$ and balanced coloring $\bowtie$
- Define *quotient network*:
  - $\mathcal{C}_{\bowtie} = \{\overline{c} : c \in \mathcal{C}\} = \mathcal{C}/\bowtie$
  - Quotient arrows are projections of $\mathcal{C}$ arrows
- Thm: Admissible DE restricts to quotient admissible DE
  Quotient admissible DE lifts to admissible DE

G., Stewart, and Török (2005)

Multiple Equilibria in LDE

- Recall **balanced** relation





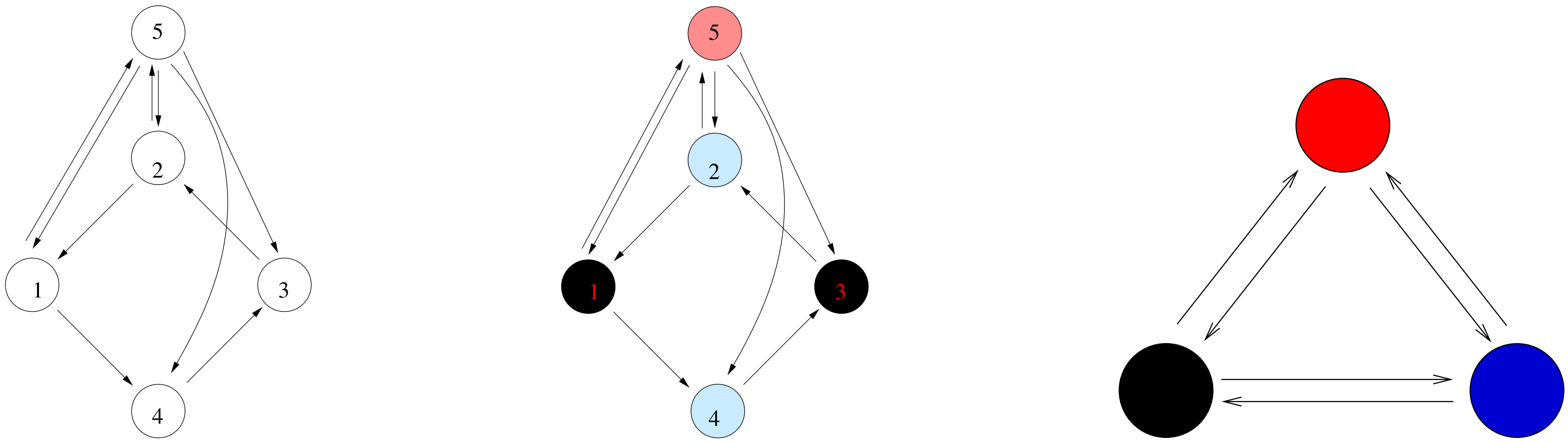
- LDE on square lattice has form

$$\dot{x}_{ij} = f(x_{ij}, \overline{x_{i+1,j}, x_{i-1,j}, x_{i,j+1}, x_{i,j-1}})$$

- Quotient network: $$\begin{aligned} \dot{B} &= f(B, \overline{B, B, W, W}) \\ \dot{W} &= f(W, \overline{W, W, B, B}) \end{aligned}$$

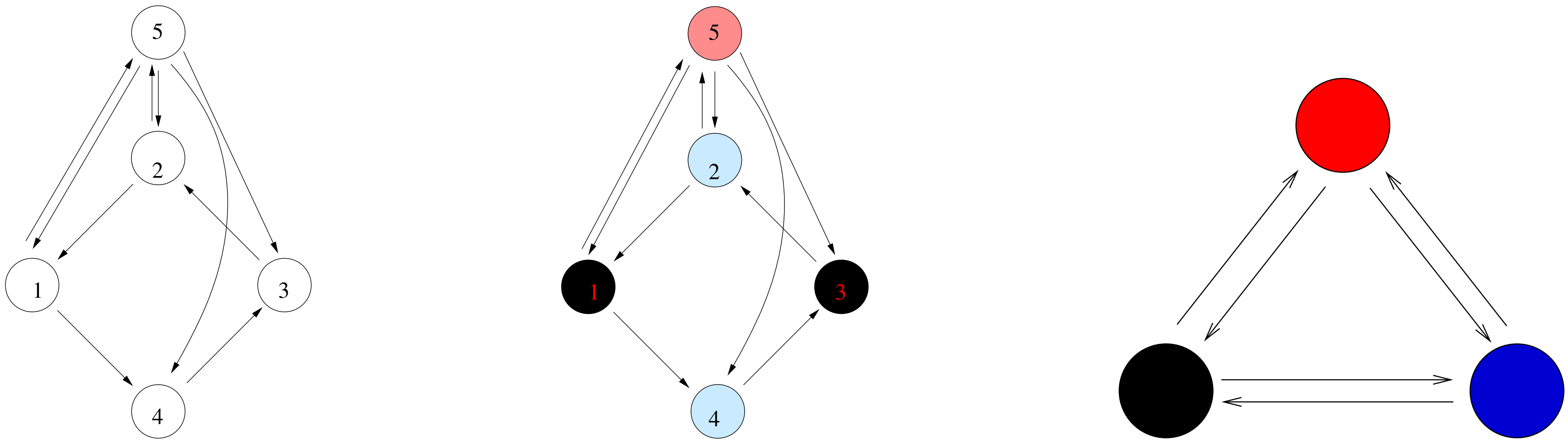
- All quotient networks in continuum are identical
  One equilibrium implies a continuum of equilibria

Asym Network; Symmetric Quotient

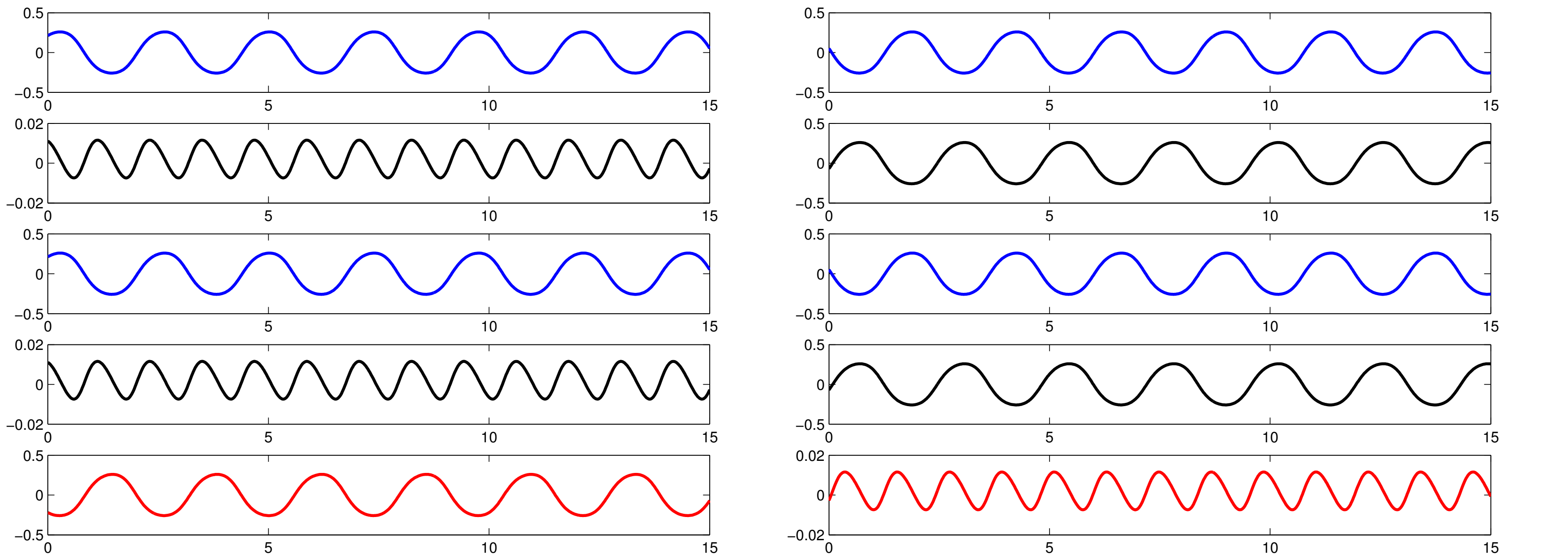


- Quotient is bidirectional 3-cell ring with D_3 symmetry

Asym Network; Symmetric Quotient



- Quotient is bidirectional 3-cell ring with $\mathbf{D}_3$ symmetry



Population Models

- Cell system is homogeneous if cells are input equivalent
- Cell system has identical edges if all arrows are equivalent
- Cell system is regular if homogeneous & identical edges

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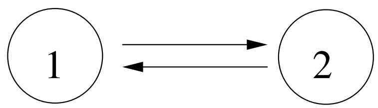
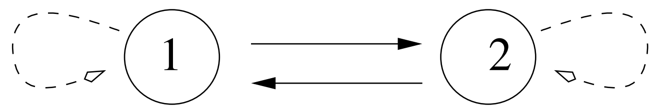
Population Models

- Cell system is homogeneous if cells are input equivalent
- Cell system has identical edges if all arrows are equivalent
- Cell system is regular if homogeneous & identical edges
- Any quotient of a regular network is regular
- Two networks are ODE-equivalent if they have the same admissible vector fields. For example





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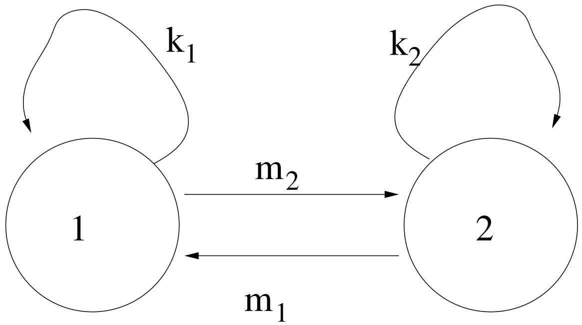
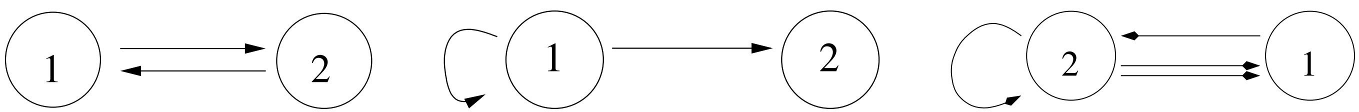
$$g(a,b,c) = f(a,c) \quad \text{and} \quad f(a,b) = g(a,a,b)$$

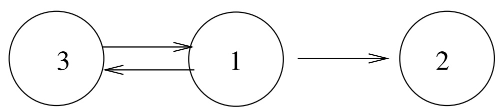
Regular Two-cell Networks



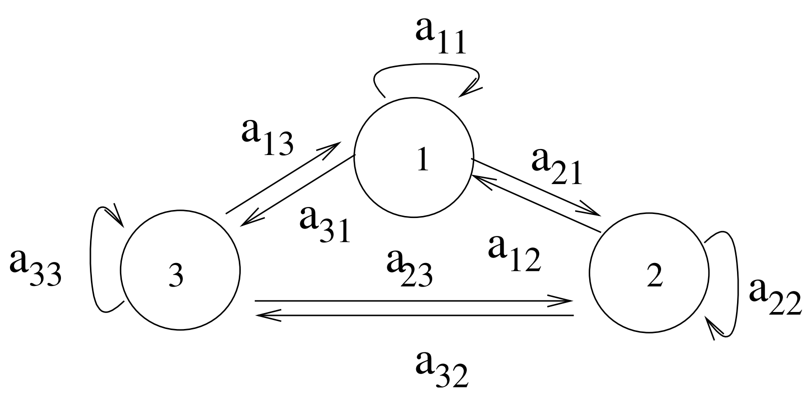
- valency = # inputs in each cell $n = k_1 + m_1 = k_2 + m_2$
- WLOG $k_1 \leq k_2$
- Dias & Stewart: Two networks are ODE-equivalent if their linear admissible vector fields are identical
- Up to ODE-equivalence, can assume $k_1 = 0$ and $m_1 = n$
- There are three two-cell networks with valency 1 or 2



- Lift



Regular Three Cell Networks



- a_{ij} = number of inputs cell i receives from cell j
- *Valency* $= n =$ total number of inputs per cell

$$a_{i1} + a_{i2} + a_{i3} = n \qquad \text{for} \quad j = 1, 2, 3$$

- Up to ODE-equivalence there are

$\boxed{\text{34 regular three-cell valency 2 networks}}$

Leite and G. (2005)

Asymptotically Stable Equilibria

- Theorem: Given balanced k-coloring with polydiagonal Δ and $X_0 \in \Delta$. Then X_0 is an asymptotically stable equilibrium for some admissible system

- Can assume homogeneous network with 1D dynamics

- X_0 has at most k distinct coordinates with distinct values $x_0^1, \ldots, x_0^\ell$. Choose interpolation polynomial g such that

$$g(x_0^i) = 0 \quad \text{and} \quad g'(x_0^i) = -1 \quad \text{for } 1 \leq i \leq \ell$$

- Then system $\dot{x}_i = g(x_i)$ has equilibrium at X_0 with Jacobian equal to $-I$.

  So X_0 is asymptotically stable equilibrium

Detection of Patterns by Equilibria

- Let $X_0 = (x_1^0, \ldots, x_N^0)$. Let

  $$\Delta_{X_0} = \{x : x_c = x_d \text{ iff } c \sim_C d \text{ and } x_c^0 = x_d^0\}$$

  Δ_{X_0} is the smallest polydiagonal that contains all points with the same pattern of synchrony.

- Let X_0 be a hyperbolic equilibrium of a C^1 admissible cell system. The pattern of synchrony defined by X_0 is rigid if in each C^1 perturbed admissible system the hyperbolic equilibrium near X_0 remains in Δ_{X_0}

- Theorem: The equilibrium X_0 is rigid if and only if the coloring associated to Δ_{X_0} is balanced

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  iff balanced
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- genericity in quotient network
  implies
  genericity in original network