京都大学

京都大学

京都大学

R.I.M.S. Workshop

Dynamical Systems and Applications: Recent Progress

京都大学

Lecture 2

Continuation of Doubly Symmetric Solutions in Reversible Systems

UNIVERSITEIT GENT

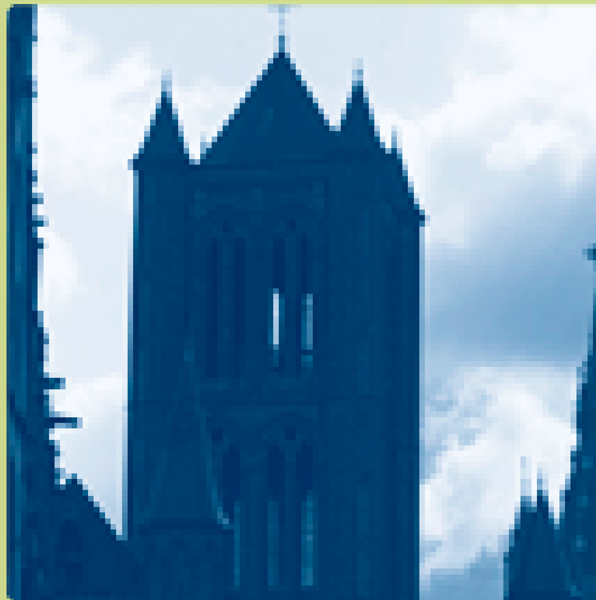
André Vanderbauwhede

Ghent University, Belgium

avdb@cage.ugent.be

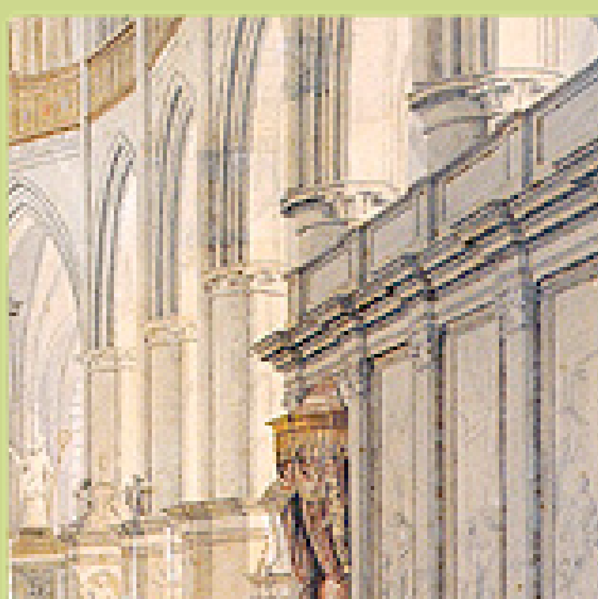
Faculty of Sciences

UNIVERSITEIT GENT



André Vanderbauwhede

Ghent University, Belgium
avdb@cage.ugent.be



Faculty of Sciences

Joint Work with

- Francisco Javier Muñoz Almaraz
- Emilio Freire
- Jorge Galán Vioque

Joint Work with

- Francisco Javier Muñoz Almaraz
- Emilio Freire
- Jorge Galán Vioque

Universidad de Sevilla

Continuation

Continuation

Implicit Function Theorem

The Problem

Given: a smooth mapping

$$f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

and a point $x_0 \in \mathbb{R}^m$.

The Problem

Given: a smooth mapping

$$f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

and a point $x_0 \in \mathbb{R}^m$.

We want to solve

$$f(x) = f(x_0)$$

locally near x_0.

The Problem

If f is a submersion at x_0, i.e. if

$$\operatorname{Im} Df(x_0) = \mathbb{R}^n,$$

(this requires $m \geq n$), then the solution set of

$$f(x) = f(x_0)$$

is locally near x_0 a smooth $(m-n)$-dimensional manifold.

The Implicit Function Theorem

Some Examples where the submersivity condition is NOT satisfied, but the solution set is still a smooth manifold

Example 1

Suppose that

$$\varphi_i(f(x)) = 0, \ \forall x \in \mathbb{R}^m, \ (1 \leq i \leq k \leq n),$$

where the $\varphi_i : \mathbb{R}^n \to \mathbb{R}$ are smooth functions.

Assume that the vectors

$$\nabla \varphi_i(f(x_0)) \quad (1 \leq i \leq k)$$

are linearly independent.

Example 1

This means that f maps $\mathbb{R}^n$ into the codimension k submanifold

$$\mathcal{N} := \{y \in \mathbb{R}^n \mid \varphi_i(y) = 0, 1 \leq i \leq k\}.$$

Example 1

This means that f maps $\mathbb{R}^n$ into the codimension k submanifold

$$\mathcal{N} := \{y \in \mathbb{R}^n \mid \varphi_i(y) = 0, 1 \leq i \leq k\}.$$

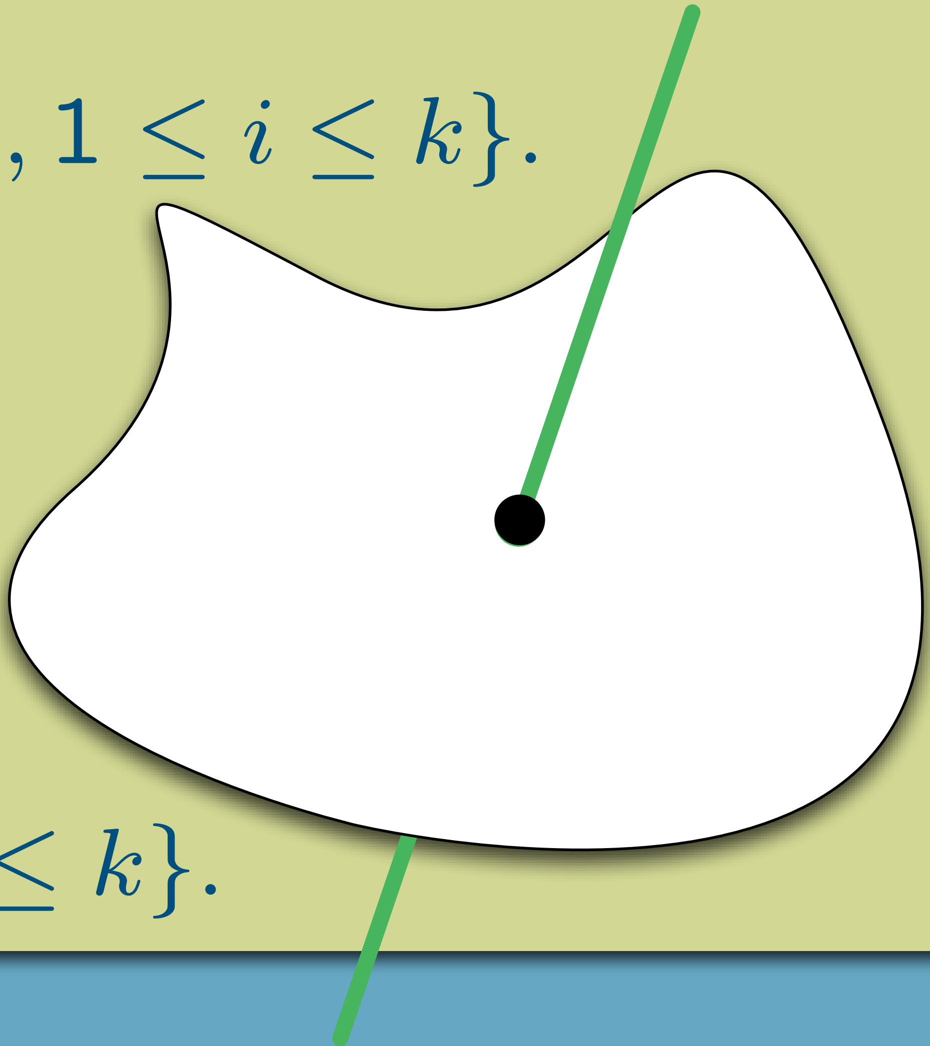
We set

$$W := \mathrm{span}_{\mathbb{R}} \{\nabla \varphi_i(f(x_0)) \mid 1 \leq i \leq k\}.$$

Example 1

This means that f maps $\mathbb{R}^n$ into the codimension k submanifold

$$\mathcal{N} := \{y \in \mathbb{R}^n \mid \varphi_i(y) = 0, 1 \leq i \leq k\}.$$



We set

$$W := \mathrm{span}_{\mathbb{R}} \{\nabla \varphi_i(f(x_0)) \mid 1 \leq i \leq k\}.$$

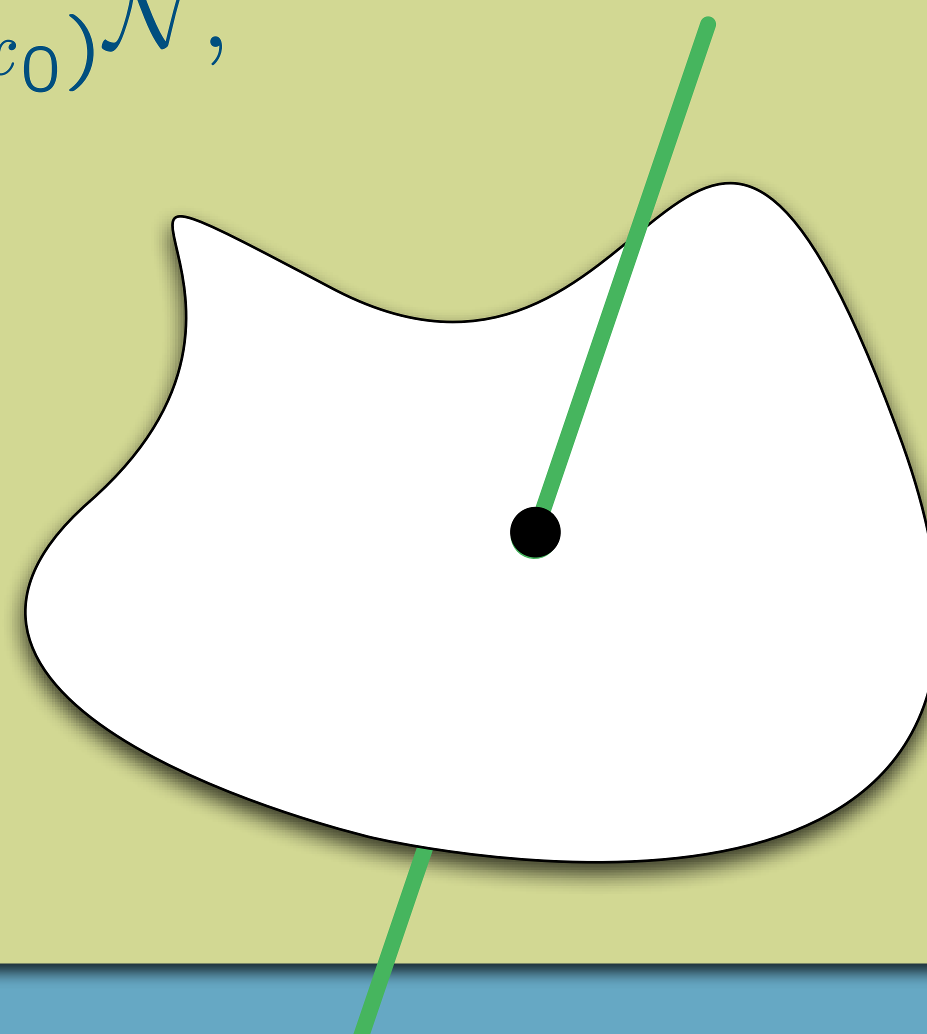
Example 1

Assume that $f : \mathbb{R}^m \to \mathcal{N}$ is a submersion at x_0, i.e.

$$\mathrm{Im} Df(x_0) = \mathsf{T}_{f(x_0)}\mathcal{N},$$

or equivalently:

$$\mathbb{R}^n = \mathrm{Im} Df(x_0) \oplus W.$$

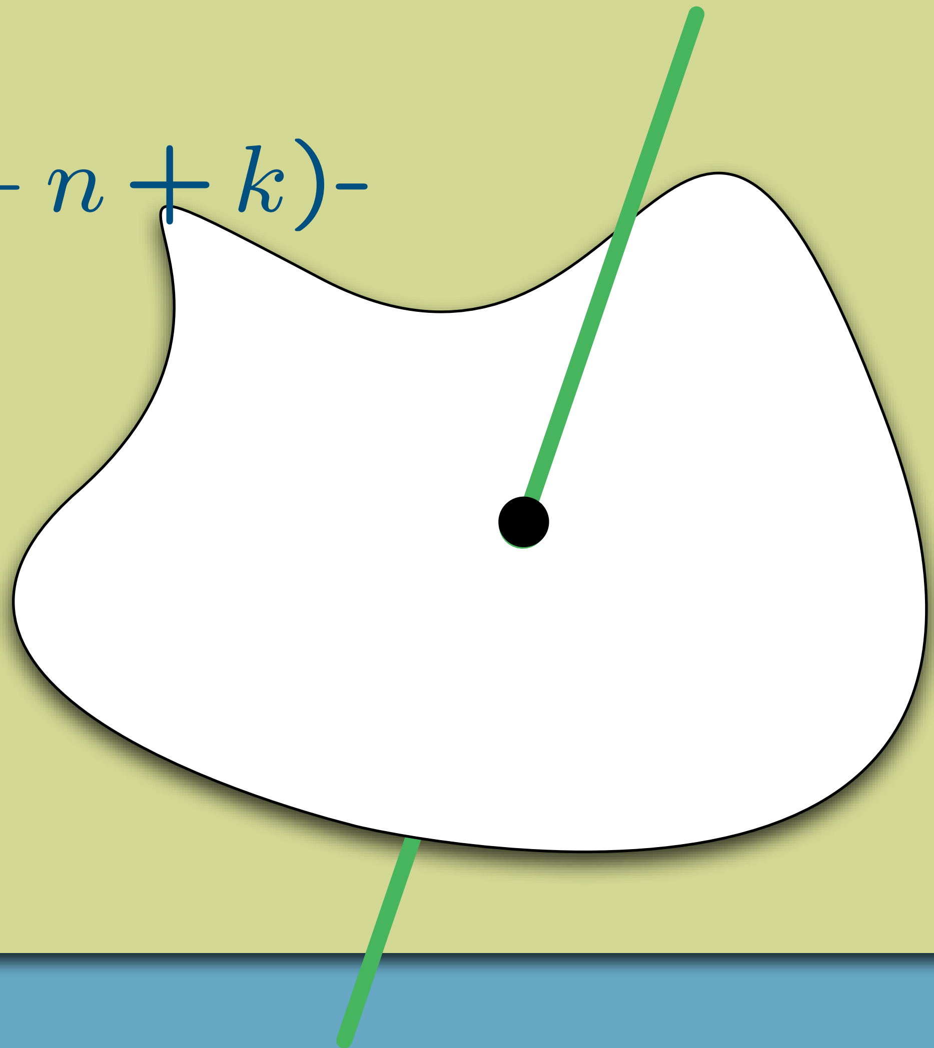


Example 1

Then the solution set of

$$f(x) = f(x_0)$$

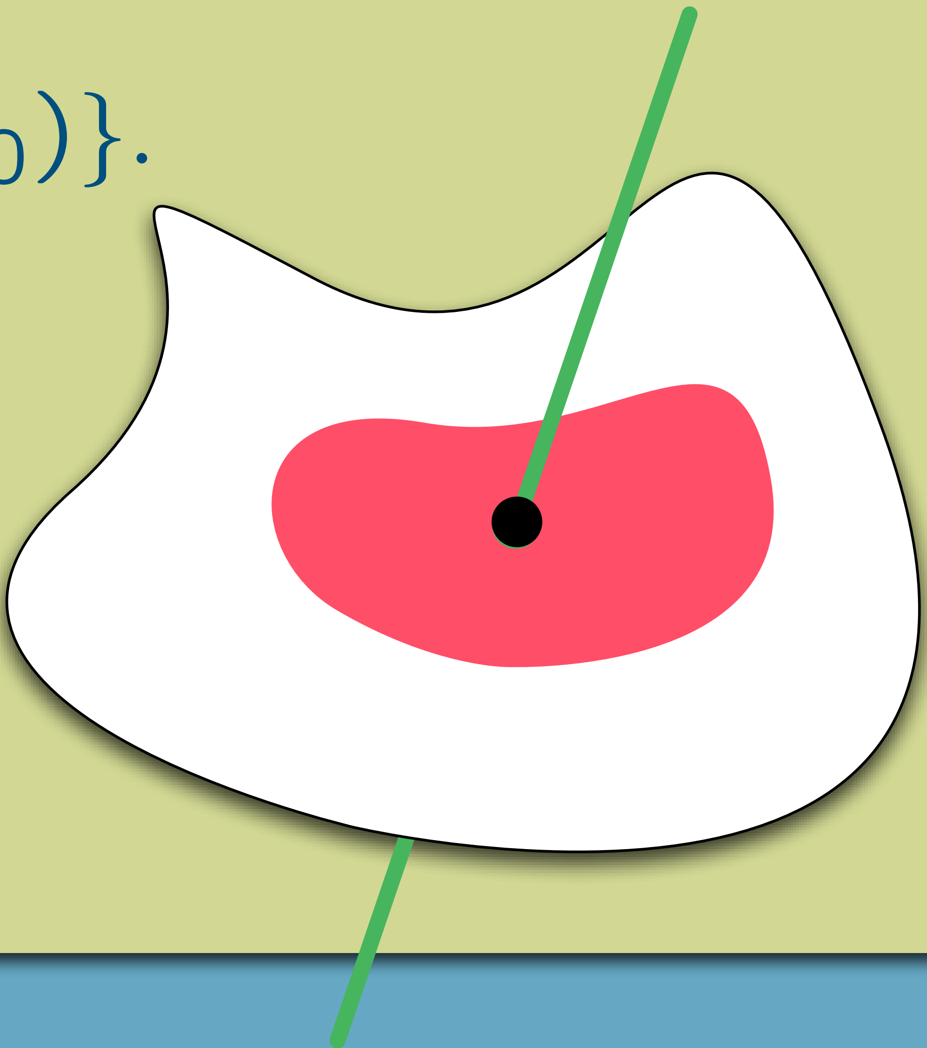
is locally near x_0 a smooth $(m-n+k)$-dimensional submanifold.



Example 1

For a sufficiently small neighborhood O of x_0 we have

$$f(O) \cap (f(x_0) + W) = \{f(x_0)\}.$$



Example 2

Here we assume that f has the form

$$f(x) = \varphi(x)g(x)$$

for some smooth mappings

$$\varphi : \mathbb{R}^m \to \mathbb{R} \quad \text{and} \quad g : \mathbb{R}^m \to \mathbb{R},$$

and such that

$$\varphi(x_0) = 0, \;\; \nabla\varphi(x_0) \neq 0 \;\text{ and }\; g(x_0) \neq 0.$$

Example 2

Here we assume that f has the form

$$f(x) = \varphi(x)g(x)$$

for some smooth mappings

$$\varphi : \mathbb{R}^m \to \mathbb{R} \quad \text{and} \quad g : \mathbb{R}^m \to \mathbb{R}^{n},$$

and such that

$$\varphi(x_0) = 0, \;\; \nabla\varphi(x_0) \neq 0 \;\text{ and }\; g(x_0) \neq 0.$$

Example 2

Clearly the equation $f(x) = f(x_0)$ reduces in this case (and for x near x_0) to the equation

$$\varphi(x) = 0,$$

and the solution set is a smooth $(m-1)$-dimensional manifold.

Example 2

Clearly the equation $f(x) = f(x_0)$ reduces in this case (and for x near x_0) to the equation

$$\varphi(x) = 0,$$

and the solution set is a smooth $(m-1)$-dimensional manifold.

Also:

$$\mathrm{Im} Df(x_0) = \mathbb{R}\, g(x_0) = 1\text{-dimensional}.$$

Example 2

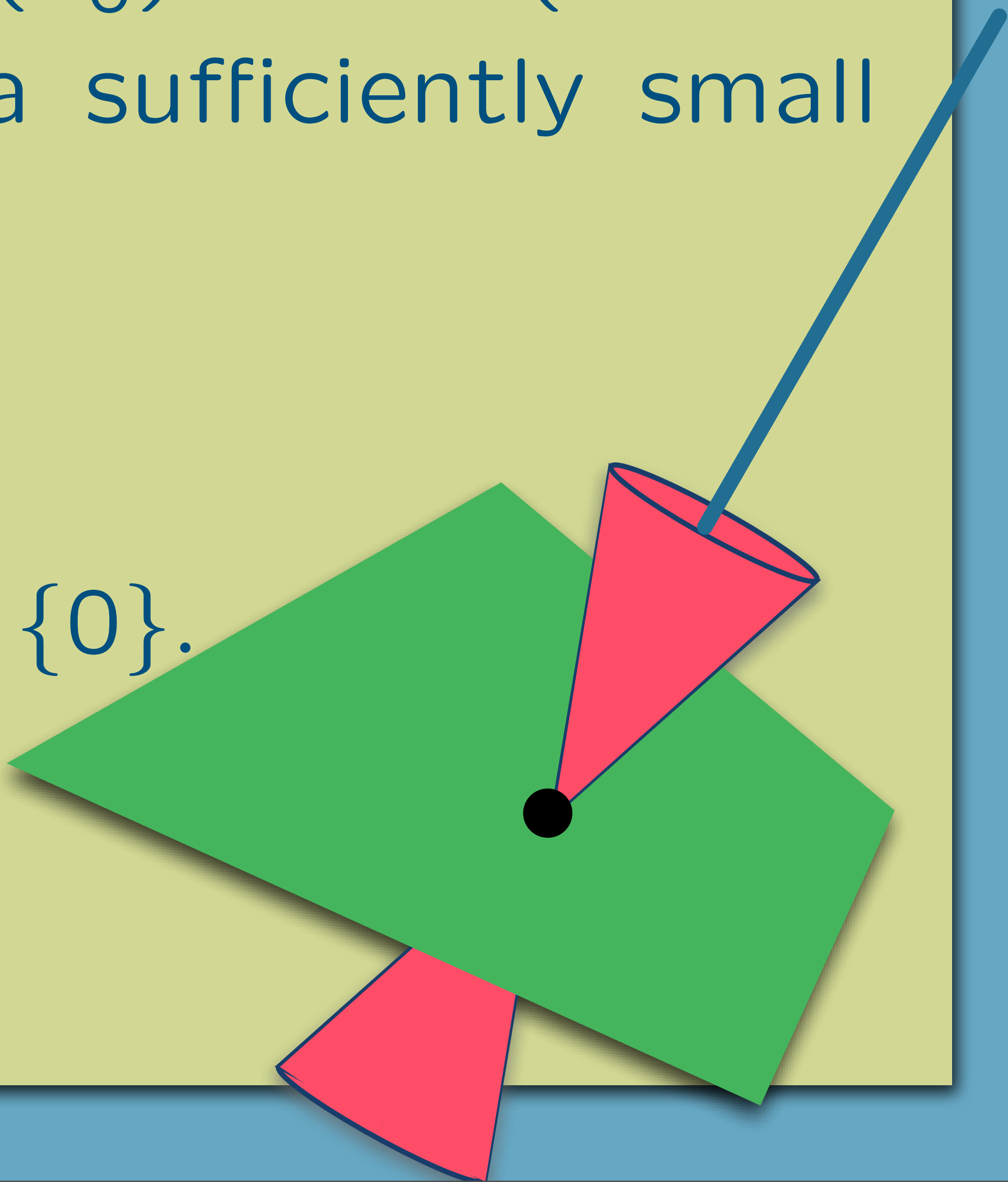
Let W be a complement of $\mathbb{R}g(x_0)$ in $\mathbb{R}^n$ (for example: $W := g(x_0)^{\perp}$) and O a sufficiently small neighborhood of x_0 in $\mathbb{R}^m$.

Example 2

Let W be a complement of $\mathbb{R}g(x_0)$ in $\mathbb{R}^n$ (for example: $W := g(x_0)^\perp$) and O a sufficiently small neighborhood of x_0 in $\mathbb{R}^m$.

Then again:

$$f(O) \cap (f(x_0) + W) = f(O) \cap W = \{0\}.$$



Example 3

In this example $f : \mathbb{R}^2 \to \mathbb{R}^2$ is explicitly given by

$$f(x_1, x_2) := (x_2 - x_1^2, e^{x_2} - e^{x_1^2}), \;\; \forall (x_1, x_2) \in \mathbb{R}^2.$$

Example 3

In this example $f : \mathbb{R}^2 \to \mathbb{R}^2$ is explicitly given by

$$f(x_1, x_2) := (x_2 - x_1^2, e^{x_2} - e^{x_1^2}), \;\; \forall (x_1, x_2) \in \mathbb{R}^2.$$

The zero's of f lie on the 1-dimensional curve

$$x_2 = x_1^2;$$

f is not a submersion at such zero:

$$\dim(\mathrm{Im} Df(x_1, x_2)) = 1.$$

Example 3

For example, at $(x_1, x_2) = (0, 0)$ we have

$$\mathrm{Im} Df(0,0) = \mathbb{R}(1,1).$$

Example 3

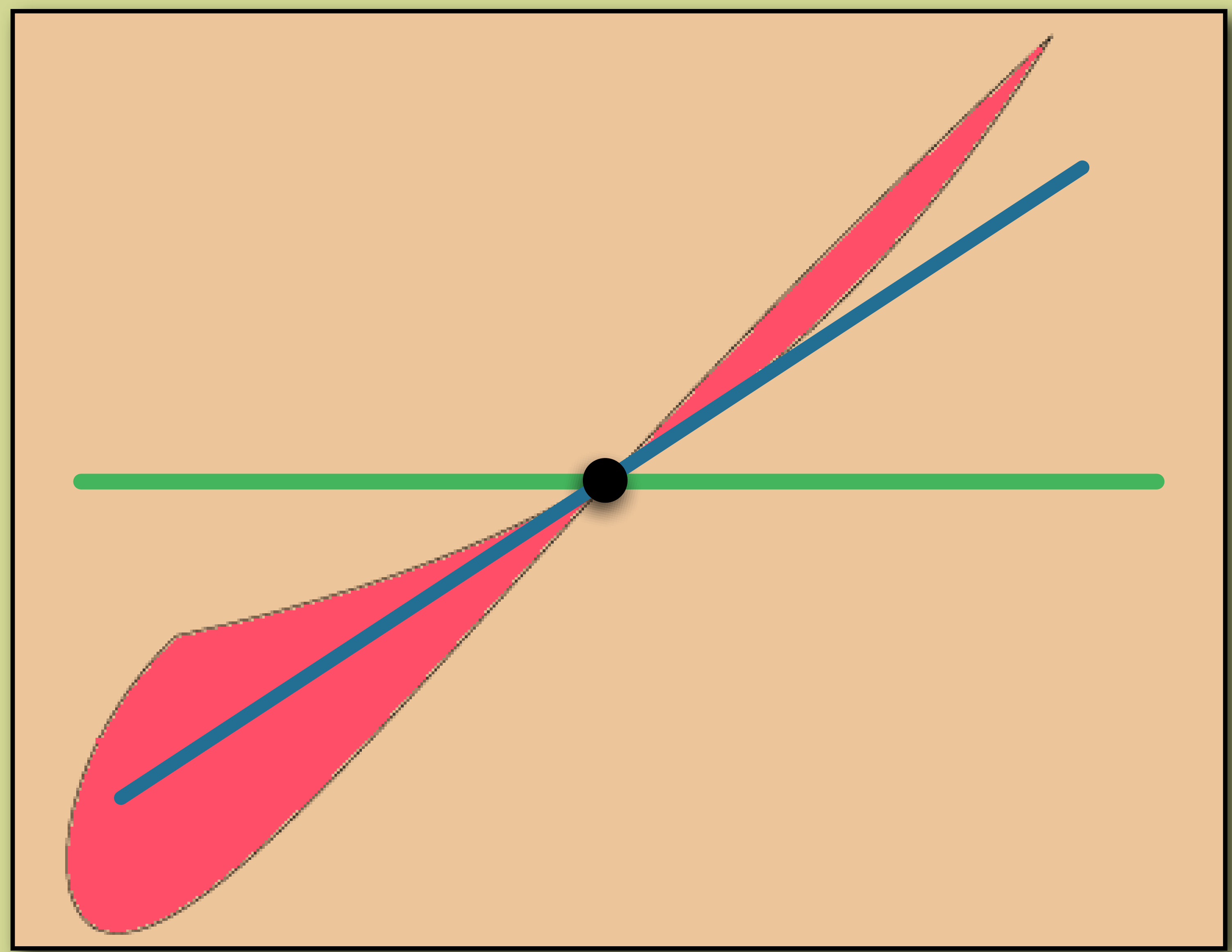
For example, at $(x_1, x_2) = (0, 0)$ we have

$$\mathrm{Im} Df(0,0) = \mathbb{R}(1,1).$$

Taking for example $W := \mathbb{R}(1,0)$ as a complement of $\mathrm{Im} Df(0,0)$ in $\mathbb{R}^2$ and setting O equal to the unit disk around the origin one can explicitly show that

$$f(0) \cap W = \{0\}.$$

Example 3



These examples bring us to the following definition

The mapping $f : \mathbb{R}^m \to \mathbb{R}^n$ is a

quasi-submersion

at some point $x_0 \in \mathbb{R}^m$ if there exist a neighborhood O of x_0 in $\mathbb{R}^m$ and a subspace W of $\mathbb{R}^n$ such that

$$\mathbb{R}^n = \mathrm{Im} Df(x_0) \oplus W$$

and

$$f(O) \cap (f(x_0) + W) = \{f(x_0)\}.$$

The mapping $f : \mathbb{R}^m \to \mathbb{R}^n$ is a

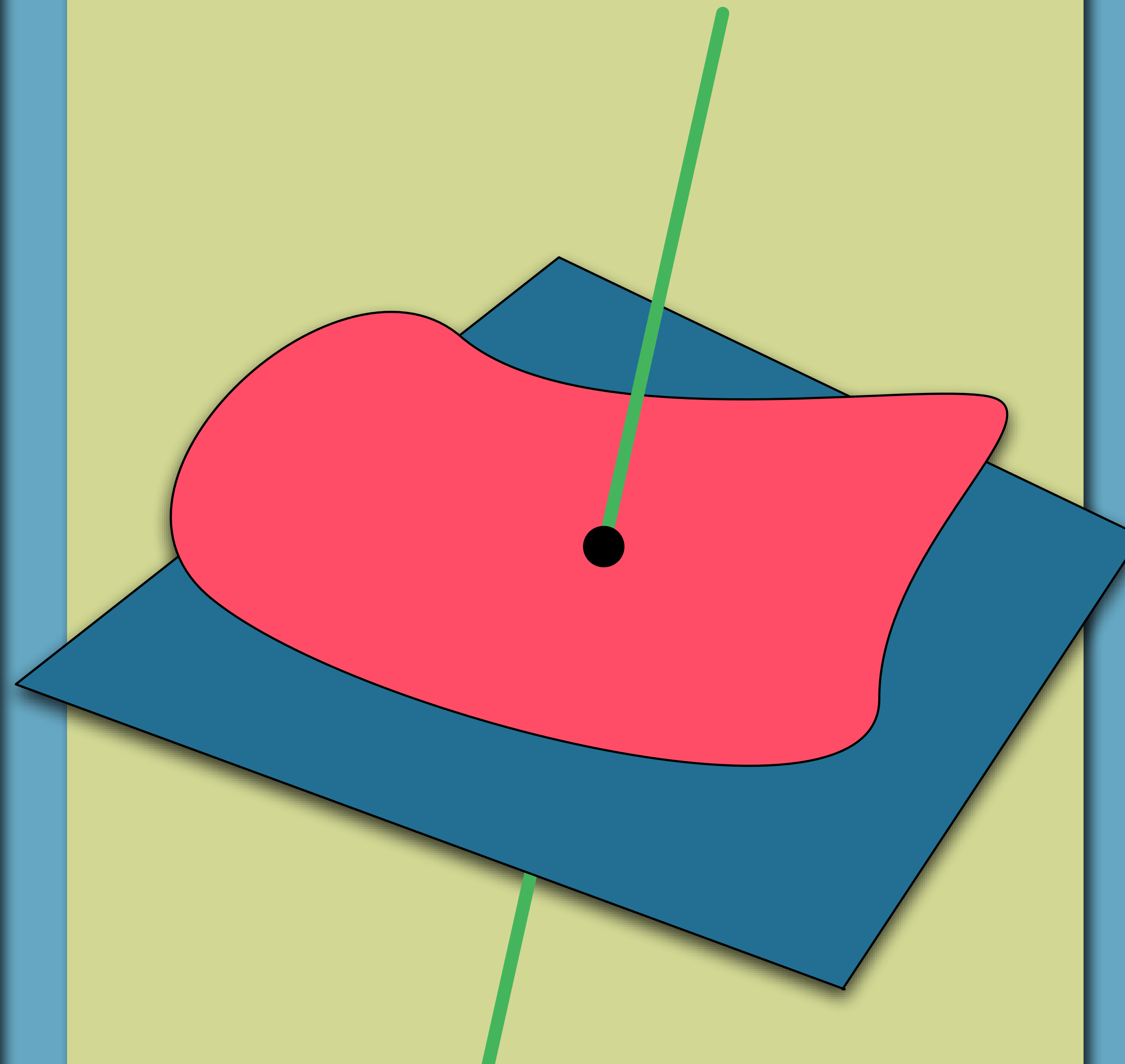
quasi-submersion

at some point $x_0 \in \mathbb{R}^m$ if there exist a neighborhood O of x_0 in $\mathbb{R}^m$ and a subspace W of $\mathbb{R}^n$ such that

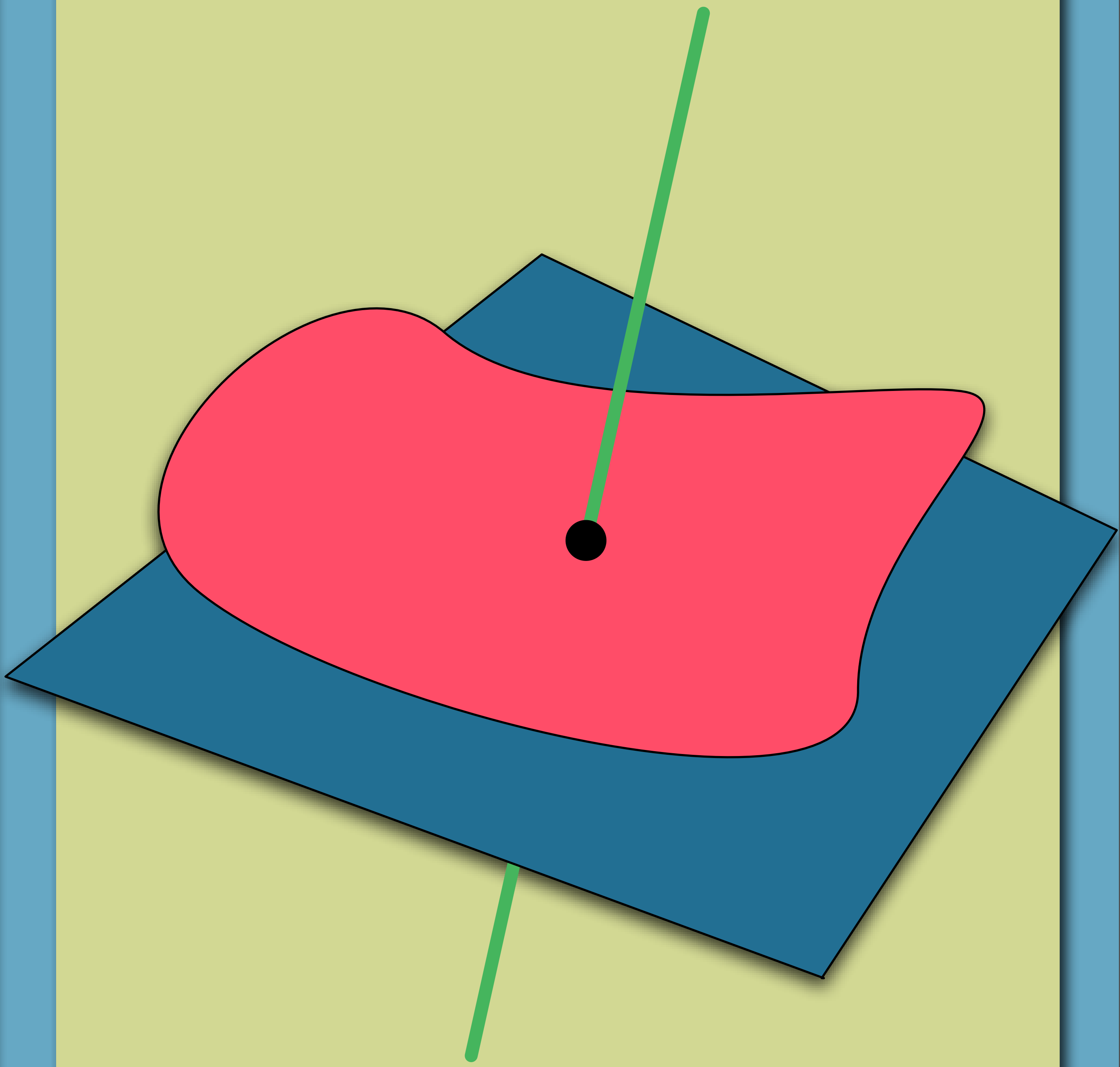
$$\mathbb{R}^n = \mathrm{Im} Df(x_0) \oplus W$$

and

$$f(O) \cap (f(x_0) + W) = \{f(x_0)\}.$$



The main result about quasi-submersions is the following



Theorem

If $f:\mathbb{R}^m \to \mathbb{R}^n$ is a quasi-submersion at $x_0 \in \mathbb{R}^m$, with

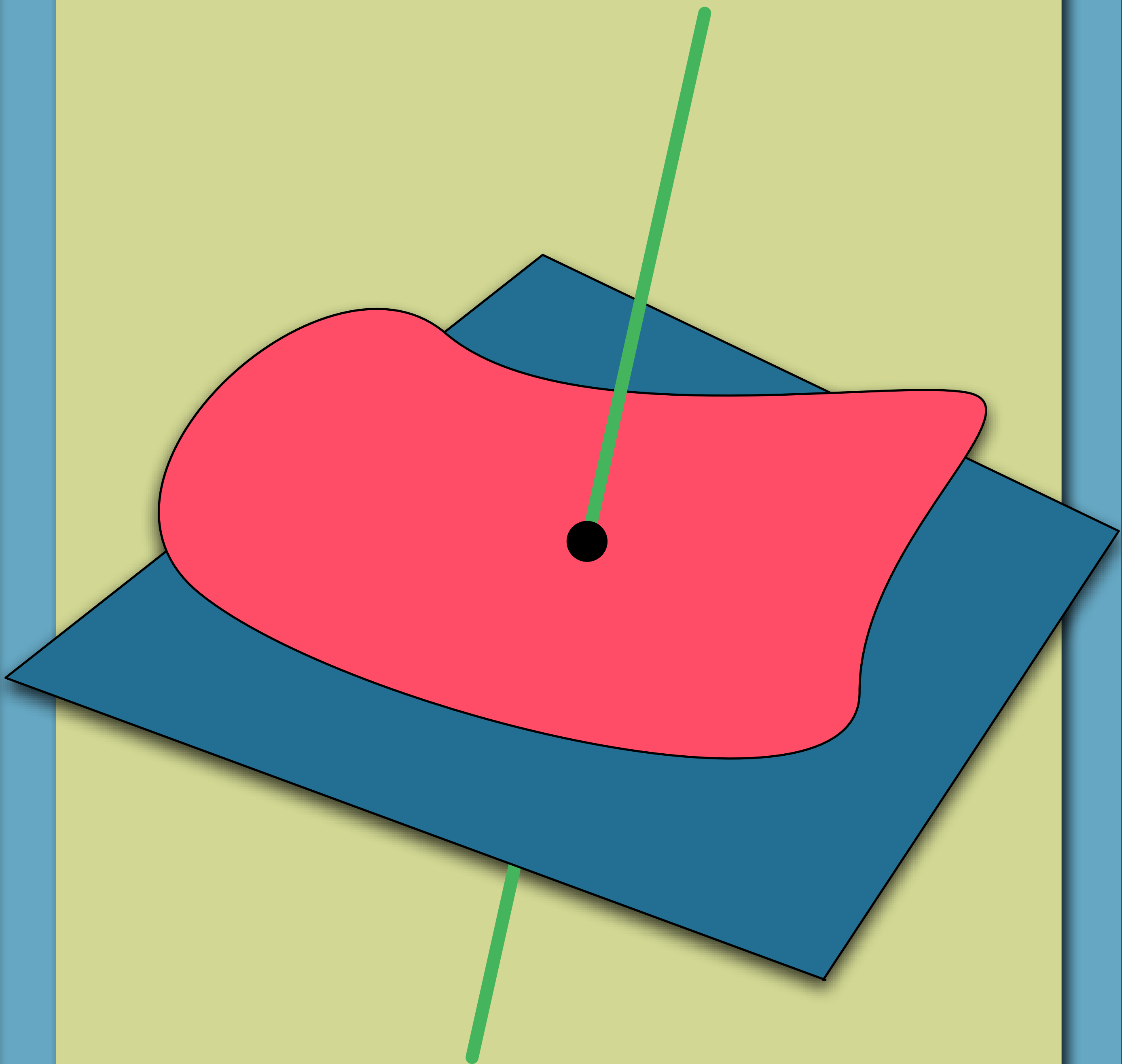
$$\dim W = \operatorname{codim} \operatorname{Im} Df(x_0) = k,$$

then the solution set of the equation

$$f(x) = f(x_0)$$

is locally near x_0 a smooth submanifold of dimension

$$m - n + k.$$



The proof is extremely simple:

The proof is extremely simple:

- Locally near x_0 the equation $f(x) = f(x_0)$ is equivalent to

$$f(x) = f(x_0) + w.$$

The proof is extremely simple:

- Locally near x_0 the equation $f(x) = f(x_0)$ is equivalent to

$$f(x) = f(x_0) + w.$$

- The mapping $F : \mathbb{R}^m \times W \to \mathbb{R}^n$ given by

$$F(x, w) := f(x) - w$$

is at $(x_0, 0)$ a submersion.

A Special Case

Constrained Mappings

A Special Case

Constrained Mappings

Assume the following:

- $f(x) = g(x) - h(x)$ for some smooth $g, h : \mathbb{R}^m \to \mathbb{R}^n$;

Constrained Mappings

Assume the following:

- $f(x) = g(x) - h(x)$ for some smooth $g, h : \mathbb{R}^m \to \mathbb{R}^n$;
- the space

$$\mathcal{F} := \{F : \mathbb{R}^n \to \mathbb{R} \mid F \circ g = F \circ h\}$$

contains some non-constant functions;

Constrained Mappings

Assume the following:

- $f(x) = g(x) - h(x)$ for some smooth $g, h : \mathbb{R}^m \to \mathbb{R}^n$;
- the space

$$\mathcal{F} := \{F : \mathbb{R}^n \to \mathbb{R} \mid F \circ g = F \circ h\}$$

  contains some non-constant functions;
- $f(x_0) = 0$, i.e. x_0 is a solution of

$$g(x) = h(x).$$

Constrained Mappings

We call such f a

constrained mapping,

and we are interested in the zero's of f, more in particular in the continuation of the solution x_0 of the equation

$$g(x) = h(x). \tag{1}$$

Constrained Mappings

It follows from the identity $F(g(x)) = F(h(x))$ (valid for all $F \in \mathcal{F}$) that

$$DF(y_0) \cdot Dg(x_0) = DF(y_0) \cdot Dh(x_0),$$

with $y_0 := g(x_0) = h(x_0)$, and hence

$$\operatorname{Im} Df(x_0) \subset W^\perp,$$

where

$$W := \{\nabla F(y_0) \mid F \in \mathcal{F}\}.$$

Constrained Mappings

We say that x_0 is a **normal zero** of the constrained mapping f if

$$\operatorname{Im} Df(x_0) = W^{\perp},$$

or equivalently, if

$$\dim(\operatorname{Im} Df(x_0)) = n - \dim W.$$

Constrained Mappings

The Main Result

A constrained mapping is quasi-submersive at each of it’s normal zero’s

A constrained mapping is quasi-submersive at each of it's normal zero's

Let x_0 be a normal zero of the constrained mapping $f = g - h$. Then, locally near x_0, the solution set of the equation

$$g(x) = h(x) \qquad (1)$$

is a smooth submanifold of dimension

$$m - n + \dim W.$$

Proof

By the normality

$$\mathbb{R}^n = \operatorname{Im} Df(x_0) \oplus W,$$

so we only have to show that

$$g(x) = h(x) + w \qquad (*)$$

implies $w = 0$ and $g(x) = h(x)$.

Proof

By the normality

$$\mathbb{R}^n = \operatorname{Im} Df(x_0) \oplus W,$$

so we only have to show that

$$g(x) = h(x) + w \tag{*}$$

implies $w = 0$ and $g(x) = h(x)$.

Let P be the orthogonal projection in $\mathbb{R}^n$ onto $W^\perp$, and let $F_i \in \mathcal{F}$ $(1 \le i \le k = \dim W)$ be such that $\{\nabla F_i(y_0) \mid 1 \le i \le k\}$ forms a basis of W.

Proof

Then $g(x) = h(x) + w$ implies

$$Pg(x) = Ph(x),$$

while also

$$F_i(g(x)) = F_i(h(x)), \quad (1 \leq i \leq k).$$

Proof

Then $g(x) = h(x) + w$ implies

$$Pg(x) = Ph(x),$$

while also

$$F_i(g(x)) = F_i(h(x)), \quad (1 \leq i \leq k).$$

But

$$y \in \mathbb{R}^n \mapsto (Py, F_1(y), \ldots, F_k(y)) \in W^\perp \times \mathbb{R}^k$$

forms a local diffeomorphism at y_0, and therefore $g(x) = h(x)$ and hence $w = 0$.

Observation:

Instead of solving

$$g(x) = h(x)$$

one can solve the "regular" equation

$$g(x) = h(x) + \sum_{1 \le i \le k} \alpha_i \nabla F_i(y_0)$$

for $(x, \alpha) = (x, \alpha_1, \ldots, \alpha_k)$.

Observation:

Instead of solving

$$g(x) = h(x)$$

one can solve the "regular" equation

$$g(x) = h(x) + \sum_{1 \le i \le k} \alpha_i \nabla F_i(y_0)$$

for $(x, \alpha) = (x, \alpha_1, \ldots, \alpha_k)$.

For all solutions (x, α) near $(x_0, 0)$ we have

$$\alpha = 0.$$

an example

Periodic Orbits in Conservative Systems

an example

Periodic Orbits in Conservative Systems

Consider

$$\dot{x} = X(x), \tag{2}$$

with $X : \mathbb{R}^n \to \mathbb{R}^n$ a smooth vectorfield such that the space

$$\mathcal{F} := \{F : \mathbb{R}^n \to \mathbb{R} \mid DF(x) \cdot X(x) \equiv 0\}$$

contains some non-constant functions.

an example

Periodic Orbits in Conservative Systems

Consider

$$\dot{x} = X(x), \tag{2}$$

with $X : \mathbb{R}^n \to \mathbb{R}^n$ a smooth vectorfield such that the space

$$\mathcal{F} := \{F : \mathbb{R}^n \to \mathbb{R} \mid DF(x) \cdot X(x) \equiv 0\}$$

contains some non-constant functions.

Denote the flow of (2) by $\tilde{x}(t, x)$.

an example

Periodic Orbits in Conservative Systems

Periodic solutions of (2) are given by solutions (T, x) of the equation

$$\tilde{x}(T, x) = x.$$

AN EXAMPLE

Periodic Orbits in Conservative Systems

Periodic solutions of (2) are given by solutions (T, x) of the equation

$$\tilde{x}(T, x) = x.$$

The mapping $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ given by

$$f(T, x) := \tilde{x}(t, x) - x$$

is a constrained mapping since

$$F(\tilde{x}(T, x)) = F(x), \quad \forall F \in \mathcal{F}.$$

an example

Periodic Orbits in Conservative Systems

Periodic solutions of (2) are given by solutions (T, x) of the equation

$$\tilde{x}(T, x) = x.$$

The mapping $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ given by

$$f(T, x) := \tilde{x}(\cancel{t}\,T, x) - x$$

is a constrained mapping since

$$F(\tilde{x}(T, x)) = F(x), \quad \forall F \in \mathcal{F}.$$

an example

Periodic Orbits in Conservative Systems

A simple calculation shows that at a zero (T_0, x_0) of f (with $T_0 > 0$ the minimal period of $\tilde{x}(t, x_0)$) we have

$$\operatorname{Im} Df(T_0, x_0) = \mathbb{R}\, X(x_0) + \operatorname{Im}(M - I),$$

with M the *monodromy matrix* of the T_0-periodic solution $\tilde{x}(t, x_0)$.

Periodic Orbits in Conservative Systems

A simple calculation shows that at a zero (T_0, x_0) of f (with $T_0 > 0$ the minimal period of $\tilde{x}(t, x_0)$) we have

$$\operatorname{Im} Df(T_0, x_0) = \mathbb{R}\, X(x_0) + \operatorname{Im}(M - I),$$

with M the *monodromy matrix* of the T_0-periodic solution $\tilde{x}(t, x_0)$.

Also: $W = \{\nabla F(x_0) \mid F \in \mathcal{F}\}$.

Periodic Orbits in Conservative Systems

Therefore (T_0, x_0) is a **normal zero** of f if

$$\mathbb{R}\,X(x_0) + \mathrm{Im}\,(M - I) = W^{\perp};$$

this coincides with the condition for a **normal periodic solution** of the conservative system $\dot{x} = X(x)$ as given in Lecture 1.

an example

Periodic Orbits in Conservative Systems

Such normal zero's belong to a $(k+1)$-parameter family of (normal) zero's of f, meaning that a normal periodic orbit belongs to a k-parameter family of normal orbits (with $k := \dim W$).

Doubly Symmetric Solutions in Reversible Systems

Reversible Systems

Reversible Systems

The n-dimensional system

$$\boxed{\dot{x} = X(x)} \tag{2}$$

is **reversible** if there exist

- a compact group $\Gamma \subset O(n)$, and
- a nontrivial character $\chi : \Gamma \to \{1, -1\}$

such that

$$X(\gamma x) = \chi(\gamma)\gamma X(x), \quad \forall \gamma \in \Gamma.$$

Reversible Systems

The flow $\tilde{x}(t,x)$ of (2) then satisfies

$$\tilde{x}(\chi(\gamma)t, \gamma x) = \gamma\tilde{x}(t,x), \quad \forall\gamma \in \Gamma.$$

Reversible Systems

The flow $\tilde{x}(t, x)$ of (2) then satisfies

$$\tilde{x}(\chi(\gamma)t, \gamma x) = \gamma \tilde{x}(t, x), \quad \forall \gamma \in \Gamma.$$

A **reversor** is an element $R \in \Gamma$ such that $\chi(R) = -1$; for such reversor we have

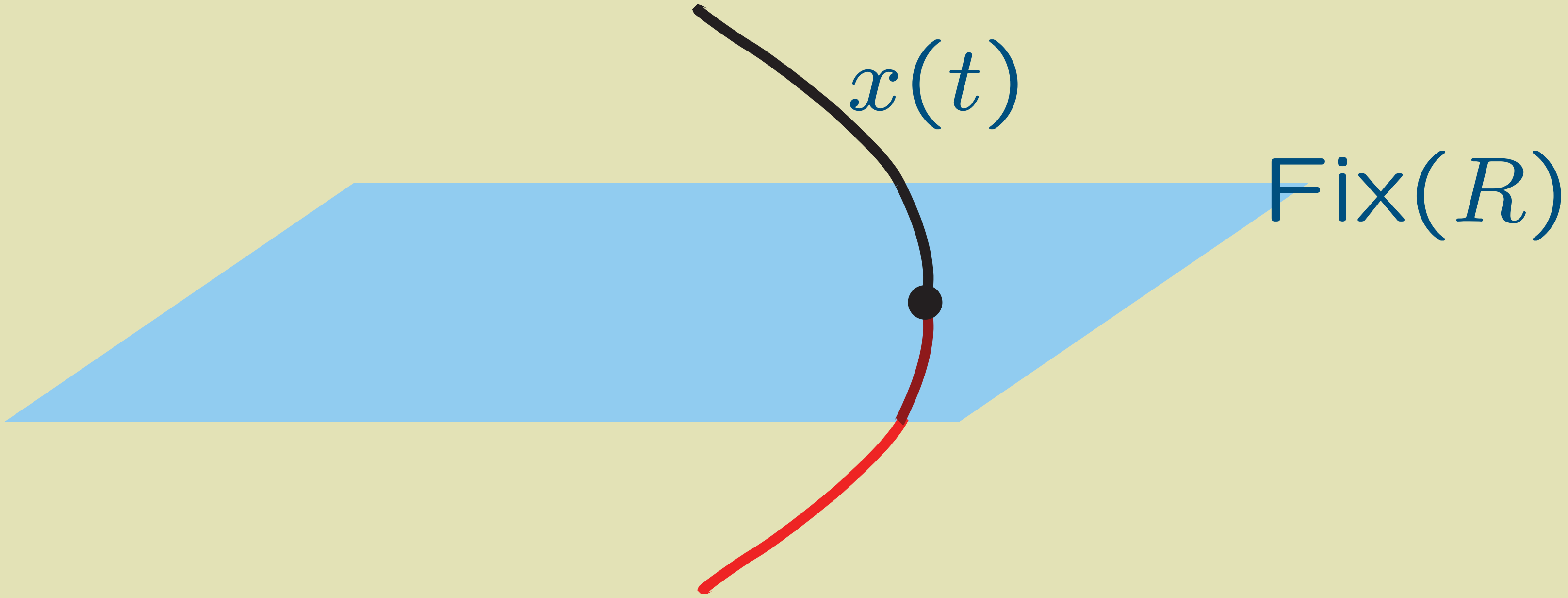
$$\tilde{x}(-t, Rx) = R\tilde{x}(t, x).$$

Reversible Systems

Let $R \in \Gamma$ be a reversor of $\dot{x} = X(x)$; a solution $x(t) = \tilde{x}(t, x(0))$ is called R-**symmetric** if its orbit intersects $\mathrm{Fix}(R)$ in at least one point:

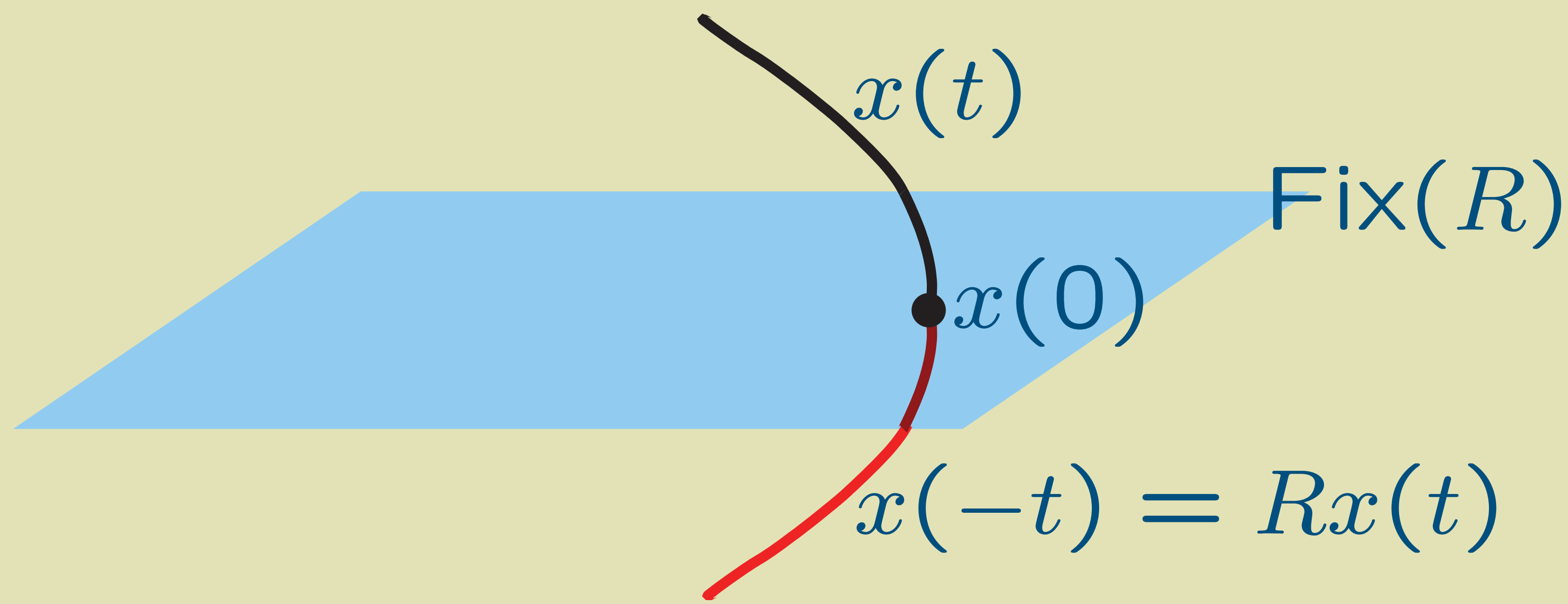
Reversible Systems

Let $R \in \Gamma$ be a reversor of $\dot{x} = X(x)$; a solution $x(t) = \tilde{x}(t, x(0))$ is called R-**symmetric** if its orbit intersects $\text{Fix}(R)$ in at least one point:



Reversible Systems

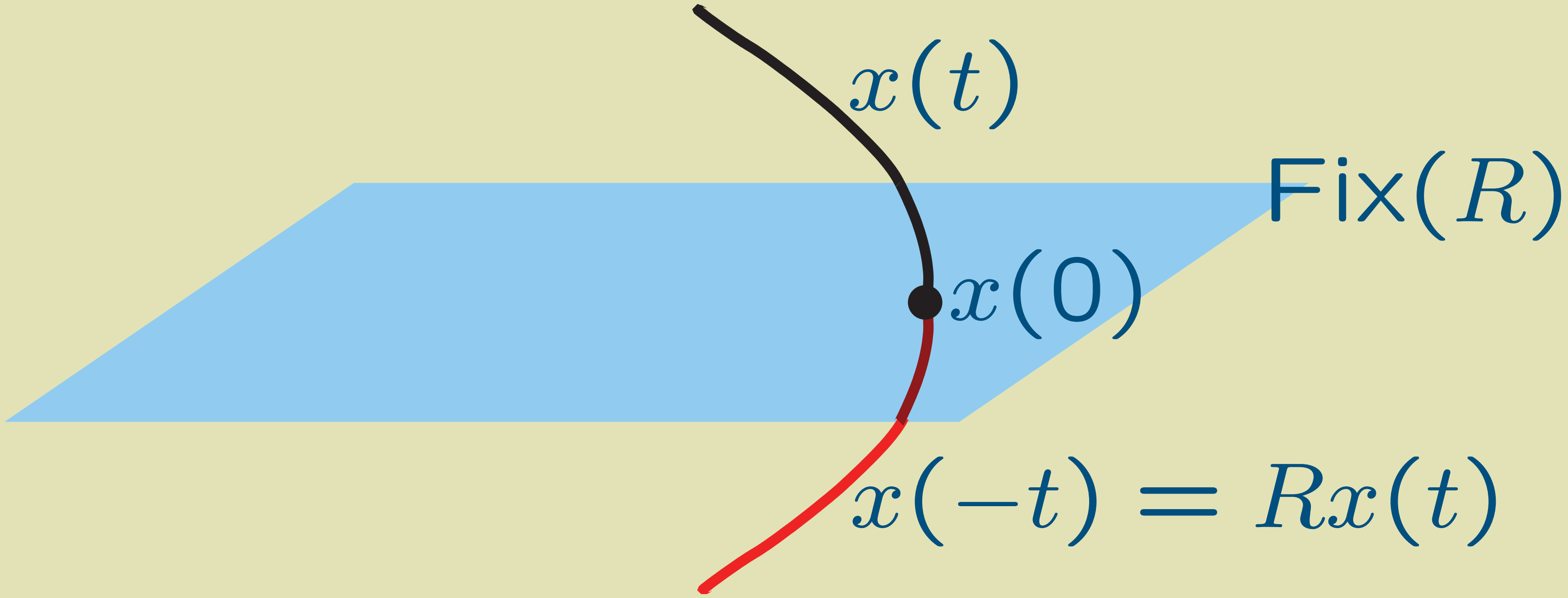
Let $R \in \Gamma$ be a reversor of $\dot{x} = X(x)$; a solution $x(t) = \tilde{x}(t, x(0))$ is called R-**symmetric** if its orbit intersects $\mathsf{Fix}(R)$ in at least one point:



Taking $t = 0$ at the intersection point we have then $x(-t) = Rx(t)$.

Reversible Systems

Such R-symmetric solution satisfies $R^2x(t) = x(t)$, i.e. when considering R-symmetric solutions we may w.l.o.g. work in $\mathrm{Fix}(R^2)$, or assume that $R^2 = I$.



Doubly Symmetric Solutions

Loosely speaking, **doubly symmetric solutions** are solutions of the reversible system $\dot{x} = X(x)$ which are symmetric with respect to two reversors R_0 and R_1 of the system.

Doubly Symmetric Solutions

Loosely speaking, **doubly symmetric solutions** are solutions of the reversible system $\dot{x} = X(x)$ which are symmetric with respect to two reversors R_0 and R_1 of the system.

The case $R_1 = R_0$ is allowed.

Doubly Symmetric Solutions

Loosely speaking, **doubly symmetric solutions** are solutions of the reversible system $\dot{x} = X(x)$ which are symmetric with respect to two reversors R_0 and R_1 of the system.

The case $R_1 = R_0$ is allowed.

As explained before we will assume that

$$R_0^2 = R_1^2 = I.$$

Definition:

A solution $x(t)$ is (R_0, R_1)-symmetric if there exist $t_0, t_1 \in \mathbb{R}$, with $t_1 > t_0$ and such that

$$x(t_0) \in \mathrm{Fix}(R_0) \quad \text{and} \quad x(t_1) \in \mathrm{Fix}(R_1).$$

Definition:

A solution $x(t)$ is (R_0, R_1)-symmetric if there exist $t_0, t_1 \in \mathbb{R}$, with $t_1 > t_0$ and such that

$$x(t_0) \in \mathrm{Fix}(R_0) \quad \text{and} \quad x(t_1) \in \mathrm{Fix}(R_1).$$

We call $[t_0, t_1]$ the **basic domain** of the doubly symmetric solution $x(t)$. Most of the time we will assume that $t_0 = 0$ and $t_1 = T > 0$.

Definition:

A solution $x(t)$ is (R_0, R_1)-symmetric if there exist $t_0, t_1 \in \mathbb{R}$, with $t_1 > t_0$ and such that

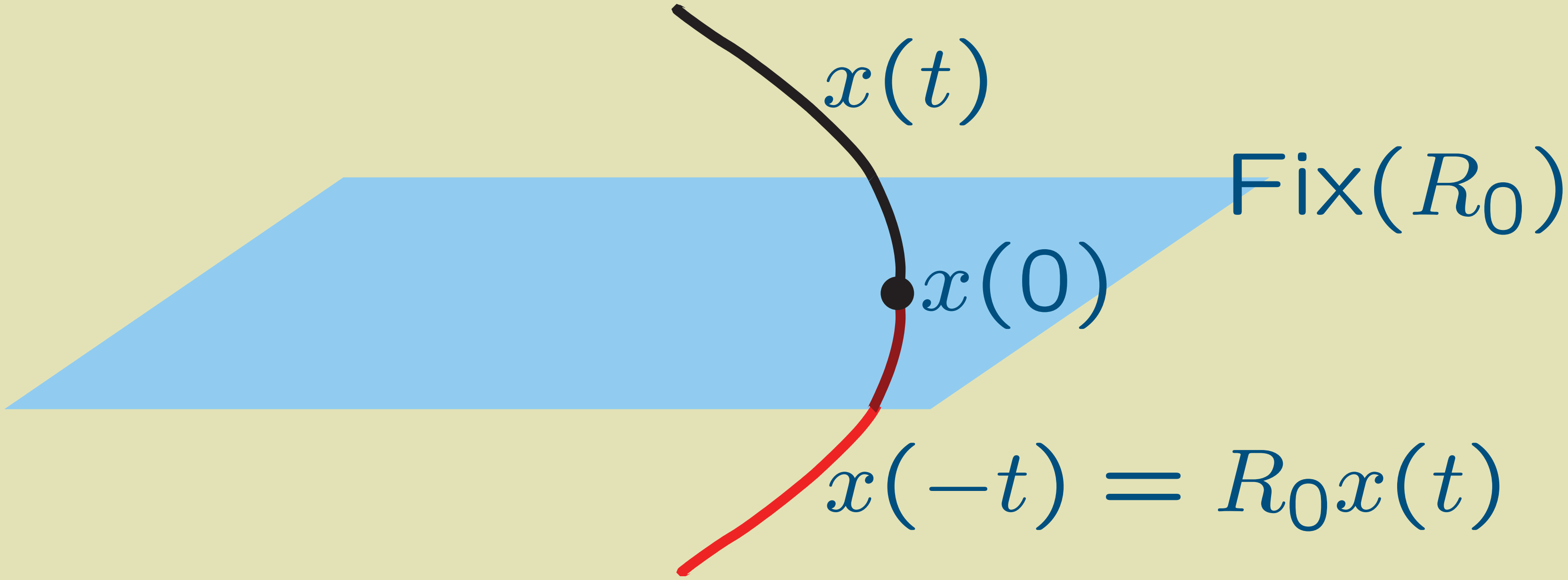
$$x(t_0) \in \mathrm{Fix}(R_0) \quad \text{and} \quad x(t_1) \in \mathrm{Fix}(R_1).$$

We call $[t_0, t_1]$ the **basic domain** of the doubly symmetric solution $x(t)$. Most of the time we will assume that $t_0 = 0$ and $t_1 = T > 0$.

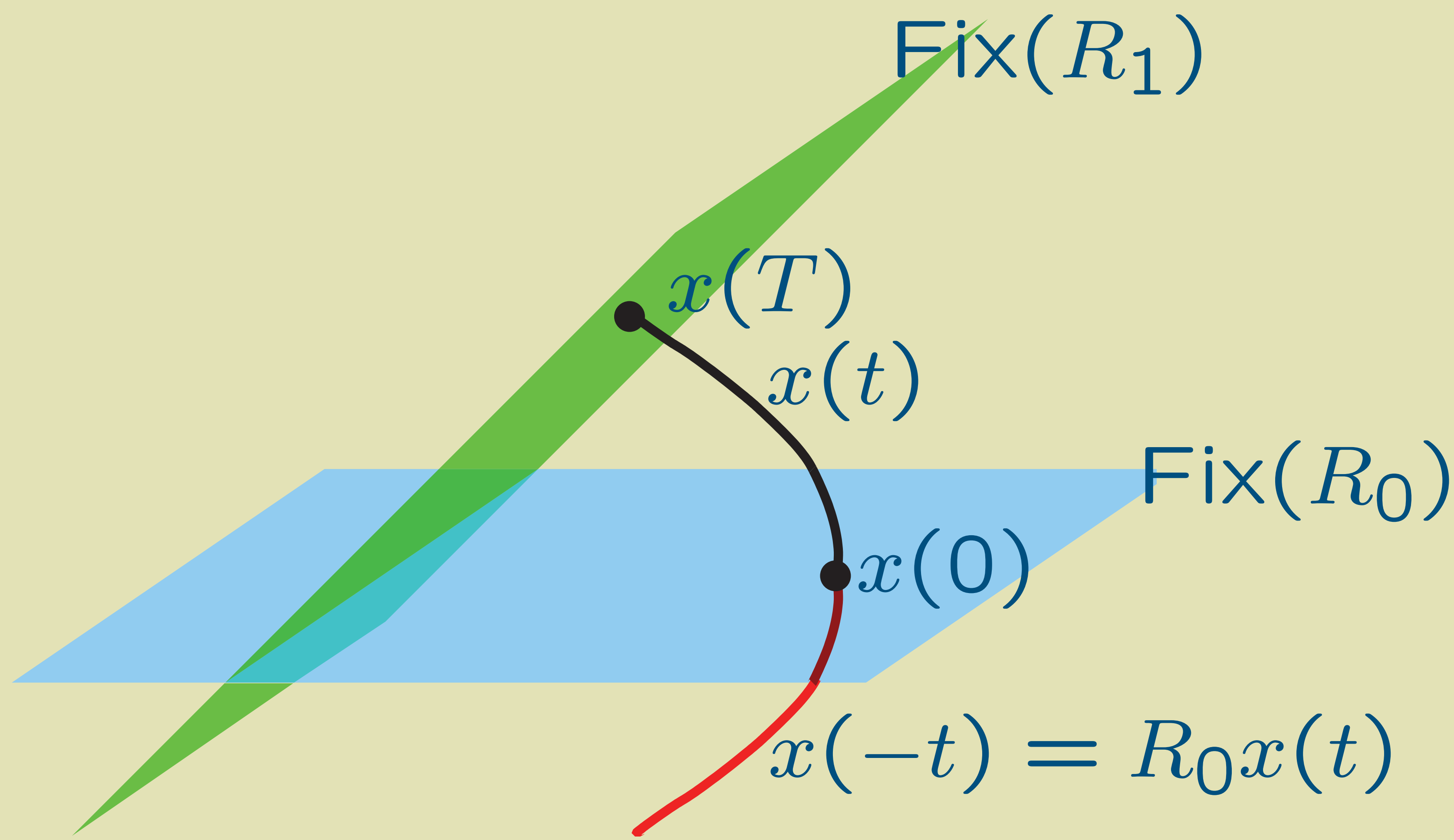
Then:

$$x(-t) = R_0 x(t) \;\text{ and }\; x(T+t) = R_1 x(T-t).$$

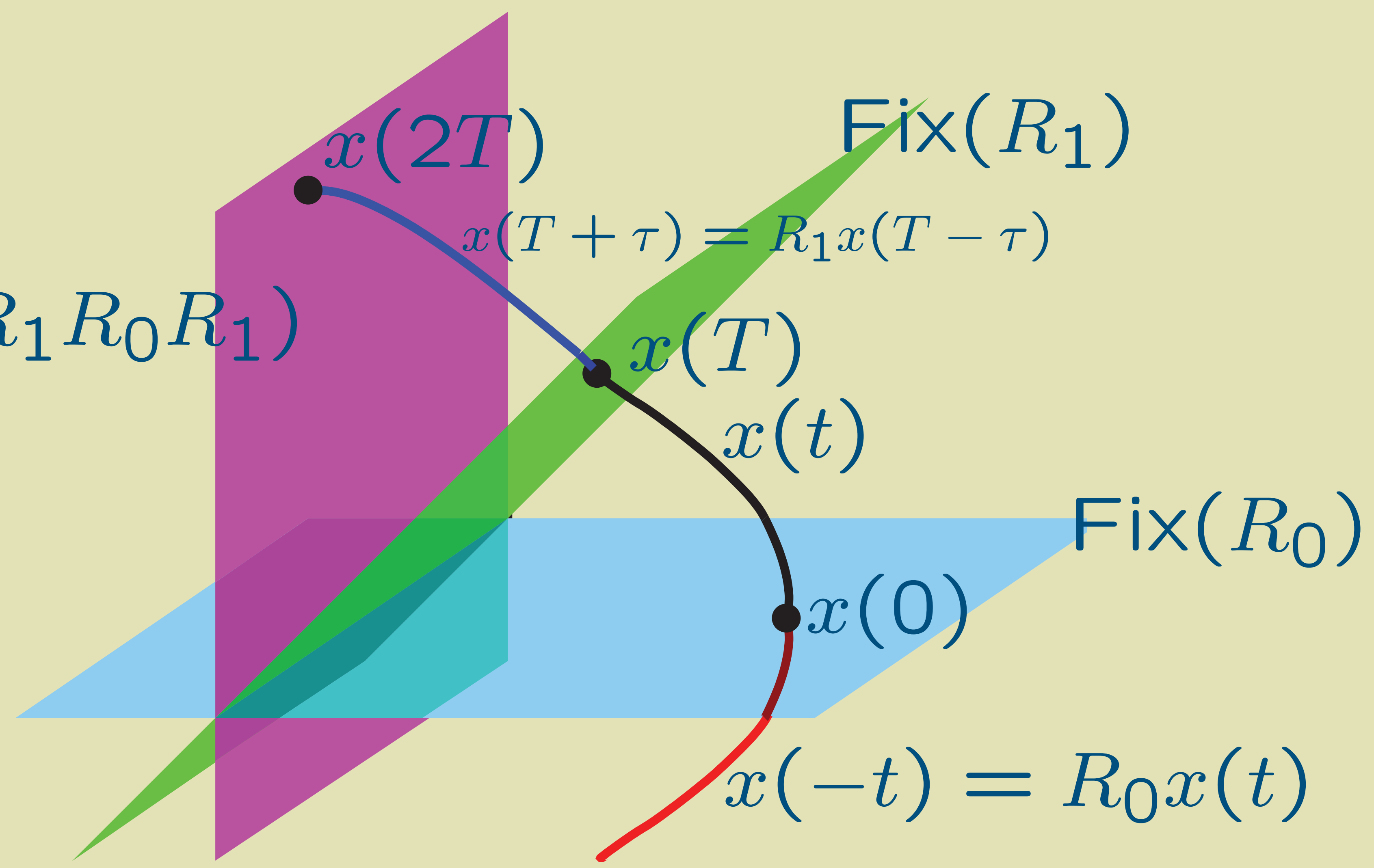
The picture looks as follows:



The picture looks as follows:



The picture looks as follows:



The main properties of such doubly symmetric solutions $x(t)$ are:

The main properties of such doubly symmetric solutions $x(t)$ are:

- they exist for all $t \in \mathbb{R}$;

The main properties of such doubly symmetric solutions $x(t)$ are:

- they exist for all $t \in \mathbb{R}$;
- $x(-t) = R_0 x(t)$ and $x(T-t) = R_1 x(T+t)$;

The main properties of such doubly symmetric solutions $x(t)$ are:

- they exist for all $t \in \mathbb{R}$;
- $x(-t) = R_0 x(t)$ and $x(T-t) = R_1 x(T+t)$;
- $x(2mT+t) = (R_1 R_0)^m x(t)$;

The main properties of such doubly symmetric solutions $x(t)$ are:

- they exist for all $t \in \mathbb{R}$;
- $x(-t) = R_0 x(t)$ and $x(T-t) = R_1 x(T+t)$;
- $x(2mT + t) = (R_1 R_0)^m x(t)$;
- $x(2mT) = (R_1 R_0)^{2m-1} R_1 x(2mT)$;

The main properties of such doubly symmetric solutions $x(t)$ are:

- they exist for all $t \in \mathbb{R}$;
- $x(-t) = R_0 x(t)$ and $x(T-t) = R_1 x(T+t)$;
- $x(2mT + t) = (R_1 R_0)^m x(t)$;
- $x(2mT) = (R_1 R_0)^{2m-1} R_1 x(2mT)$;
- $x((2m+1)T) = (R_1 R_0)^m R_1 x((2m+1)T)$.

The main properties of such doubly symmetric solutions $x(t)$ are:

- they exist for all $t \in \mathbb{R}$;
- $x(-t) = R_0 x(t)$ and $x(T-t) = R_1 x(T+t)$;
- $x(2mT + t) = (R_1 R_0)^m x(t)$;
- $x(2mT) = (R_1 R_0)^{2m-1} R_1 x(2mT)$;
- $x((2m+1)T) = (R_1 R_0)^m R_1 x((2m+1)T)$.

In particular, if $(R_1 R_0)^M = I$ then $x(t)$ is

- $2MT$-periodic;

The main properties of such doubly symmetric solutions $x(t)$ are:

- they exist for all $t \in \mathbb{R}$;
- $x(-t) = R_0 x(t)$ and $x(T-t) = R_1 x(T+t)$;
- $x(2mT+t) = (R_1 R_0)^m x(t)$;
- $x(2mT) = (R_1 R_0)^{2m-1} R_1 x(2mT)$;
- $x((2m+1)T) = (R_1 R_0)^m R_1 x((2m+1)T)$.

In particular, if $(R_1 R_0)^M = I$ then $x(t)$ is

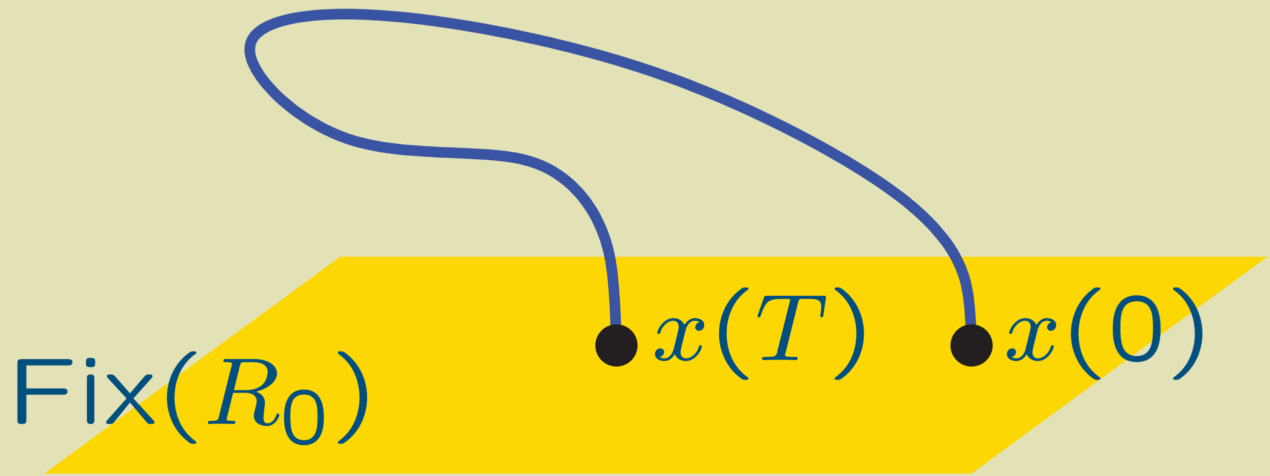
- $2MT$-periodic;
- (R_0, R_0)-symm. with basic domain $[0, MT]$.

Special case $R_1 = R_0$

A (R_0, R_0)-symmetric solution with basic domain $[0, T]$ is automatically $2T$-periodic:

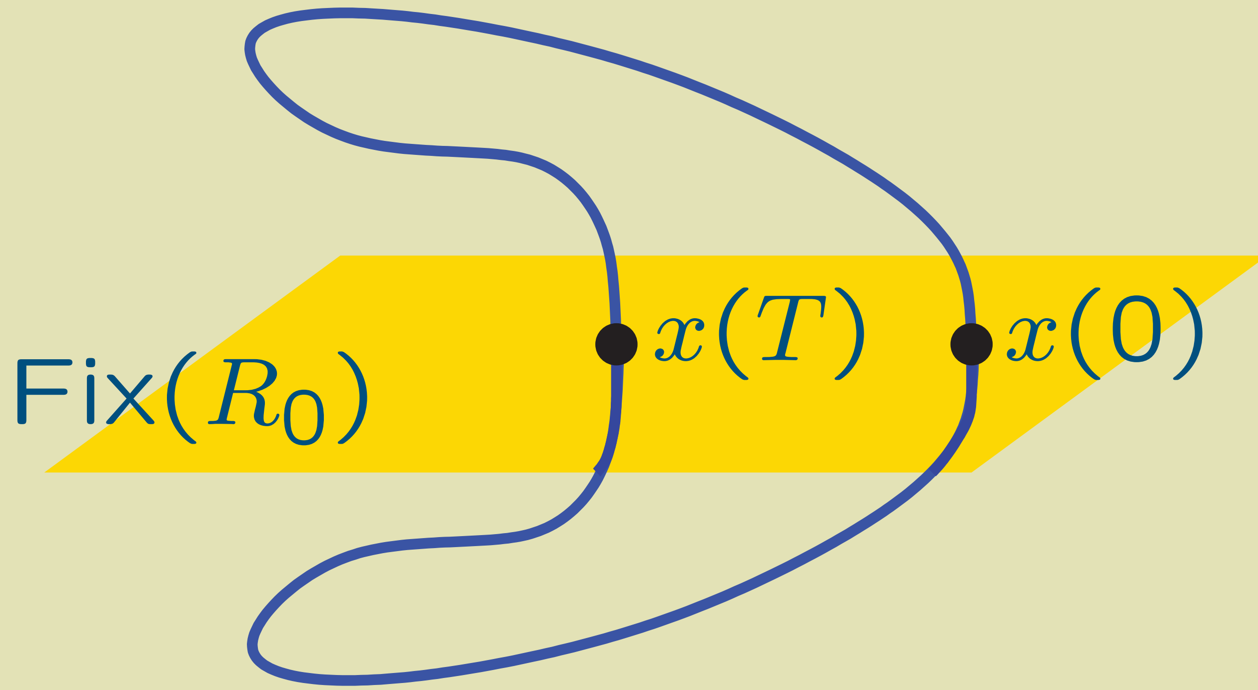
Special case $R_1 = R_0$

A (R_0, R_0)-symmetric solution with basic domain $[0, T]$ is automatically $2T$-periodic:



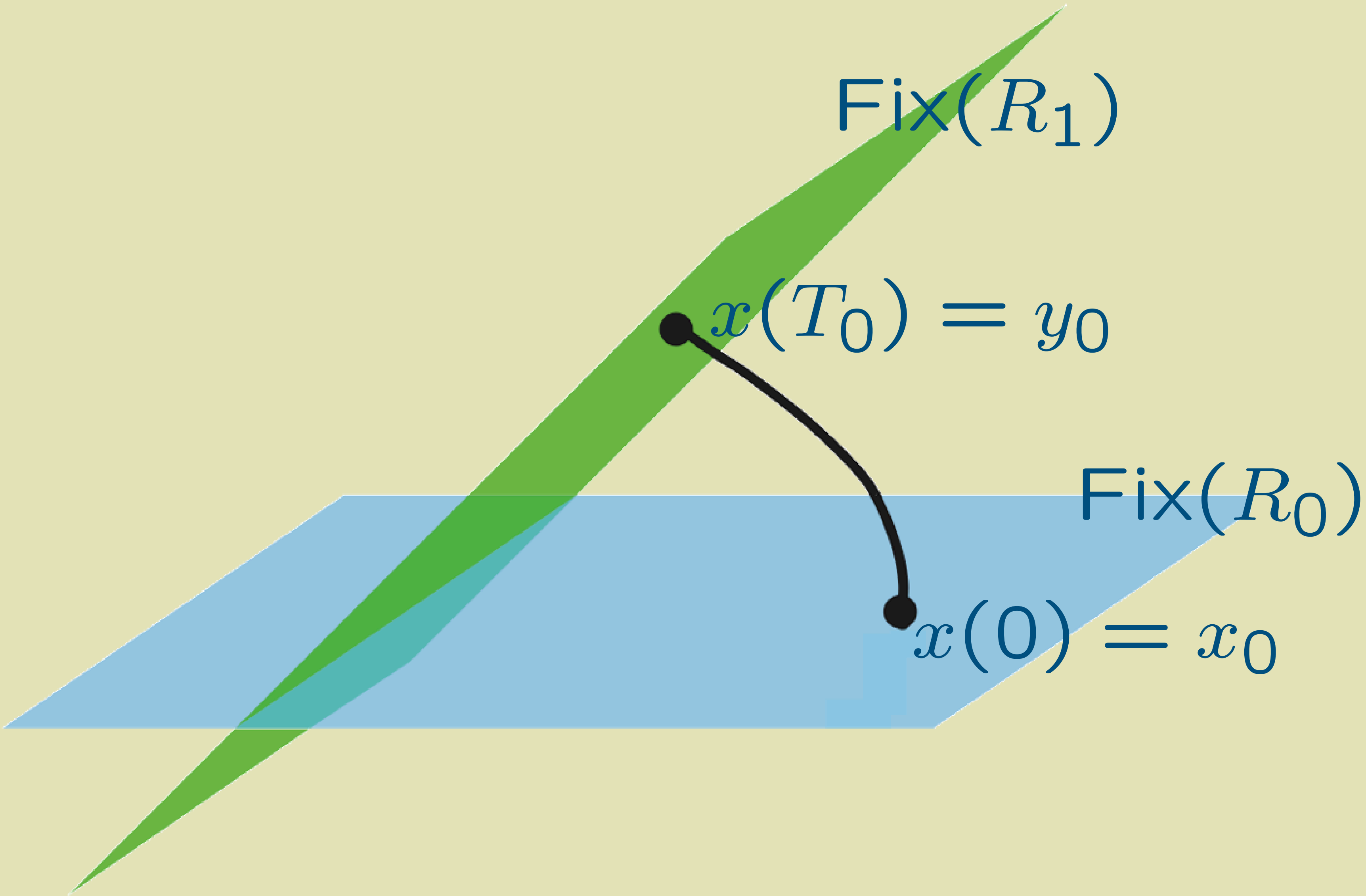
Special case $R_1 = R_0$

A (R_0, R_0)-symmetric solution with basic domain $[0, T]$ is automatically $2T$-periodic:

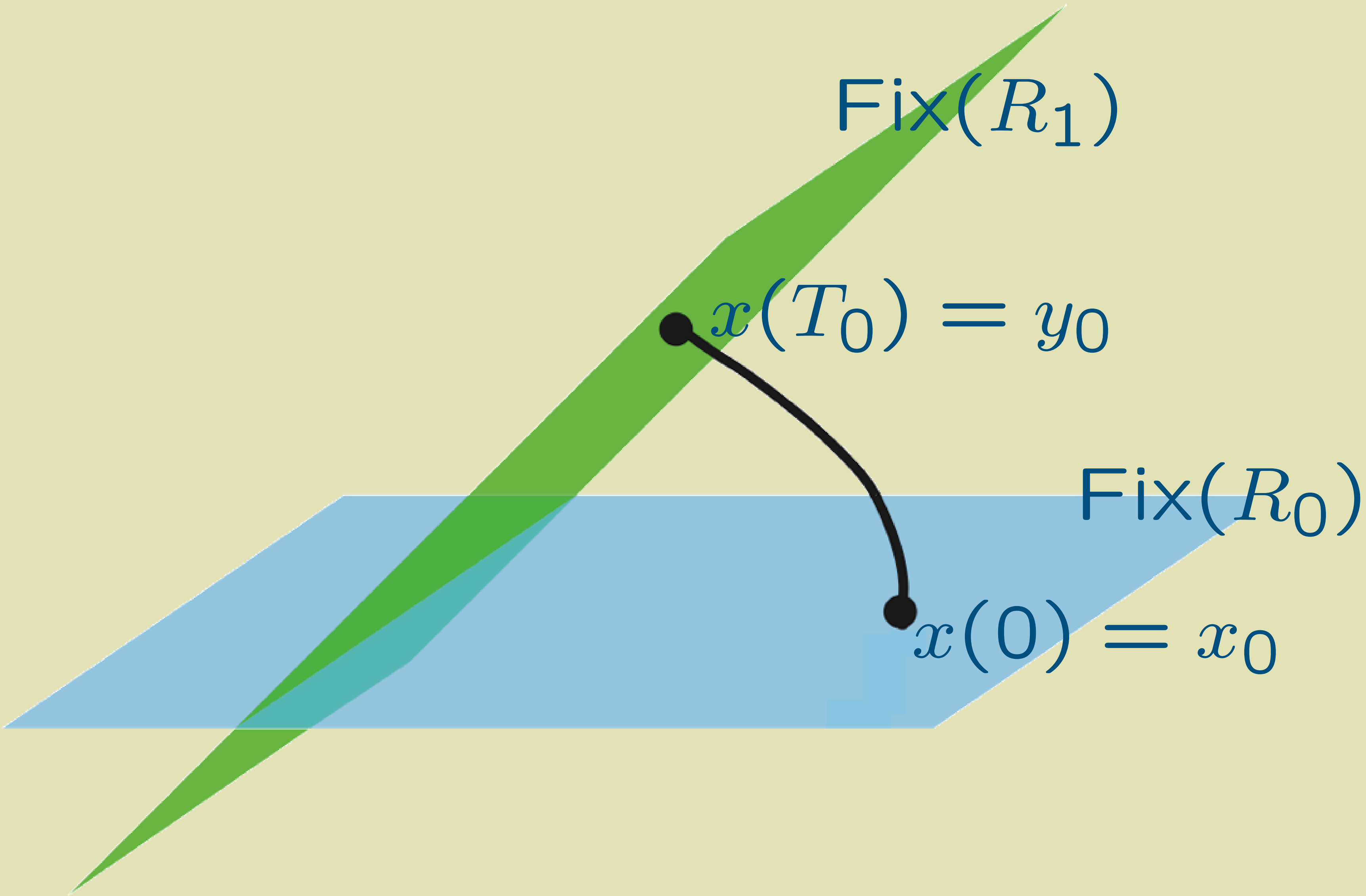


The Continuation Problem

The Continuation Problem



The Continuation Problem



How to find nearby doubly symmetric solutions?

The Continuation Problem

Geometrically we need to find the intersection points near y_0 of the subspace $\mathsf{Fix}(R_1)$ with the submanifold

$$\mathcal{M}_0 := \{\tilde{x}(t,x) \mid t \in \mathbb{R}, x \in \mathsf{Fix}(R_0)\}.$$

The Continuation Problem

Geometrically we need to find the intersection points near y_0 of the subspace $\mathsf{Fix}(R_1)$ with the submanifold

$$\mathcal{M}_0 := \{\tilde{x}(t,x) \mid t \in \mathbb{R}, x \in \mathsf{Fix}(R_0)\}.$$

If at y_0 the manifold $\mathcal{M}_0$ is **transversal** to $\mathsf{Fix}(R_1)$ then the intersection will locally be a submanifold of dimension

$$1 + \dim \mathsf{Fix}(R_0) + \dim \mathsf{Fix}(R_1) - n.$$

The Continuation Problem

Typically in applications we have

$$n = 2 \dim \mathrm{Fix}(R_0) = 2 \dim \mathrm{Fix}(R_1); \qquad (\dagger)$$

then doubly symmetric orbits appear along **one-dimensional branches**.

The Continuation Problem

Typically in applications we have

$$n = 2\dim\mathrm{Fix}(R_0) = 2\,\dim\,\mathrm{Fix}(R_1); \qquad (\dagger)$$

then doubly symmetric orbits appear along **one-dimensional branches**.

For simplicity we assume from now on that ($\dagger$) holds.

The Continuation Problem

To express the transversality condition analytically we denote by

$$\pi_0^{\pm} := \frac{1}{2}(I \pm R_0) \text{ and } \pi_1^{\pm} := \frac{1}{2}(I \pm R_1)$$

the projections in $\mathbb{R}^n$ on respectively $\mathrm{Fix}(\pm R_0)$ and $\mathrm{Fix}(\pm R_1)$. Remember that

$$\begin{aligned} \mathbb{R}^n &= \mathrm{Fix}(R_0) \oplus \mathrm{Fix}(-R_0) \\ &= \mathrm{Fix}(R_1) \oplus \mathrm{Fix}(-R_1). \end{aligned}$$

The Continuation Problem

Also, we denote by $V(t, t_0)$ the transition matrix for the variational equation

$$\dot{x} = DX(\tilde{x}(t, x_0)) \cdot x.$$

The Continuation Problem

Also, we denote by $V(t, t_0)$ the transition matrix for the variational equation

$$\dot{x} = DX(\tilde{x}(t, x_0)) \cdot x.$$

By lack of a better name we call

$$M := V(T_0, 0)$$

the **momodromy matrix** of the doubly symmetric solution $x(t) = \tilde{x}(t, x_0)$.

The Continuation Problem

The transversality condition then takes the form

$$\boxed{\mathrm{Im}(\pi_1^- M \pi_0^+) + \mathbb{R}X(y_0) = \mathrm{Fix}(-R_1)}$$

The Continuation Problem

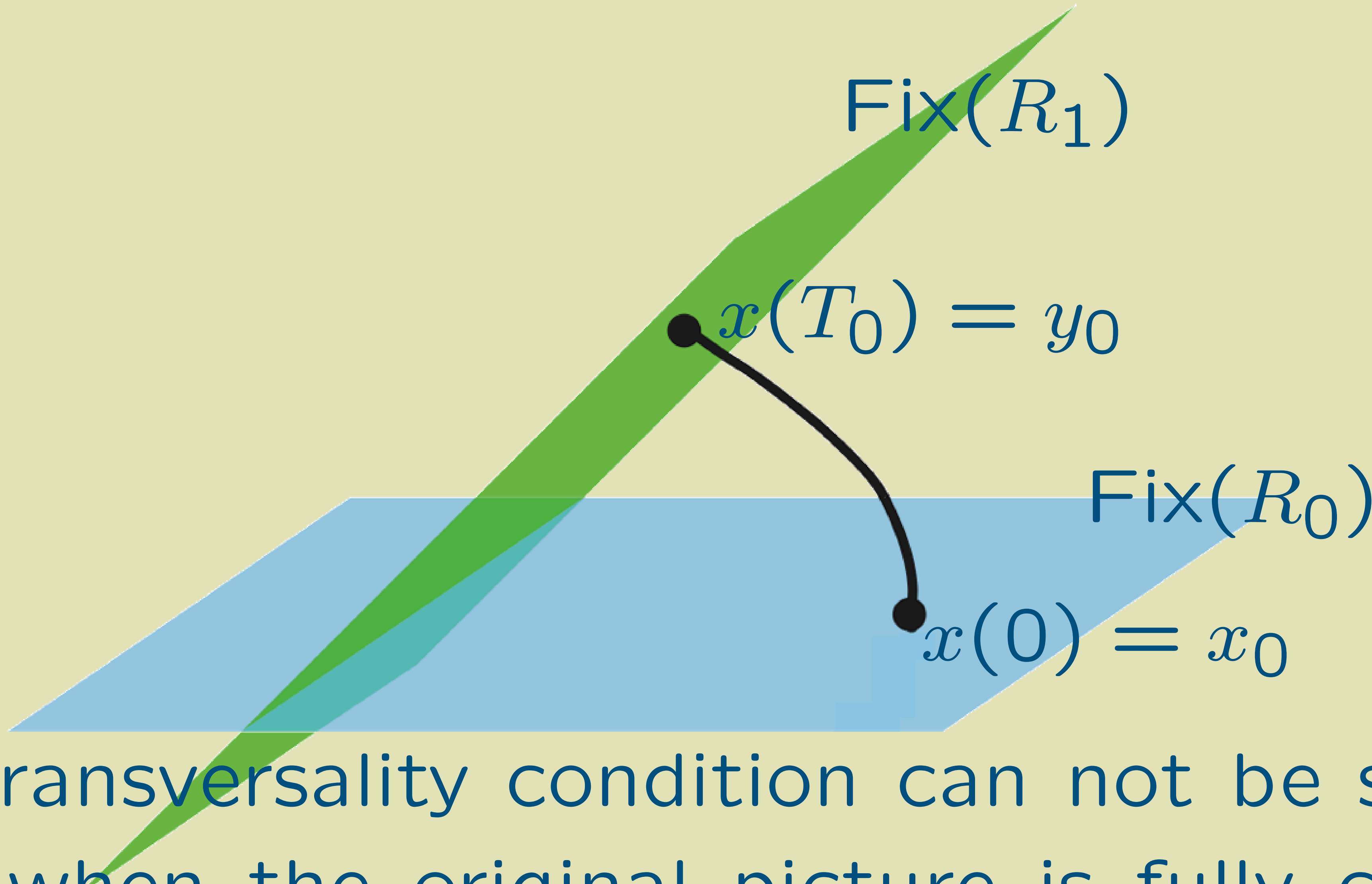
The transversality condition then takes the form

$$\boxed{\operatorname{Im}(\pi_1^- M \pi_0^+) + \mathbb{R}X(y_0) = \operatorname{Fix}(-R_1)}$$

Observe:

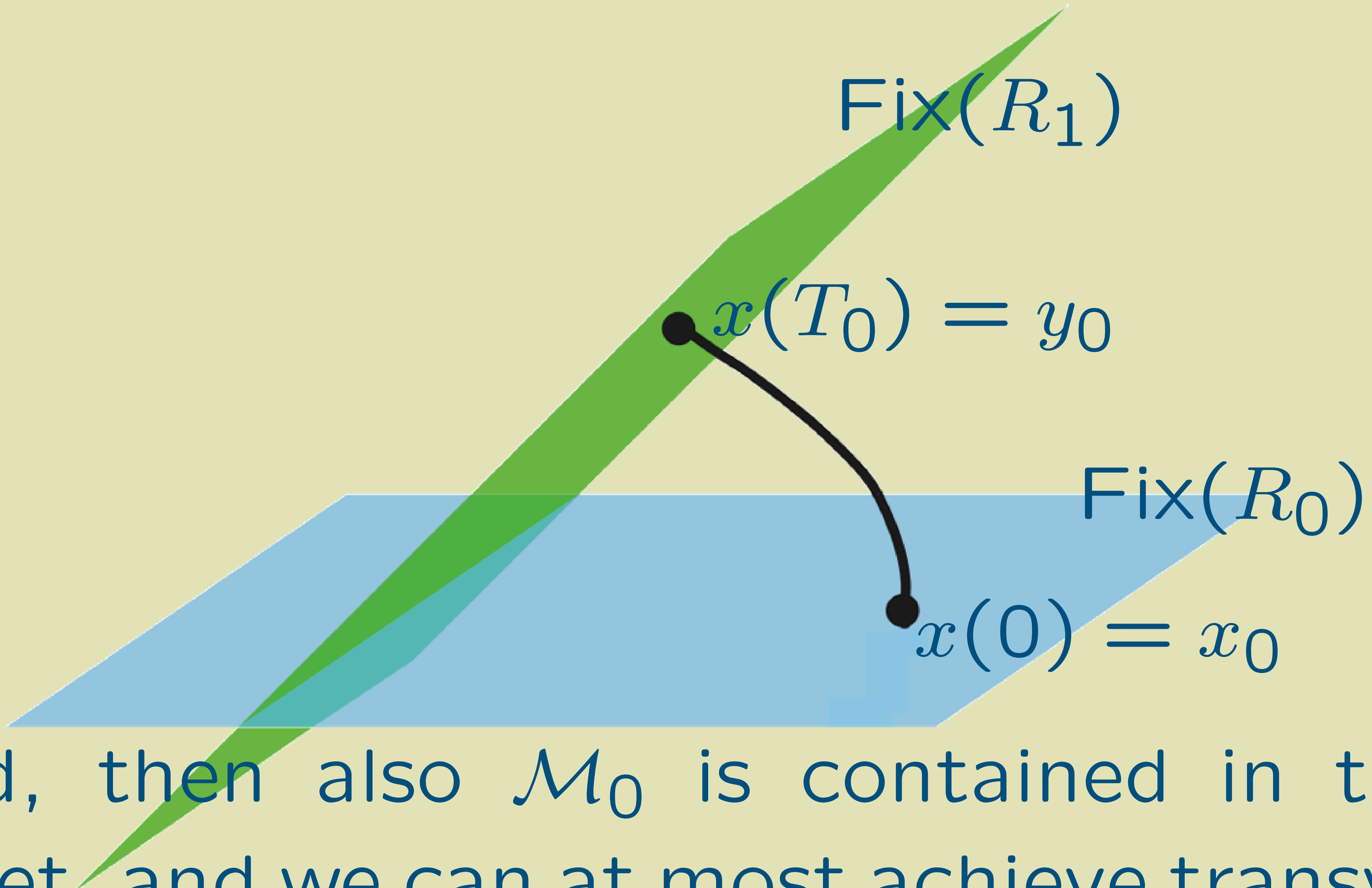
$$X(y_0) \in \operatorname{Fix}(-R_1) \quad (\text{since } y_0 \in \operatorname{Fix}(R_1)).$$

The Continuation Problem



The transversality condition can not be satisfied when the original picture is fully contained in a level set of a first integral of (2).

The Continuation Problem



Indeed, then also $\mathcal{M}_0$ is contained in that level set, and we can at most achieve transversality within the (codimension one) level set.

More in general, (R_0, R_1)-symmetric solutions are generated by the solutions $(T, x) \in \mathbb{R} \times \mathrm{Fix}(R_0)$ of the equation

$$f(T, x) := \pi_1^- \tilde{x}(T, x) = 0.$$

More in general, (R_0, R_1)-symmetric solutions are generated by the solutions $(T, x) \in \mathbb{R} \times \mathrm{Fix}(R_0)$ of the equation

$$f(T, x) := \pi_1^- \tilde{x}(T, x) = 0.$$

The mapping $f : \mathbb{R} \times \mathrm{Fix}(R_0) \to \mathbb{R}^n$ is a constrained mapping, as follows.

More in general, (R_0, R_1)-symmetric solutions are generated by the solutions $(T, x) \in \mathbb{R} \times \mathrm{Fix}(R_0)$ of the equation

$$f(T, x) := \pi_1^- \tilde{x}(T, x) = 0.$$

The mapping $f : \mathbb{R} \times \mathrm{Fix}(R_0) \to \mathbb{R}^n$ is a constrained mapping, as follows.

$$f(T, x) = \tilde{x}(T, x) - \pi_1^+ \tilde{x}(T, x)$$

More in general, (R_0, R_1)-symmetric solutions are generated by the solutions $(T, x) \in \mathbb{R} \times \mathrm{Fix}(R_0)$ of the equation

$$f(T, x) := \pi_1^- \tilde{x}(T, x) = 0.$$

The mapping $f : \mathbb{R} \times \mathrm{Fix}(R_0) \to \mathbb{R}^n$ is a constrained mapping, as follows.

$$f(T, x) = \underbrace{\tilde{x}(T, x)}_{g(T, x)} - \pi_1^+ \tilde{x}(T, x)$$

More in general, (R_0, R_1)-symmetric solutions are generated by the solutions $(T, x) \in \mathbb{R} \times \mathrm{Fix}(R_0)$ of the equation

$$f(T, x) := \pi_1^- \tilde{x}(T, x) = 0.$$

The mapping $f : \mathbb{R} \times \mathrm{Fix}(R_0) \to \mathbb{R}^n$ is a constrained mapping, as follows.

$$f(T, x) = \underbrace{\tilde{x}(T, x)}_{g(T, x)} - \underbrace{\pi_1^+ \tilde{x}(T, x)}_{h(T, x)}$$

More in general, (R_0, R_1)-symmetric solutions are generated by the solutions $(T, x) \in \mathbb{R} \times \mathrm{Fix}(R_0)$ of the equation

$$f(T, x) := \pi_1^- \tilde{x}(T, x) = 0.$$

The mapping $f : \mathbb{R} \times \mathrm{Fix}(R_0) \to \mathbb{R}^n$ is a constrained mapping, as follows.

$$f(T, x) = \underbrace{\tilde{x}(T, x)}_{g(T,x)} - \underbrace{\pi_1^+ \tilde{x}(T, x)}_{h(T,x)}$$

Clearly $\pi_1^+ g(T, x) = \pi_1^+ h(T, x)$.

Continuation of DS Solutions

Moreover, let

$$\mathcal{F} := \left\{ F : \mathbb{R}^n \to \mathbb{R} \;\middle|\; \begin{array}{l} \nabla F(x) \cdot X(x) = 0 \\ \text{and } F \text{ is constant on} \\ \mathrm{Fix}(R_0) \cup \mathrm{Fix}(R_1) \end{array} \right\}$$

Moreover, let

$$\mathcal{F} := \left\{ F : \mathbb{R}^n \to \mathbb{R} \;\middle|\; \begin{array}{l} \nabla F(x) \cdot X(x) = 0 \\ \text{and } F \text{ is constant on} \\ \text{Fix}(R_0) \cup \text{Fix}(R_1) \end{array} \right\}$$

We have then for each $F \in \mathcal{F}$ that

$$\begin{aligned} F(g(T,x)) &= F(\tilde{x}(T,x)) \\ &= F(x) = F(\pi_1^+ \tilde{x}(T,x)) \\ &= F(h(T,x)). \end{aligned}$$

Moreover, let

$$\mathcal{F} := \left\{ F : \mathbb{R}^n \to \mathbb{R} \;\middle|\; \begin{array}{l} \nabla F(x) \cdot X(x) = 0 \\ \text{and } F \text{ is constant on} \\ \mathrm{Fix}(R_0) \cup \mathrm{Fix}(R_1) \end{array} \right\}$$

We have then for each $F \in \mathcal{F}$ that

$$\begin{aligned} F(g(T,x)) &= F(\tilde{x}(T,x)) \\ &= F(x) = F(\pi_1^+ \tilde{x}(T,x)) \\ &= F(h(T,x)). \end{aligned}$$

We set

$$W := \{\nabla F(y_0) \mid F \in \mathcal{F}\}.$$

Our general results on constrained mappings show that

$$\mathrm{Im}(Df(T_0, x_0)) \subset W^{\perp} \cap \mathrm{Fix}(-R_1).$$

Our general results on constrained mappings show that

$$\mathrm{Im}(Df(T_0, x_0)) \subset W^{\perp} \cap \mathrm{Fix}(-R_1).$$

We say that the (R_0, R_1)-symmetric solution $\tilde{x}(t, x_0)$ is **normal** if we have equality, i.e. if

$$\mathrm{Im}(\pi_1^- M \pi_0^+) + \mathbb{R}X(y_0) = W^{\perp} \cap \mathrm{Fix}(-R_1).$$

Our general results on constrained mappings show that

$$\mathrm{Im}(Df(T_0, x_0)) \subset W^{\perp} \cap \mathrm{Fix}(-R_1).$$

We say that the (R_0, R_1)-symmetric solution $\tilde{x}(t, x_0)$ is **normal** if we have equality, i.e. if

$$\mathrm{Im}(\pi_1^- M \pi_0^+) + \mathbb{R}X(y_0) = W^{\perp} \cap \mathrm{Fix}(-R_1).$$

Such normal doubly symmetric solutions appear in

$$(1+k)\text{-dimensional families,}$$

where $k := \dim W$.

How can we calculate this manifold of doubly symmetric solutions?

General theory learns us that we can apply the implicit function theorem to the equation

$$\pi_1^- \tilde{x}(T, x) = \sum_{i=1}^{k} \alpha_i \nabla F_i(y_0),$$

where the $F_i \in \mathcal{F}$ are chosen such that

$$\{\nabla F_i(y_0) \mid 1 \leq i \leq k\}$$

forms a basis of W.

How can we calculate this manifold of doubly symmetric solutions?

However, there is a different approach which leads to the same result but which is better suited for numerical calculations; it is based on the following

Lemma

Let $F \in \mathcal{F}$, and let $\hat{x}(t)$ be a solution of

$$\dot{x} = X(x) + \nabla F(x)$$

such that

$$\hat{x}(t_0) \in \mathrm{Fix}(R_0) \ \text{ and } \ \hat{x}(t_1) \in \mathrm{Fix}(R_1)$$

for some $t_0 < t_1$. Then

$$\nabla F(\hat{x}(t)) = 0, \quad \forall t \in [t_0, t_1],$$

i.e. $\hat{x}(t)$ is a solution of

$$\dot{x} = X(x).$$

Proof

$$\begin{aligned}
&\int_{t_0}^{t_1} \langle \nabla F(\widehat{x}(t)), \nabla F(\widehat{x}(t)) \rangle \, dt \\
&\quad = \int_{t_0}^{t_1} \langle \nabla F(\widehat{x}(t)), X(\widehat{x}(t)) + \nabla F(\widehat{x}(t)) \rangle \, dt \\
&\quad = F(\widehat{x}(t_1)) - F(\widehat{x}(t_0)) \\
&\quad = 0.
\end{aligned}$$

Calculation of doubly symmetric solutions

Calculation of doubly symmetric solutions

Denote by $\tilde{x}_{mod}(t, x, \alpha)$ the flow of the modified equation

$$\dot{x} = X(x) + \sum_{i=1}^{k} \alpha_i \nabla F_i(x).$$

Calculation of doubly symmetric solutions

Denote by $\tilde{x}_{mod}(t, x, \alpha)$ the flow of the modified equation

$$\dot{x} = X(x) + \sum_{i=1}^{k} \alpha_i \nabla F_i(x).$$

Then we can apply the IFT to find solutions $(T, x, \alpha) \in \mathbb{R} \times \mathsf{Fix}(R_0) \times \mathbb{R}^k$ near $(T_0, x_0, 0)$ of the equation

$$\pi_1^- \tilde{x}_{mod}(T, x, \alpha) = 0.$$

Calculation of doubly symmetric solutions

One obtains (under the normality condition) a $(1+k)$-dimensional solution manifold along which

$$\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0,$$

i.e. all points on this solution manifold generate (normal) (R_0, R_1)-symmetric solutions of $\dot{x} = X(x)$.

Calculation of doubly symmetric solutions

Numerical people prefer to have 1-dimensional solution curves — then they can use e.g. the pseudo-arclength method to calculate these branches.

Calculation of doubly symmetric solutions

Numerical people prefer to have 1-dimensional solution curves — then they can use e.g. the pseudo-arclength method to calculate these branches.

Question: is it possible to add k further conditions without losing information on the full solution manifold?

Calculation of doubly symmetric solutions

Numerical people prefer to have 1-dimensional solution curves — then they can use e.g. the pseudo-arclength method to calculate these branches.

Question: is it possible to add k further conditions without losing information on the full solution manifold?

Answer: **yes**, in the **Hamiltonian** case.

The Hamiltonian case

The Hamiltonian case

We set:

- $n = 2N$;

The Hamiltonian case

We set:

- $n = 2N$;
- $x = (p, q)$, with $p, q \in \mathbb{R}^N$;

The Hamiltonian case

We set:

- $n = 2N$;
- $x = (p, q)$, with $p, q \in \mathbb{R}^N$;
- $J \in \mathcal{L}(\mathbb{R}^{2N})$ the standard symplectic matrix defined by $J(p, q) = (-q, p)$;

The Hamiltonian case

We set:

- $n = 2N$;
- $x = (p, q)$, with $p, q \in \mathbb{R}^N$;
- $J \in \mathcal{L}(\mathbb{R}^{2N})$ the standard symplectic matrix defined by $J(p, q) = (-q, p)$;
- $H : \mathbb{R}^{2N} \to \mathbb{R}$ a smooth function;

The Hamiltonian case

We set:

- $n = 2N$;
- $x = (p, q)$, with $p, q \in \mathbb{R}^N$;
- $J \in \mathcal{L}(\mathbb{R}^{2N})$ the standard symplectic matrix defined by $J(p, q) = (-q, p)$;
- $H : \mathbb{R}^{2N} \to \mathbb{R}$ a smooth function;
- $X(x) = X_H(x) := J\nabla H(x)$ the Hamiltonian vectorfield corresponding to H;

The Hamiltonian case

We set:

- $n = 2N$;
- $x = (p, q)$, with $p, q \in \mathbb{R}^N$;
- $J \in \mathcal{L}(\mathbb{R}^{2N})$ the standard symplectic matrix defined by $J(p, q) = (-q, p)$;
- $H : \mathbb{R}^{2N} \to \mathbb{R}$ a smooth function;
- $X(x) = X_H(x) := J\nabla H(x)$ the Hamiltonian vectorfield corresponding to H;
- $\tilde{x}_H(t, x)$ the corresponding Hamiltonian flow.

The Hamiltonian case

An operator $S \in O(2N)$ is a **symmetry** for X_H and $\tilde{x}_H$ if

$$JS = SJ \quad \text{and} \quad H(Sx) = H(x).$$

The Hamiltonian case

An operator $S \in O(2N)$ is a **symmetry** for X_H and $\tilde{x}_H$ if

$$JS = SJ \quad \text{and} \quad H(Sx) = H(x).$$

An operator $R \in O(2N)$ is a **reversor** for X_H and $\tilde{x}_H$ if

$$JR = -RJ \quad \text{and} \quad H(Rx) = H(x).$$

The Hamiltonian case

Noether's Theorem:

In Hamiltonian systems there is a relation between first integrals and (continuous) symmetries.

The Hamiltonian case

$$F : \mathbb{R}^{2N} \to \mathbb{R} \text{ is a first integral for } X_H$$

$$\Updownarrow$$

$$\{H, F\}(x) := \langle \nabla H(x), J \nabla F(x) \rangle = 0$$

$$\Downarrow$$

the flows $\tilde{x}_H$ and $\tilde{x}_F$ commute

The Hamiltonian case

Suppose:

- R_0 and R_1 are reversors of X_H;
- $\tilde{x}_H(t, x_0)$ is a (R_0, R_1)-symmetric solution of $\dot{x} = X_H(x)$, with basic domain $[0, T_0]$;
- $F : \mathbb{R}^{2N} \to \mathbb{R}$ is a first integral.

The Hamiltonian case

Suppose:

- R_0 and R_1 are reversors of X_H;
- $\tilde{x}_H(t, x_0)$ is a (R_0, R_1)-symmetric solution of $\dot{x} = X_H(x)$, with basic domain $[0, T_0]$;
- $F : \mathbb{R}^{2N} \to \mathbb{R}$ is a first integral.

Then

$$\tilde{x}_H(t, \tilde{x}_F(s, x_0)) = \tilde{x}_F(s, \tilde{x}_H(t, x_0)),\ s \in \mathbb{R},$$

forms a one-parameter family of solutions of $\dot{x} = X_H(x)$.

The Hamiltonian case

These solutions will also be (R_0, R_1)-symmetric (with the same basic domain $[0, T_0]$) if the flow $\tilde{x}_F$ leaves the subspaces $\mathrm{Fix}(R_0)$ and $\mathrm{Fix}(R_1)$ invariant.

The Hamiltonian case

These solutions will also be (R_0, R_1)-symmetric (with the same basic domain $[0, T_0]$) if the flow $\tilde{x}_F$ leaves the subspaces $\mathrm{Fix}(R_0)$ and $\mathrm{Fix}(R_1)$ invariant.

Easy result:

If $R \in O(2N)$ is such that $JR = -RJ$, then the flow $\tilde{x}_F$ leaves $\mathrm{Fix}(R)$ invariant if and only if F is constant on $\mathrm{Fix}(R)$.

The Hamiltonian case

These solutions will also be (R_0, R_1)-symmetric (with the same basic domain $[0, T_0]$) if the flow $\tilde{x}_F$ leaves the subspaces $\mathrm{Fix}(R_0)$ and $\mathrm{Fix}(R_1)$ invariant.

Easy result:
If $R \in O(2N)$ is such that $JR = -RJ$, then the flow $\tilde{x}_F$ leaves $\mathrm{Fix}(R)$ invariant if and only if F is constant on $\mathrm{Fix}(R)$.

So, for each $F \in \mathcal{F}$,

$$\tilde{x}_F(s, \tilde{x}_H(t, x_0)), \qquad s \in \mathbb{R},$$

forms a one-parameter family of (R_0, R_1)-symmetric solutions.

Set-up for numerical continuation

Denote by $\tilde{x}_{mod}(t, x, \alpha)$ the flow of

$$\dot{x} = X_H(x) + \sum_{i=1}^{k} \alpha_i \nabla F_i(x);$$

then find solutions $(T, x) \in \mathbb{R} \times \mathrm{Fix}(R_0)$ of

$$\pi_1^- \tilde{x}_{mod}(T, x, \alpha) = 0,$$

subject to k additional **phase conditions** of the form

$$\langle X_{F_i}(x_0), x - x_0 \rangle = 0, \quad (1 \leq i \leq k).$$

Set-up for numerical continuation

One can show that this is a regular problem, suitable for pseudo-arclength continuation, and leading to one-dimensional solution branches along which $\alpha = 0$.

The phase conditions prevent the recalculation of those doubly symmetric solutions which can be obtained from $\tilde{x}_H(t, x_0)$ or its continuation by application of the symmetries $\tilde{x}_{F_i}(s, \cdot)$ ($s \in \mathbb{R}$, $1 \leq i \leq k$).

Set-up for numerical continuation

In practice the phase conditions

$$\langle X_{F_i}(x_0), x - x_0 \rangle = 0, \quad (1 \le i \le k),$$

are replaced by some "averaged" version, such as

$$\int_0^1 \langle X_{F_i}(\tilde{x}_H(T\tau, x_0)), \tilde{x}_{mod}(T\tau, x, \alpha) - \tilde{x}_H(T\tau, x_0)) \rangle \, d\tau = 0;$$

such integral conditions seem to give much better numerical results.

More details:

Continuation of Normal Doubly Symmetric Orbits in Conservative Reversible Systems.

F.J. Muñoz-Almaraz*, E. Freire†, J. Galán† and A. Vanderbauwhede‡
*Departamento de Ciencias Físicas, Matemáticas y de la Computación. Universidad Cardenal Herrera-CEU. 46115 Alfara del Patriarca, Spain.
†Departamento de Matemática Aplicada II. Escuela Superior de Ingenieros de Sevilla. Camino de los Descubrimientos s/n. Sevilla 41092, Spain.
‡Department of Pure Mathematics and Computer Algebra, University of Gent. Krijgslaan 281, B-9000 Gent, Belgium.

Abstract. In this paper we introduce the concept of a quasi-submersive mapping between two finite-dimensional spaces, we obtain the main properties of such mappings, and we introduce "normality conditions" under which a particular class of so-called "constrained mappings" are quasi-submersive at their zeros. Our main application is concerned with the continuation properties of normal doubly symmetric orbits in time-reversible systems with one or more first integrals. As examples we study the continuation of the figure-eight and the supereight choreographies in the N-body problem.

MORE DETAILS:

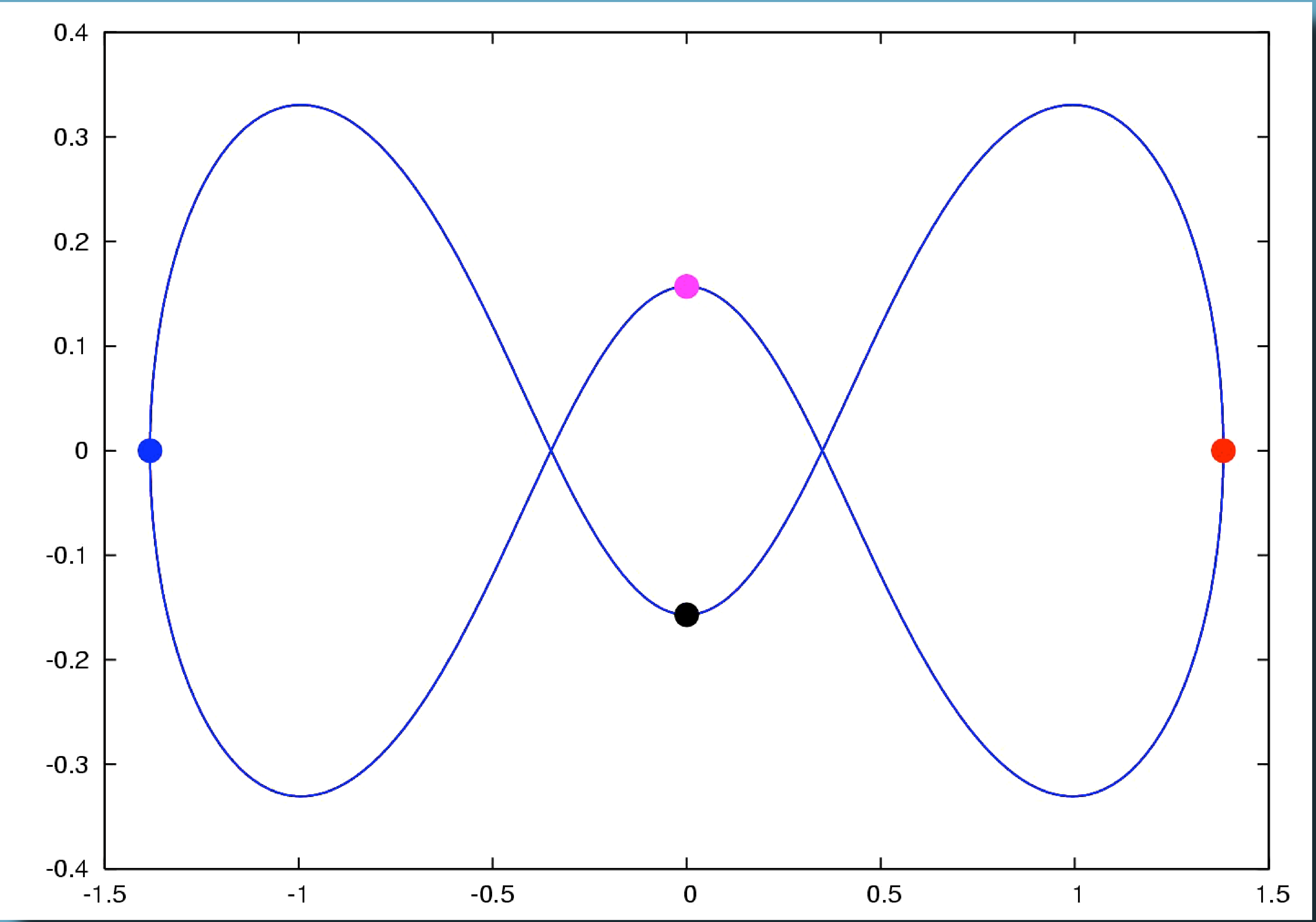
Continuation of Normal Doubly Symmetric Orbits in Conservative Reversible Systems.

F.J. Muñoz-Almaraz⋆, E. Freire†, J. Galán† and A. Vanderbauwhede‡
⋆ *Departamento de Ciencias Físicas, Matemáticas y de la Computación. Universidad Cardenal Herrera-CEU. 46115 Alfara del Patriarca, Spain.*
† *Departamento de Matemática Aplicada II. Escuela Superior d... ...nieros de Sevilla. Camino de los Descubrimientos s/n. Sevilla ...*
‡ *Department of Pure Mathematics and Co... ...of Gent. Krijgslaan 281, B-9000 Gent, Bel...*

Abstract. In this paper we ... mapping between two finite-dimensional ... ...ch mappings, and we introduce "norm... ...class of so-called "constrained mappings"... ...in application is concerned with the co... ...metric orbits in time-reversible systems w... ...examples we study the continuation of the figure-... ...choreographies in the N-body problem.

Application to n-body problems

Gerver's supereight



The n-body problem

- We work in the plane;
- we consider $N \geq 3$ bodies with masses

$$m_1, m_2, \ldots, m_N;$$

- phase space is $\mathbb{R}^{4N}$;
- $x = (p_1, p_2, \ldots, p_N, q_1, q_2, \ldots, q_N)$, with

$$p_j \in \mathbb{R}^2 = \text{ momentum of body } j$$

and

$$q_j \in \mathbb{R}^2 = \text{ position of body } j.$$

The n-body problem

$$H(x) = \sum_{j=1}^{N} \frac{1}{2m_j}\|p_j\|^2 - \sum_{1 \le i < j \le N} \frac{m_i m_j}{\|q_i - q_j\|}.$$

The N-body problem

$$H(x) = \sum_{j=1}^{N} \frac{1}{2m_j}\|p_j\|^2 - \sum_{1\le i<j\le N} \frac{m_i m_j}{\|q_i - q_j\|}.$$

The Hamiltonian H is invariant under the symplectic action of rotations in the plane given by

$$\begin{aligned}&\Psi_\theta(p_1,\dots,p_N,q_1,\dots,q_N)\\ &\quad := \ (e^{A\theta}p_1,\dots,e^{A\theta}p_N,e^{A\theta}q_1,\dots,e^{A\theta}q_N) \quad (\theta\in S^1),\end{aligned}$$

with $A \in \mathcal{L}(\mathbb{R}^2)$ given by

$$A := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The n-body problem

$$H(x) = \sum_{j=1}^{N} \frac{1}{2m_j}\|p_j\|^2 - \sum_{1\leq i<j\leq N} \frac{m_i m_j}{\|q_i - q_j\|}.$$

The corresponding first integral is the **total angular momentum**

$$L_0(x) := \sum_{j=1}^{N} q_j \cdot (Ap_j).$$

The n-body problem

$$H(x) = \sum_{j=1}^{N} \frac{1}{2m_j}\|p_j\|^2 - \sum_{1\le i<j\le N} \frac{m_i m_j}{\|q_i - q_j\|}.$$

The Hamiltonian H is also invariant under the symplectic action of translations in the plane given by

$$\begin{aligned} &T_b(p_1, \ldots, p_N, q_1, \ldots, q_n) \\ &\quad := \ (p_1, \ldots, p_N, q_1 + b, \ldots, q_N + b) \quad (b \in \mathbb{R}^2). \end{aligned}$$

The N-body problem

$$H(x) = \sum_{j=1}^{N} \frac{1}{2m_j} \|p_j\|^2 - \sum_{1 \le i < j \le N} \frac{m_i m_j}{\|q_i - q_j\|}.$$

The corresponding first integrals are the two components

$$P_1(x) := e_1 \cdot P(x) \quad \text{and} \quad P_2(x) := e_2 \cdot P(x)$$

of the **total linear momentum**

$$P(x) := \sum_{j=1}^{N} p_j.$$

The n-body problem

$$H(x) = \sum_{j=1}^{N} \frac{1}{2m_j}\|p_j\|^2 - \sum_{1\le i<j\le N} \frac{m_i m_j}{\|q_i - q_j\|}.$$

Since the total linear momentum $P(x)$ is constant one can use a uniformly moving frame in $\mathbb{R}^2$ such that $P(x) = 0$, which then implies that the **center of mass**

$$Q(x) := \sum_{j=1}^{N} m_j q_j$$

is constant.

The N-body problem

$$H(x) = \sum_{j=1}^{N} \frac{1}{2m_j}\|p_j\|^2 - \sum_{1\le i<j\le N} \frac{m_i m_j}{\|q_i - q_j\|}.$$

The Hamiltonian system X_H is also equivariant with respect to reflections in the plane, given by $\Phi \circ \Psi_\theta$ $(\theta \in S^1)$, where

$$\begin{aligned}\Phi(p_1, \ldots, p_N, q_1, \ldots, q_N) \\ := \ (Sp_1, \ldots, Sp_N, Sq_1, \ldots, Sq_N),\end{aligned}$$

with $S \in \mathcal{L}(\mathbb{R}^2)$ given by

$$Se_1 := e_1 \quad \text{and} \quad Se_2 = -e_2.$$

The n-body problem

$$H(x) = \sum_{j=1}^{N} \frac{1}{2m_j}\|p_j\|^2 - \sum_{1\le i<j\le N} \frac{m_i m_j}{\|q_i - q_j\|}.$$

If $m_i = m_j$ $(1 \le i < j \le N$ then we also have the **exchange symmetry**

$$\begin{aligned}\Sigma_{i,j}(\ldots, p_i, \ldots, p_j, \ldots, q_i, \ldots, q_j, \ldots) \\ := \ (\ldots, p_j, \ldots, p_i, \ldots, q_j, \ldots, q_i, \ldots).\end{aligned}$$

The n-body problem

$$H(x) = \sum_{j=1}^{N} \frac{1}{2m_j}\|p_j\|^2 - \sum_{1\le i<j\le N} \frac{m_i m_j}{\|q_i - q_j\|}.$$

The Hamiltonian system X_H has a **natural time reversor** given by

$$R(p_1, \ldots, p_N, q_1, \ldots, q_N) := (-p_1, \ldots, -p_N, q_1, \ldots, q_N).$$

The N-body problem

$$H(x) = \sum_{j=1}^{N} \frac{1}{2m_j}\|p_j\|^2 - \sum_{1 \le i < j \le N} \frac{m_i m_j}{\|q_i - q_j\|}.$$

Moreover, each of the compositions

$$\Psi_\theta \circ R, \quad \Phi \circ R \quad \text{and} \quad \Sigma_{i,j} \circ R$$

forms a reversor. Also

$$R^2 = (\Psi_\pi \circ R)^2 = (\Phi \circ R)^2 = (\Sigma_{i,j} \circ R)^2 = I.$$

The n-body problem

$$H(x) = \sum_{j=1}^{N} \frac{1}{2m_j}\|p_j\|^2 - \sum_{1\le i<j\le N} \frac{m_i m_j}{\|q_i - q_j\|}.$$

Finally, the system X_H has a **scaling symmetry**:

$$H(\lambda p, \lambda^{-2} q) = \lambda^2 H(p,q), \quad \forall \lambda \in \mathbb{R}\setminus\{0\}.$$

The N-body problem

$$H(x) = \sum_{j=1}^{N} \frac{1}{2m_j}\|p_j\|^2 - \sum_{1\le i<j\le N} \frac{m_i m_j}{\|q_i - q_j\|}.$$

Finally, the system X_H has a **scaling symmetry**:

$$H(\lambda p, \lambda^{-2} q) = \lambda^2 H(p,q), \quad \forall \lambda \in \mathbb{R} \setminus \{0\}.$$

This implies that for each solution $x(t) = (p(t), q(t))$ of $\dot{x} = X_H(x)$ and for each $\lambda \neq 0$ also

$$x_\lambda(t) := (\lambda p(\lambda^3 t), \lambda^{-2} q(\lambda^3 t))$$

is a solution.

Gerver's supereight

$$H(x) = \frac{1}{2}\sum_{j=1}^{4} \|p_j\|^2 - \sum_{1\le i<j\le 4} \frac{1}{\|q_i - q_j\|}.$$

Next we turn to the special case of **Gerver's supereight choreography**, where $N = 4$ and $m_1 = m_2 = m_3 = m_4 = 1$.

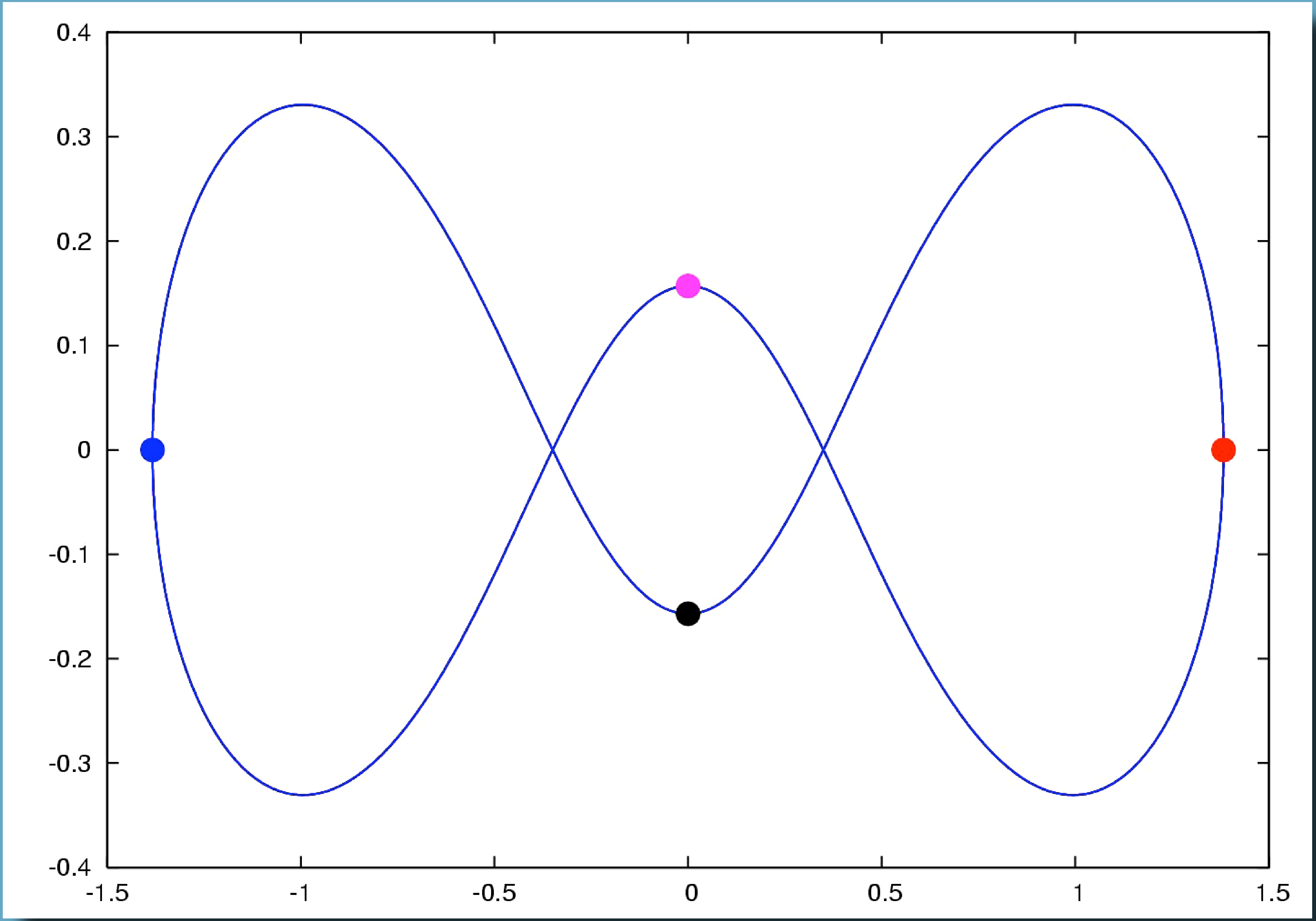
Gerver's supereight

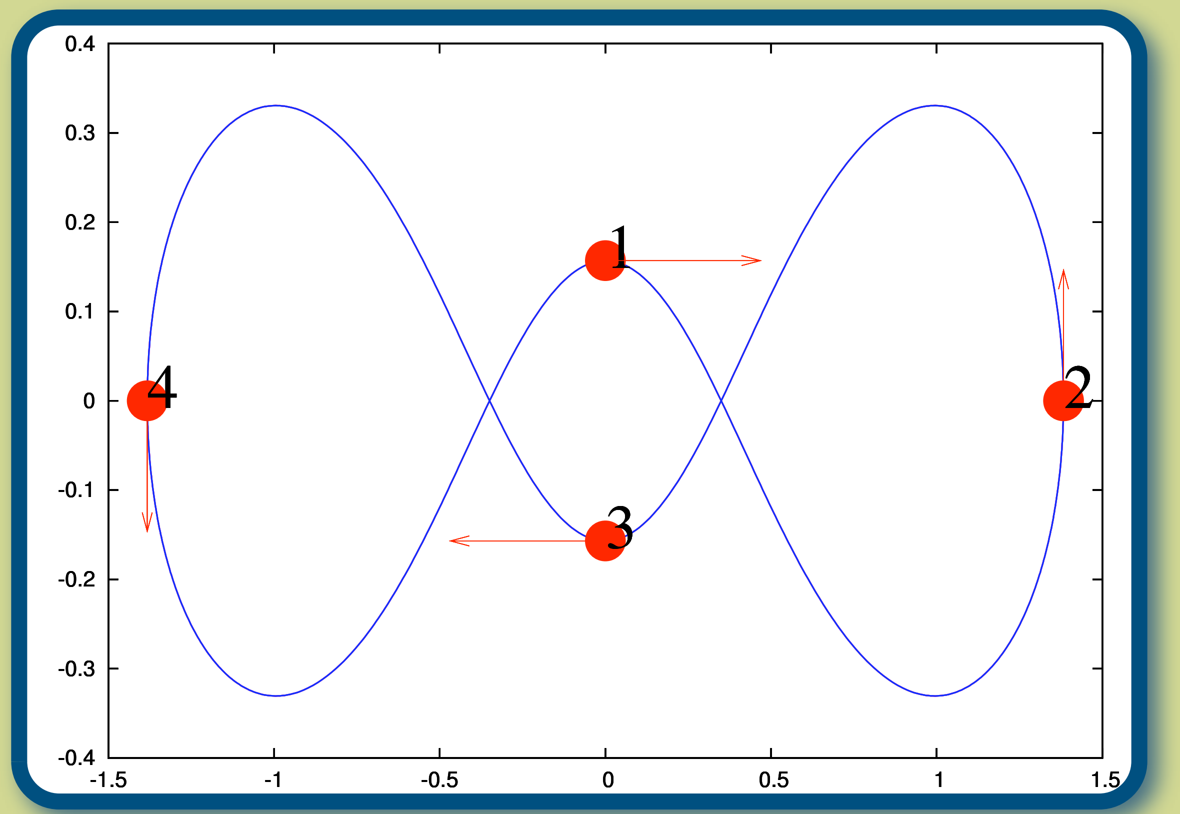
$$H(x) = \frac{1}{2}\sum_{j=1}^{4} \|p_j\|^2 - \sum_{1\le i<j\le 4} \frac{1}{\|q_i - q_j\|}.$$

Next we turn to the special case of **Gerver's supereight choreography**, where $N = 4$ and $m_1 = m_2 = m_3 = m_4 = 1$.

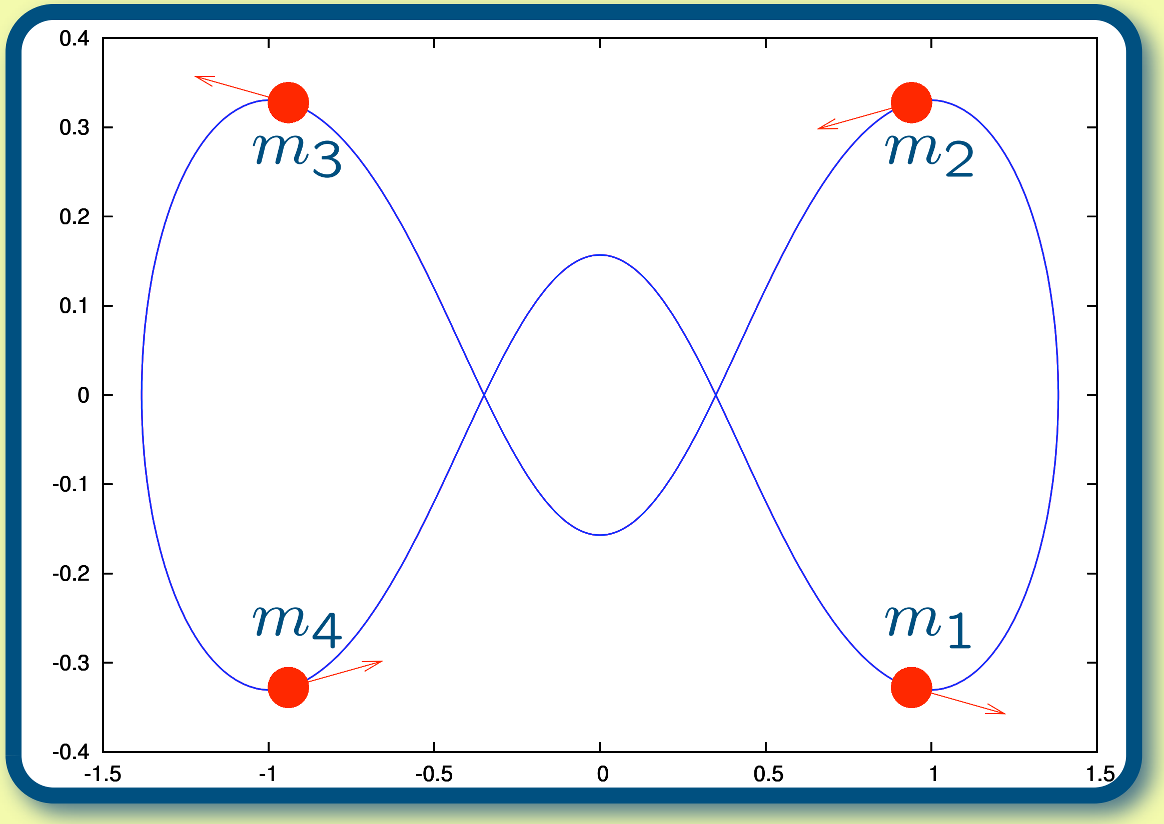
We want to find out how this choreography can be considered as a doubly symmetric solution and how it can be continued, not only within the system itself, but also when we change some external parameters which we will introduce.

Gerver's supereight

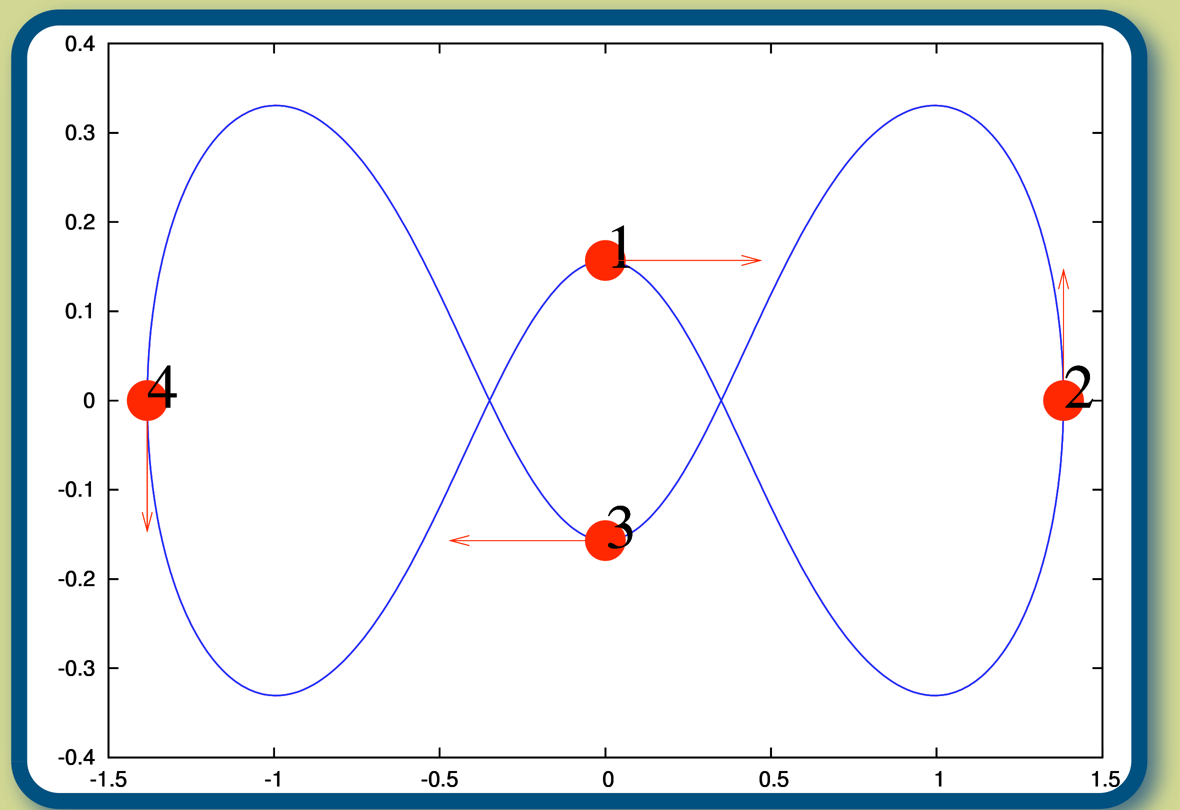


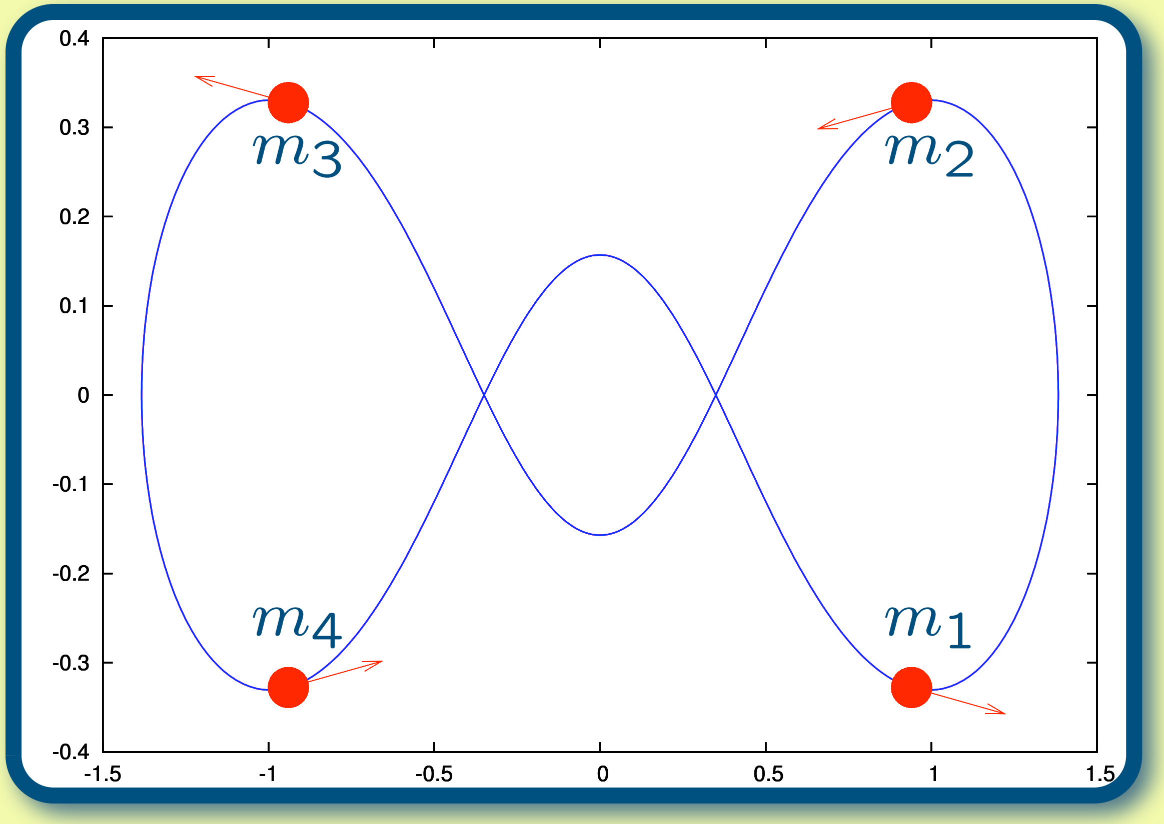


1
2
3
4



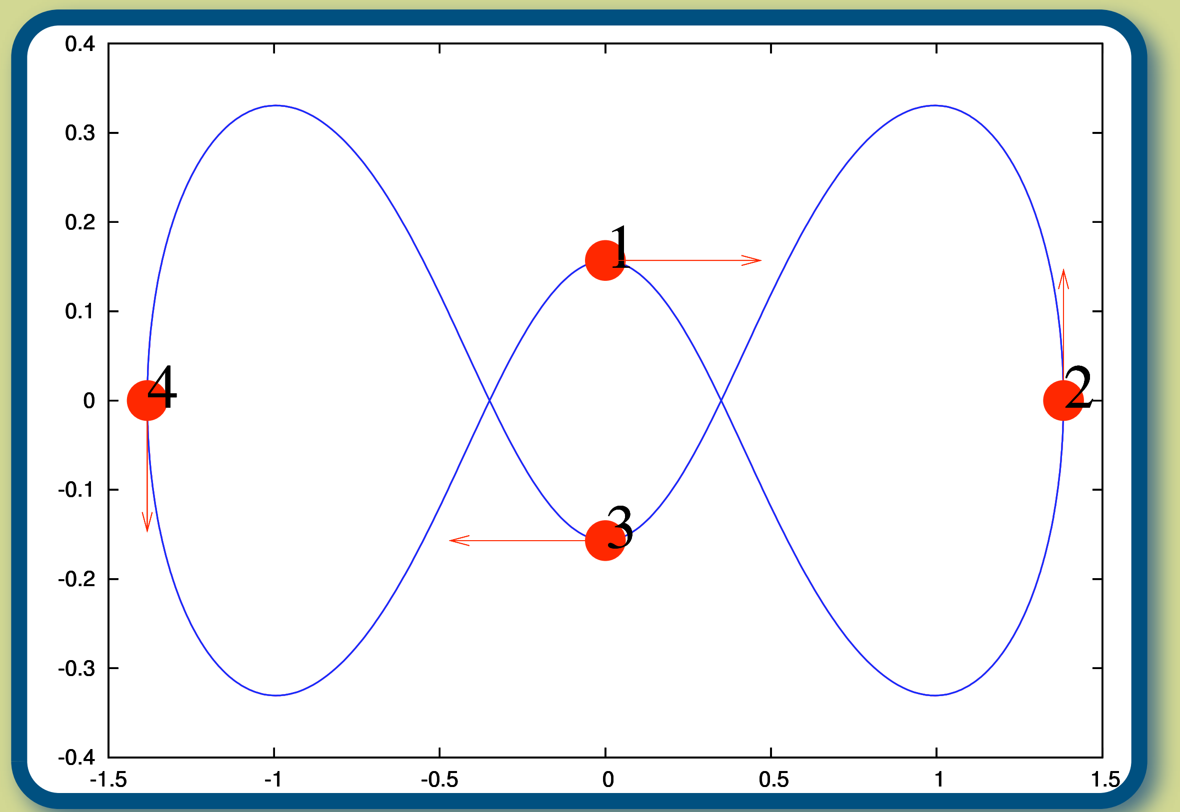
m_1
m_2
m_3
m_4





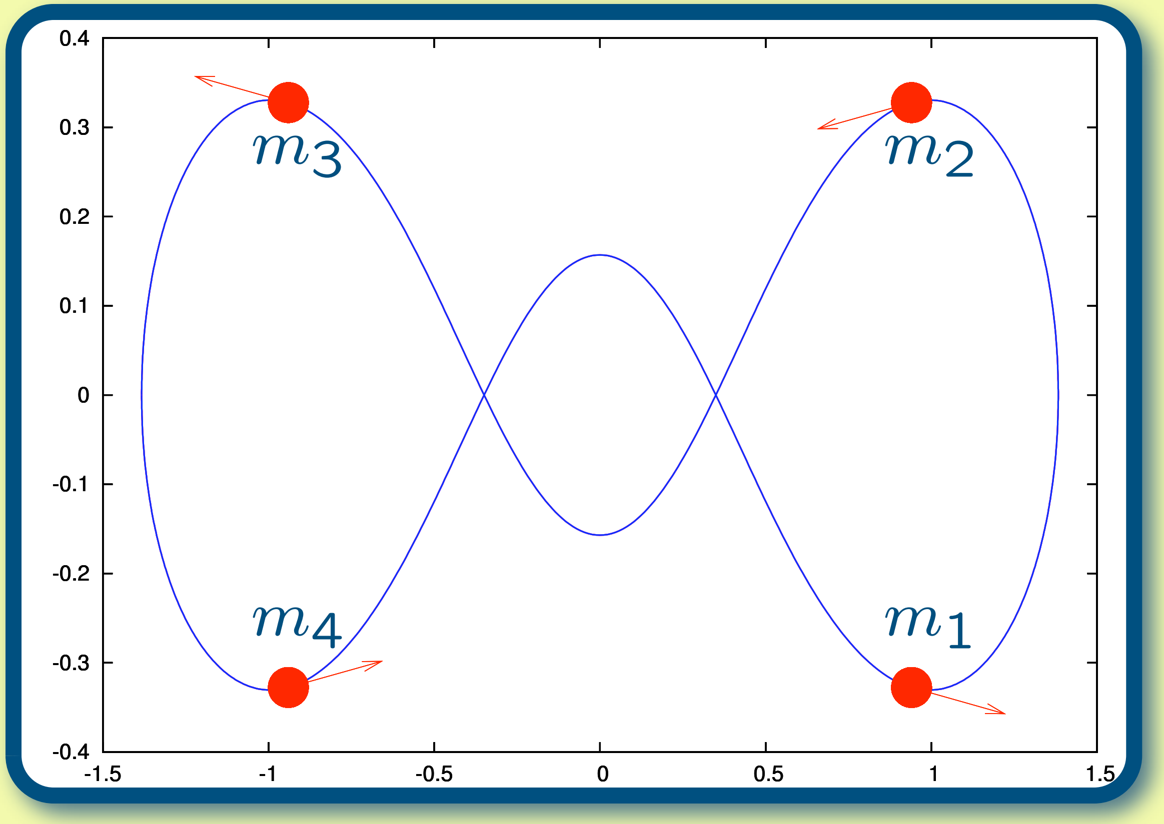
$$x(0) \in \mathrm{Fix}(R_0)$$

$$R_0 = R \circ \Sigma_{1,3} \circ \Phi$$



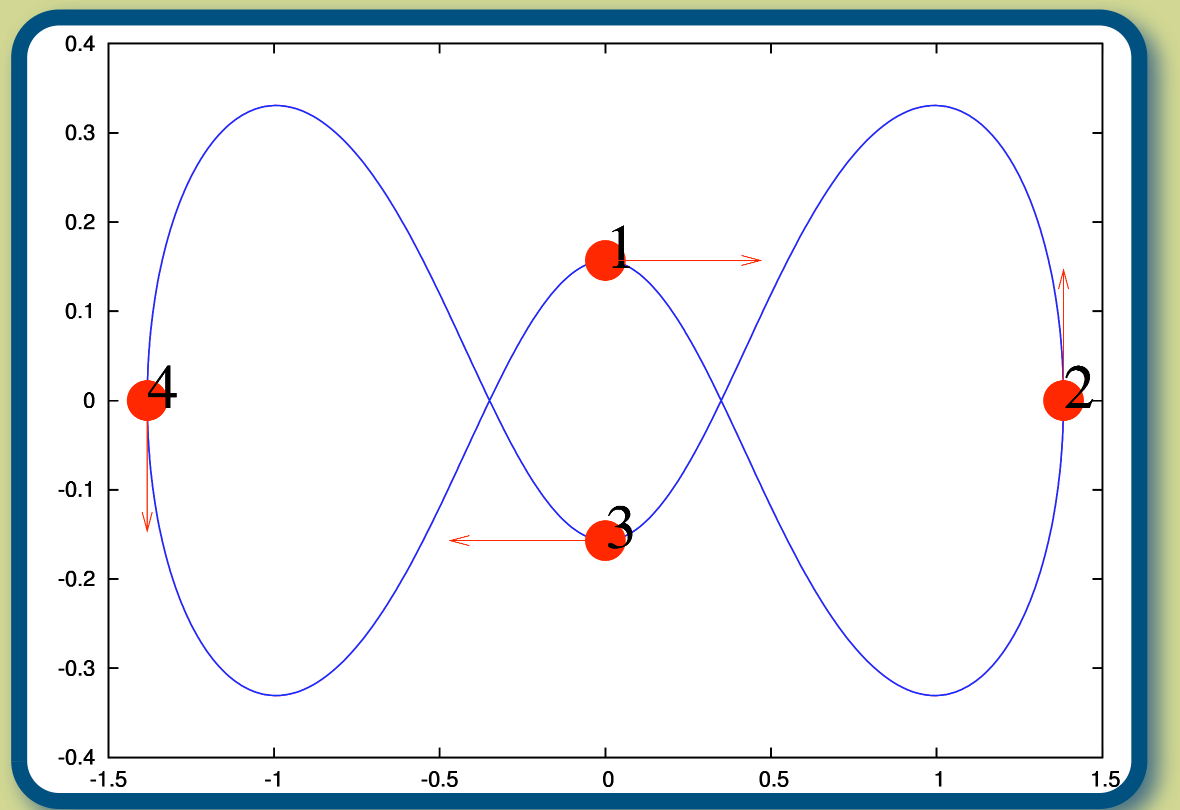
$$x(0) \in \mathsf{Fix}(R_0)$$

$$R_0 = R \circ \Sigma_{1,3} \circ \Phi$$



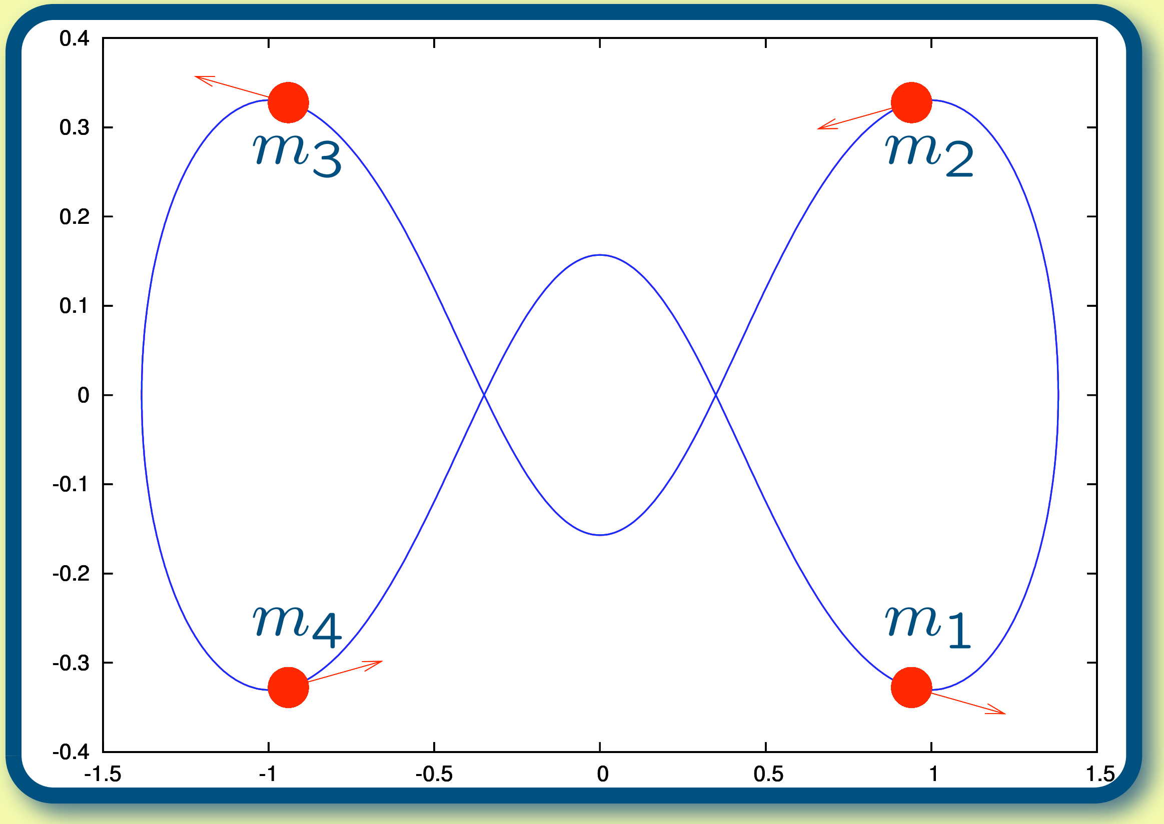
$$x(T_0) \in \mathsf{Fix}(R_1)$$

$$R_1 = R \circ \Sigma_{1,2} \circ \Sigma_{3,4} \circ \Phi$$



$$x(0) \in \mathsf{Fix}(R_0)$$

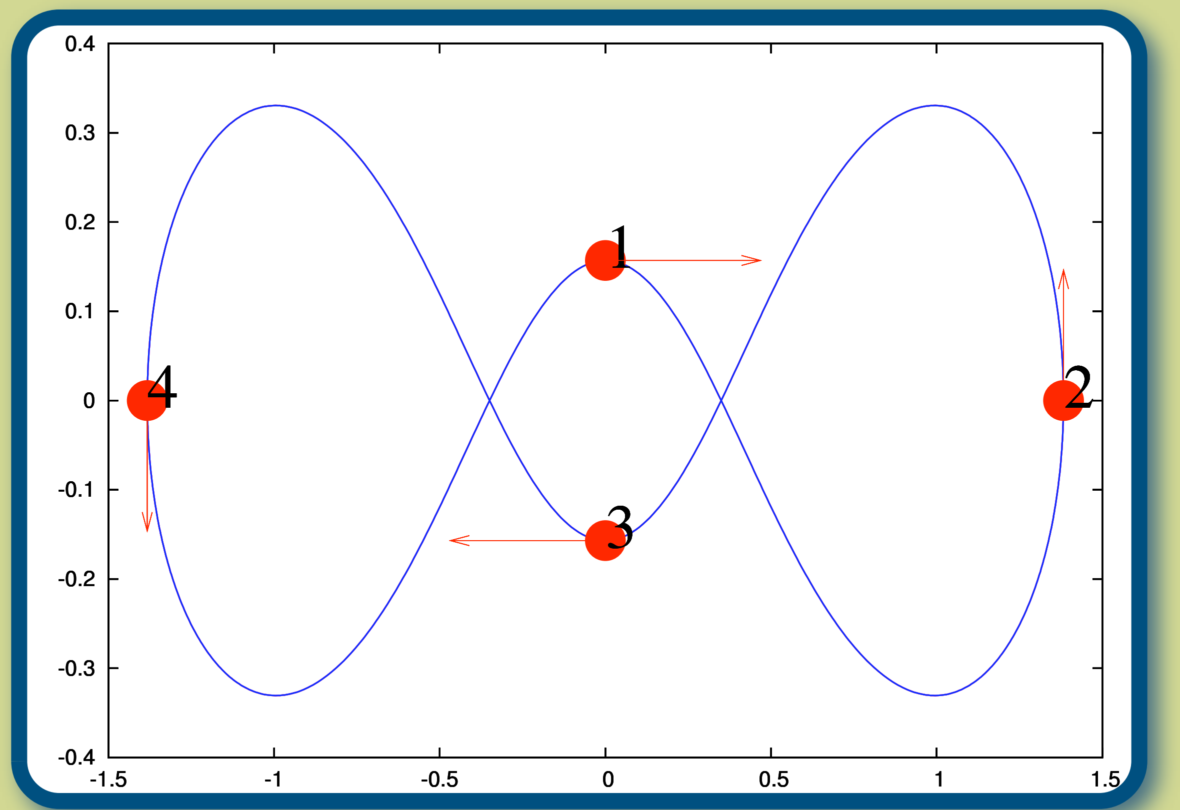
$$R_0 = R \circ \Sigma_{1,3} \circ \Phi$$



$$x(T_0) \in \mathsf{Fix}(R_1)$$

$$R_1 = R \circ \Sigma_{1,2} \circ \Sigma_{3,4} \circ \Phi$$

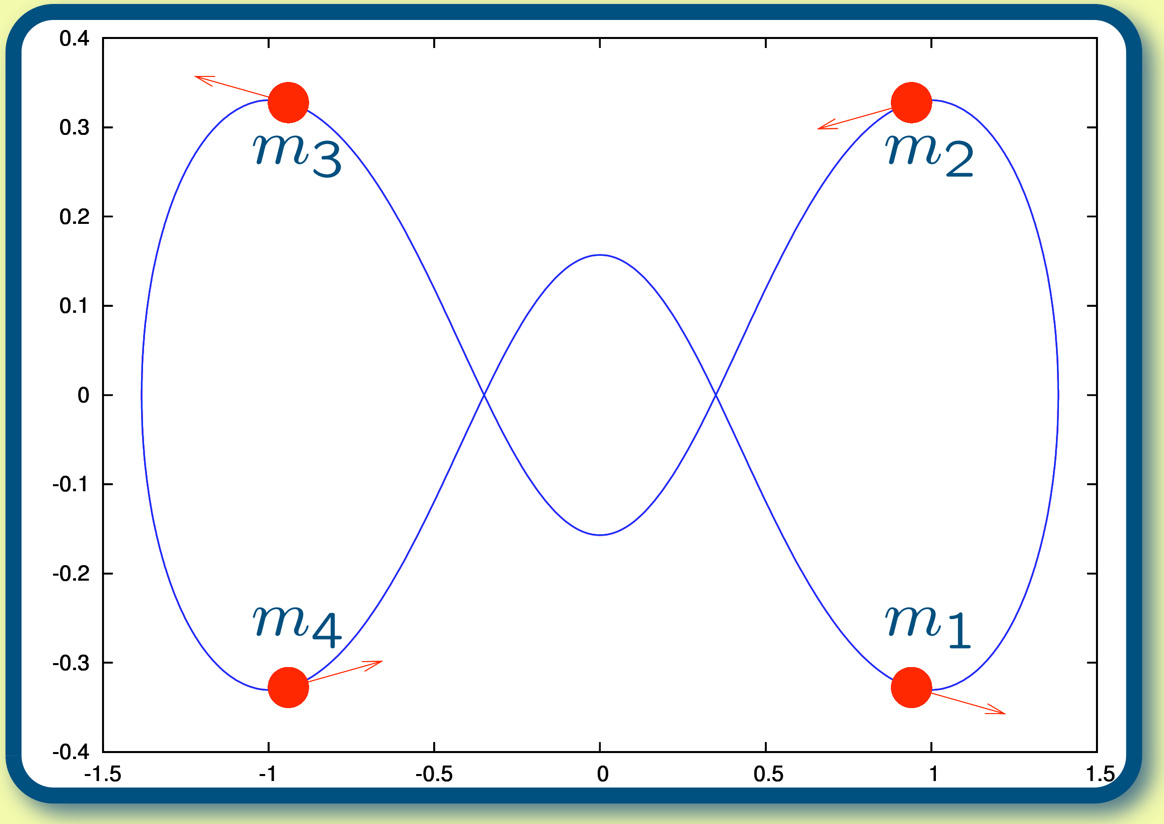
$$T_0 = \frac{1}{8} \times \text{full period}$$



$$x(0) \in \mathsf{Fix}(R_0)$$

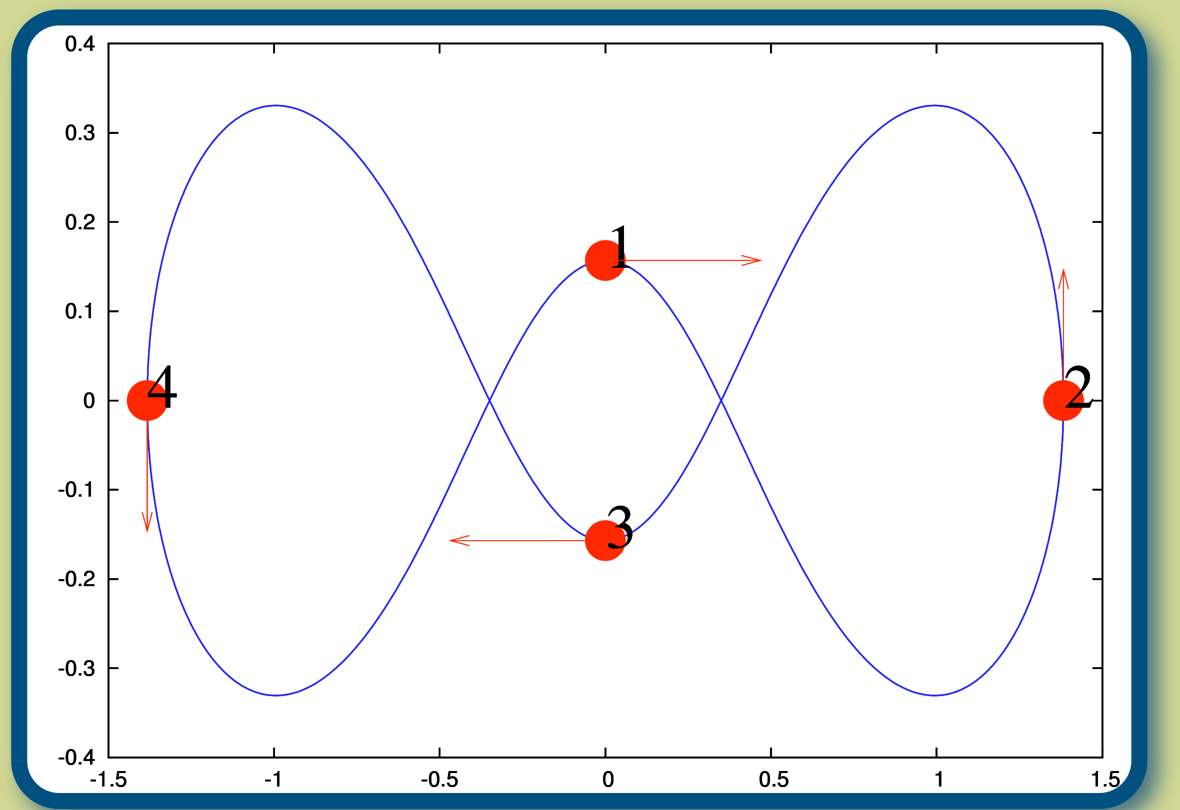
$$R_0 = R \circ \Sigma_{1,3} \circ \Phi$$

$$(R_1 R_0)^4 = I \qquad \Rightarrow$$



$$x(T_0) \in \mathsf{Fix}(R_1)$$

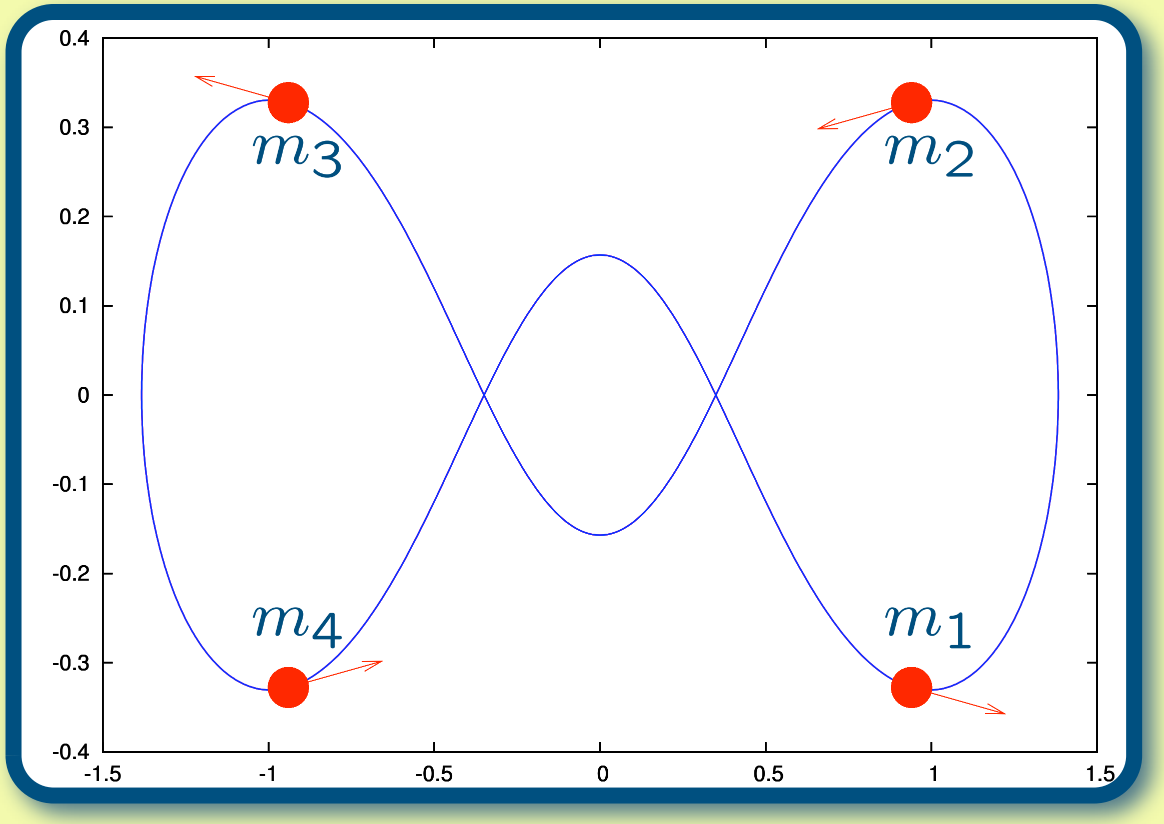
$$R_1 = R \circ \Sigma_{1,2} \circ \Sigma_{3,4} \circ \Phi$$



$x(0) \in \mathrm{Fix}(R_0)$

$R_0 = R \circ \Sigma_{1,3} \circ \Phi$

$(R_1 R_0)^4 = I \quad \Rightarrow$



$x(T_0) \in \mathrm{Fix}(R_1)$

$R_1 = R \circ \Sigma_{1,2} \circ \Sigma_{3,4} \circ \Phi$

(R_0, R_1)-symmetric solutions with basic domain $[0, T]$ are $8T$-periodic.

Continuation of the supereight

Of the 4 nontrivial first integrals (H, L_0, P_1 and P_2) only P_1 is constant on $\mathrm{Fix}(R_0) \cup \mathrm{Fix}(R_1)$, so $k = 1$.

Continuation of the supereight

Of the 4 nontrivial first integrals (H, L_0, P_1 and P_2) only P_1 is constant on $\mathrm{Fix}(R_0) \cup \mathrm{Fix}(R_1)$, so $k = 1$.

Therefore, the supereight belongs to a two-parameter family of (R_0, R_1)-symmetric solutions (normality was checked numerically). Each member of this family can be obtained from any other member by using the scaling symmetry and translation in the e_1-direction.

Continuation of the supereight

So, in order to obtain some non-trivial continuation of the supereight as a (R_0, R_1)-symmetric solution we need to introduce some external parameters in the Hamiltonian; this must be done in such a way that both R_0 and R_1 remain reversors.

This last condition prevents us from changing any of the masses, leaving us with the alternative to change the potential; we take

$$H_\gamma(x) := \frac{1}{2}\sum_{j=1}^{4} \|p_j\|^2 - \sum_{1\le i<j\le 4} \frac{1}{\|q_i - q_j\|^\gamma}.$$

Continuation of the supereight

$$H_\gamma(x) := \frac{1}{2}\sum_{j=1}^{4}\|p_j\|^2 - \sum_{1\le i<j\le 4}\frac{1}{\|q_i - q_j\|^\gamma}$$

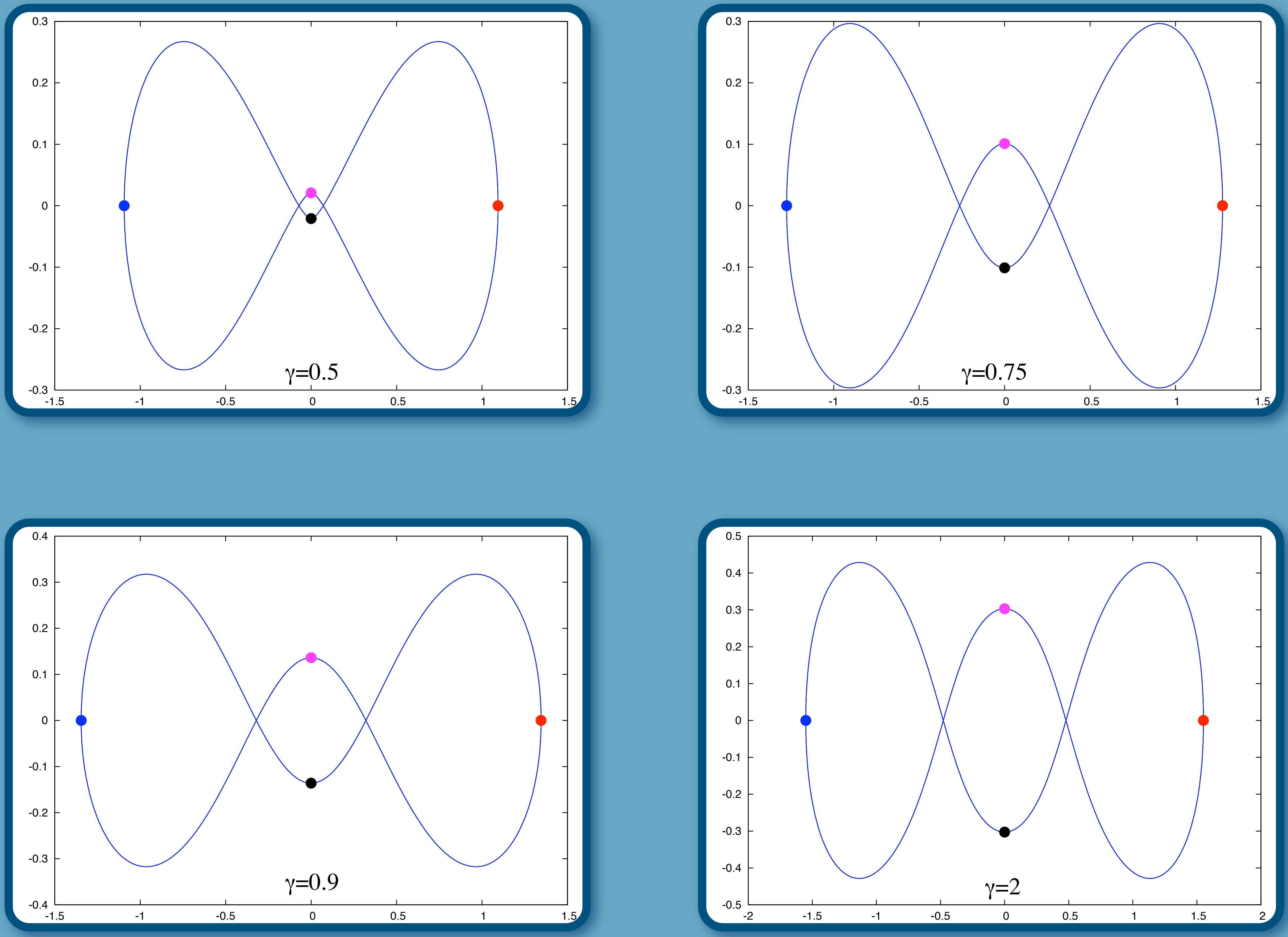
Continuation of the supereight

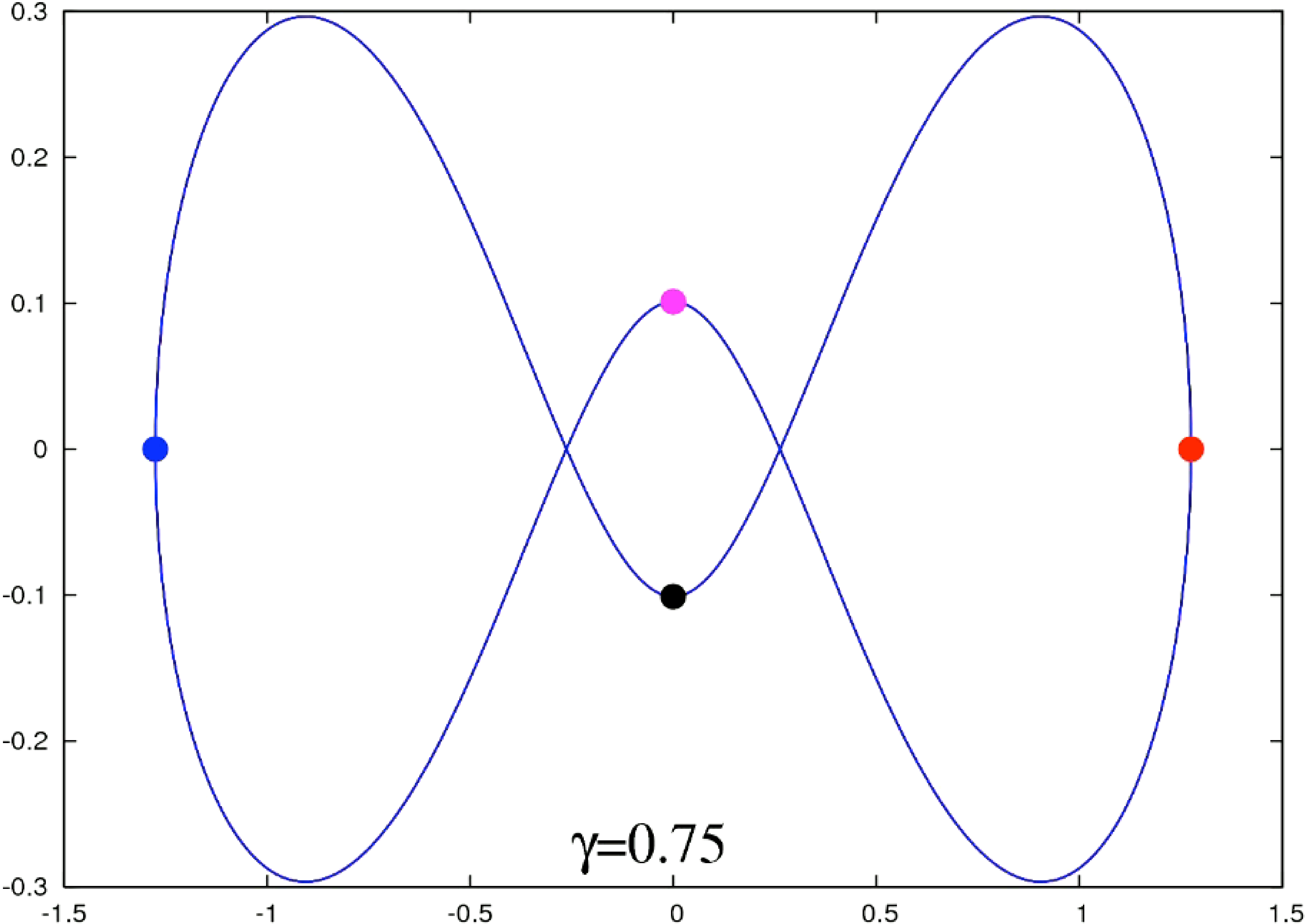
$$H_\gamma(x) := \frac{1}{2}\sum_{j=1}^{4} \|p_j\|^2 - \sum_{1\le i<j\le 4} \frac{1}{\|q_i - q_j\|^\gamma}$$

We consider the system

$$\dot{x} = X_{H_\gamma}(x)$$

and use our continuation techniques to continue the supereight (which appears for $\gamma = 1$) as a (R_0, R_1)-symmetric solution; we fix the basic domain (to prevent scaling), add a phase condition corresponding to P_1, and do continuation in the parameter γ.


γ=0.5
γ=0.75
γ=0.9
γ=2


0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-1.5
-1
-0.5
0
0.5
1
1.5
γ=0.75

Continuation of the supereight

The supereight can also be considered as a R_0-symmetric solution, that is as a (R_0, R_0)-symmetric solution with basic domain $[0, 4T_0]$. Again P_1 is the only first integral which is constant on $\mathrm{Fix}(R_0)$, and so we get by continuation a 2-dimensional family of R_0-symmetric solutions, which coincides with the 2-dimensional family of (R_0, R_1)-symmetric solutions which we found before.

Continuation of the supereight

The supereight can also be considered as a R_0-symmetric solution, that is as a (R_0, R_0)-symmetric solution with basic domain $[0, 4T_0]$. Again P_1 is the only first integral which is constant on Fix(R_0), and so we get by continuation a 2-dimensional family of R_0-symmetric solutions, which coincides with the 2-dimensional family of (R_0, R_1)-symmetric solutions which we found before.

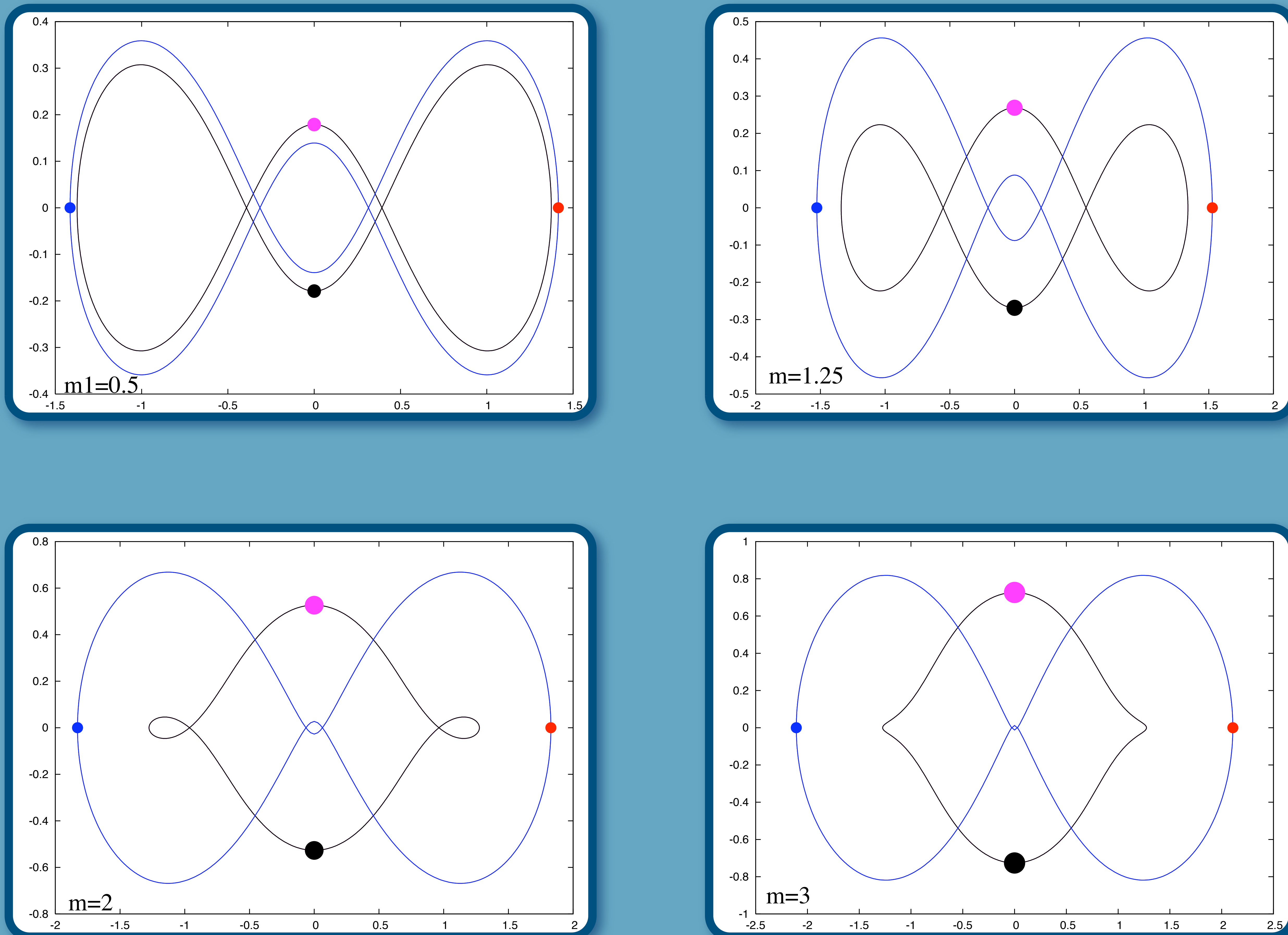
However, this time we can use the masses as external parameters: since only R_0 has to remain a reversor, the only condition is that $m_1 = m_3$.

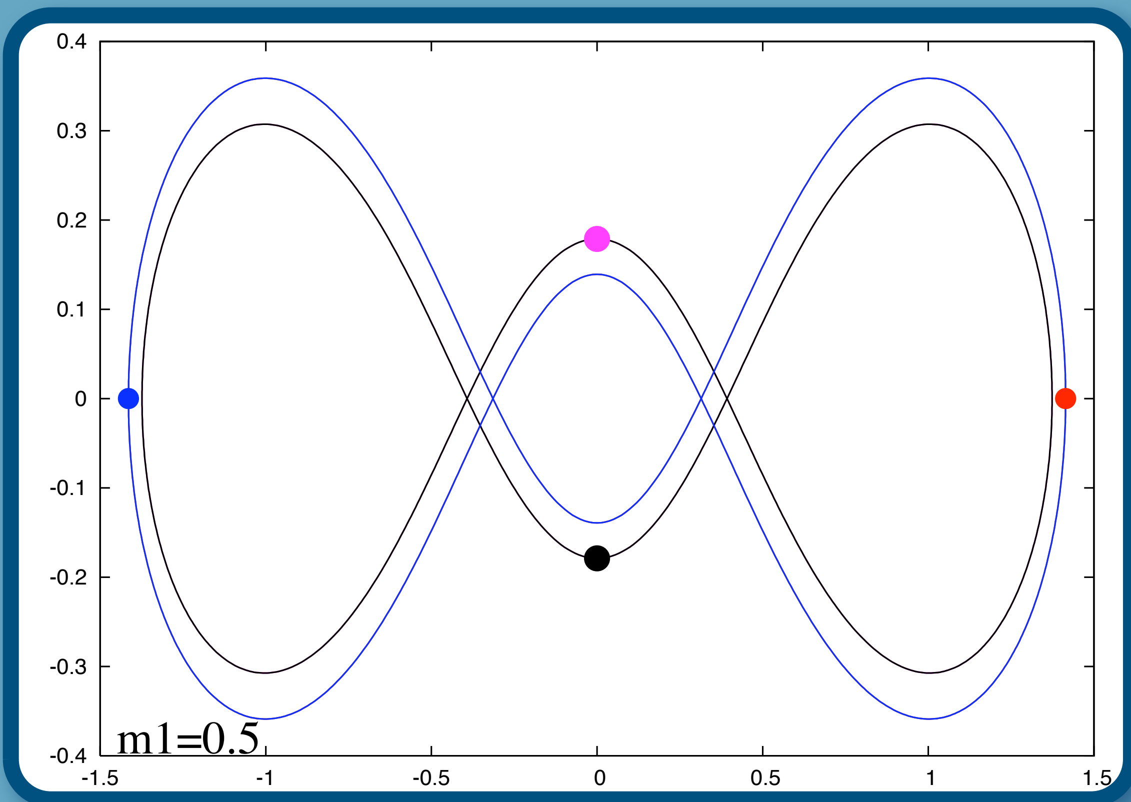
Continuation of the supereight

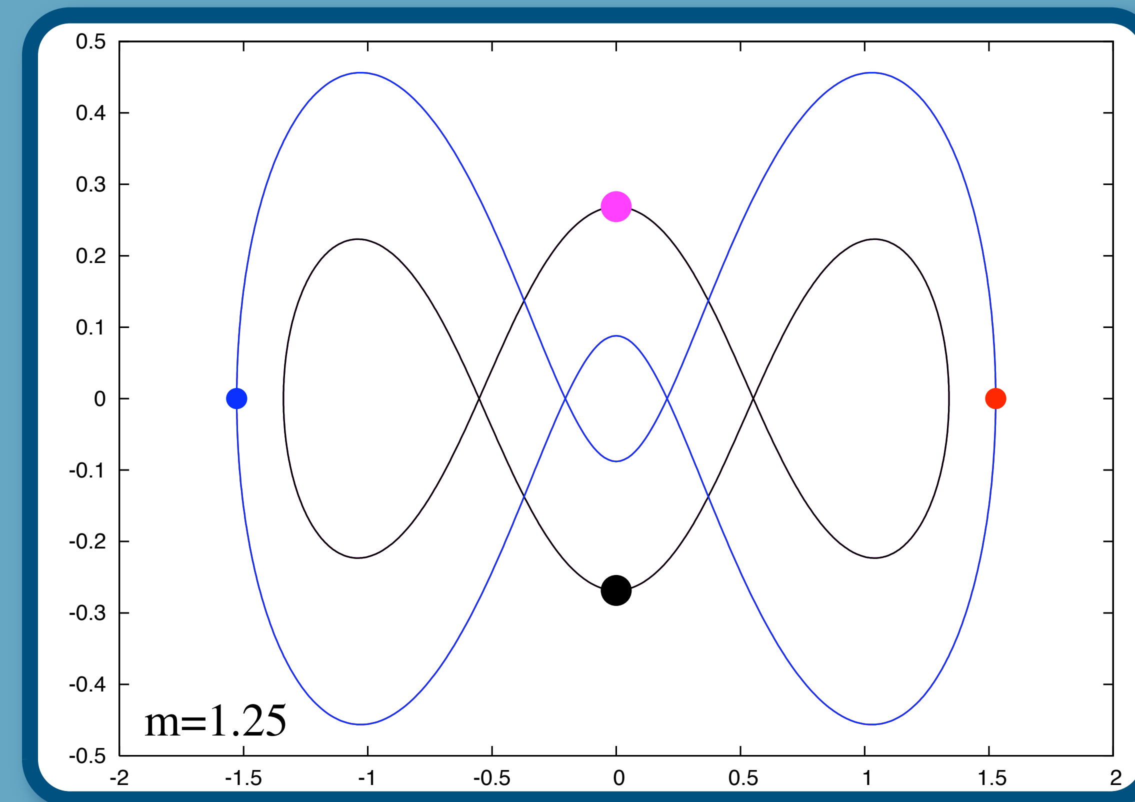
We take the simplest possible case:

$$m_1 = m_3 = m \quad \text{and} \quad m_2 = m_4 = 1,$$

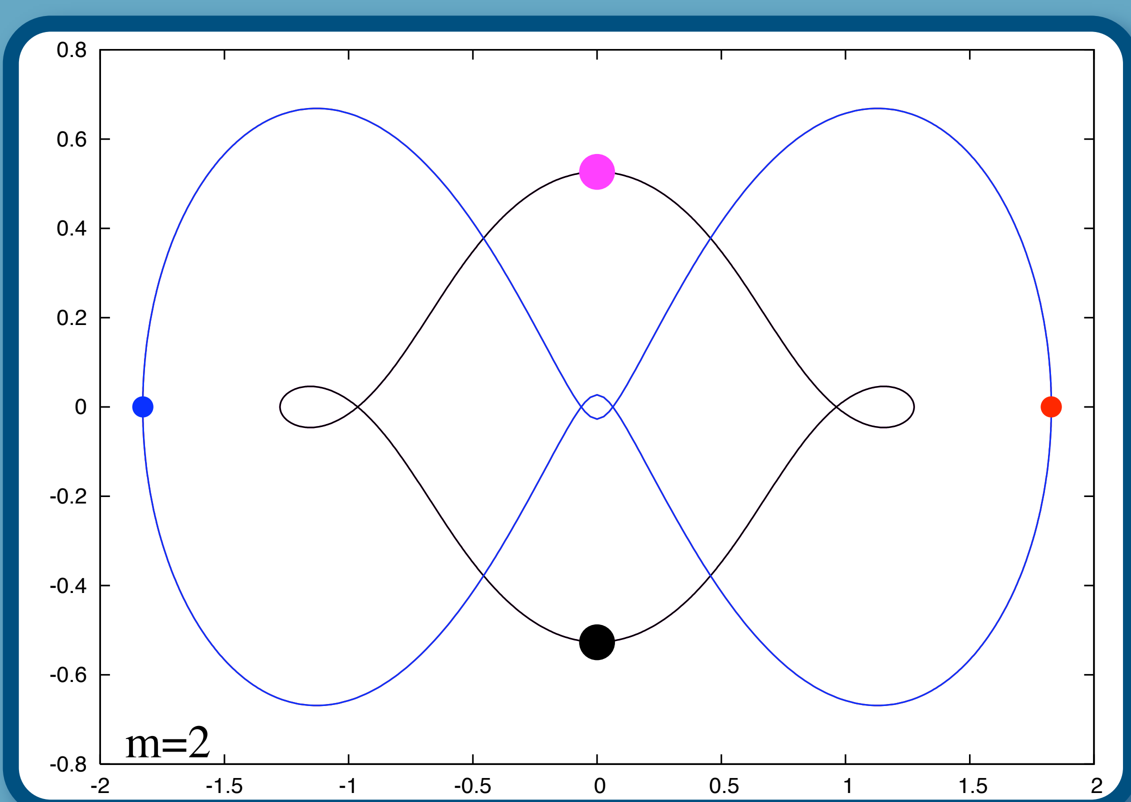
and use m as the continuation parameter. Again, we fix the basic domain to prevent scaling, and add a phase condition corresponding P_1 to prevent translations in the e_1-direction.

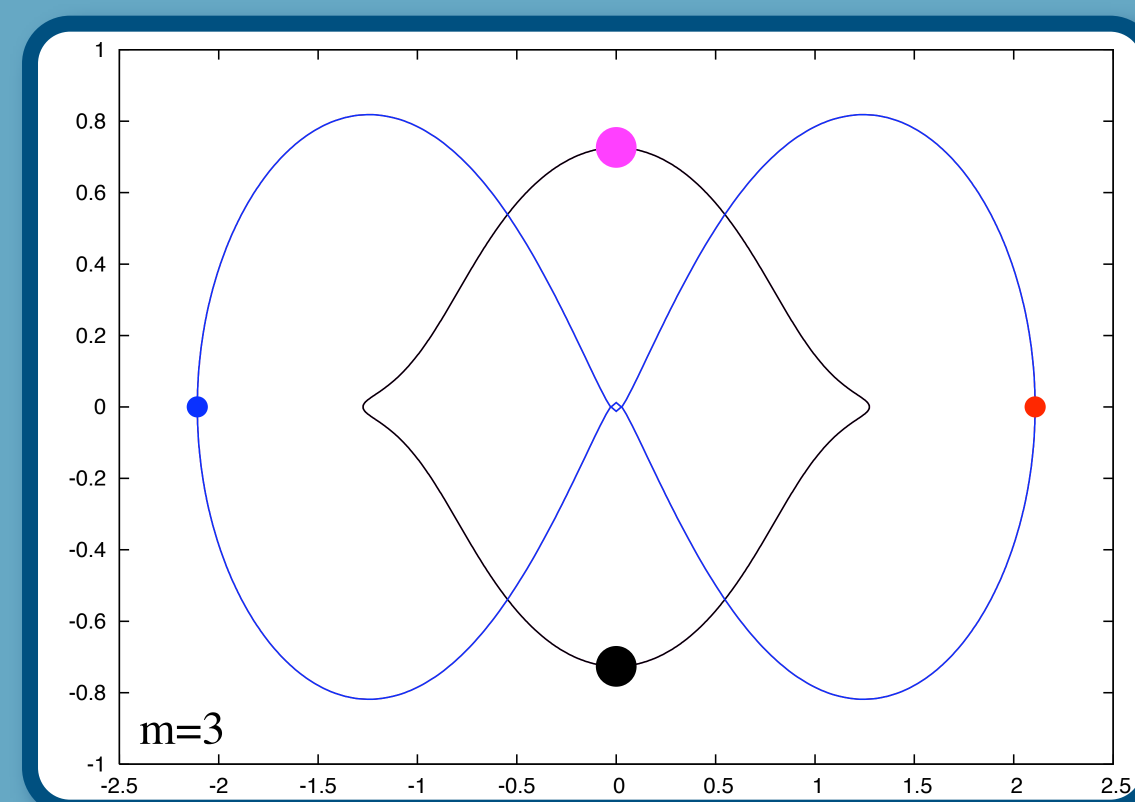

m1=0.5
m=1.25
m=2
m=3





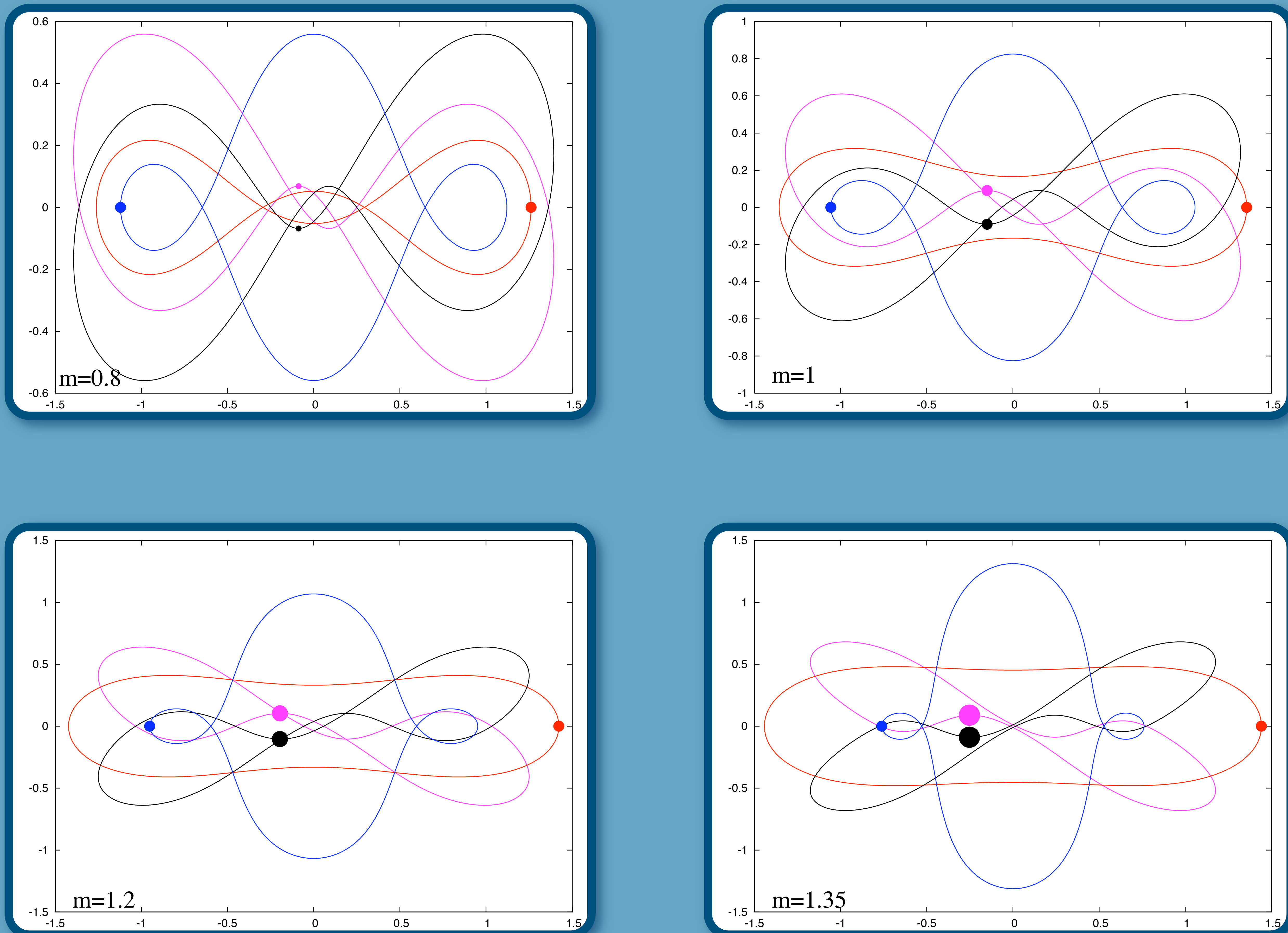
These are only partial choreographies

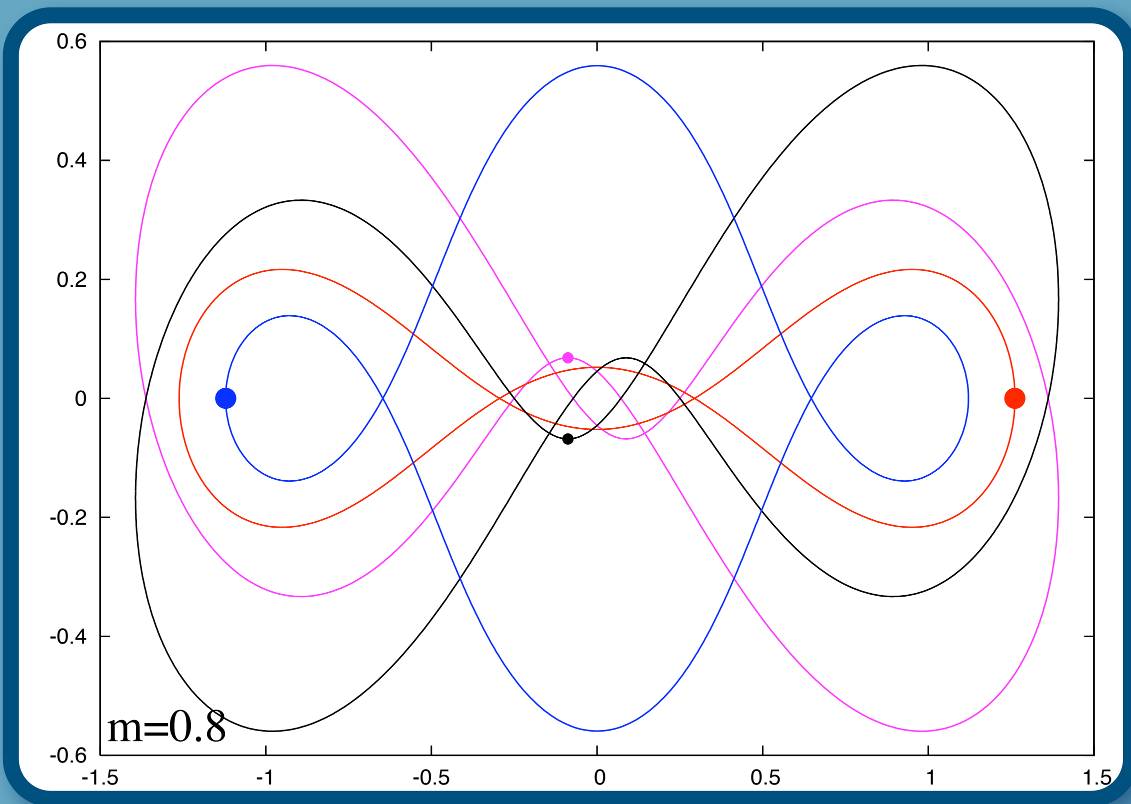


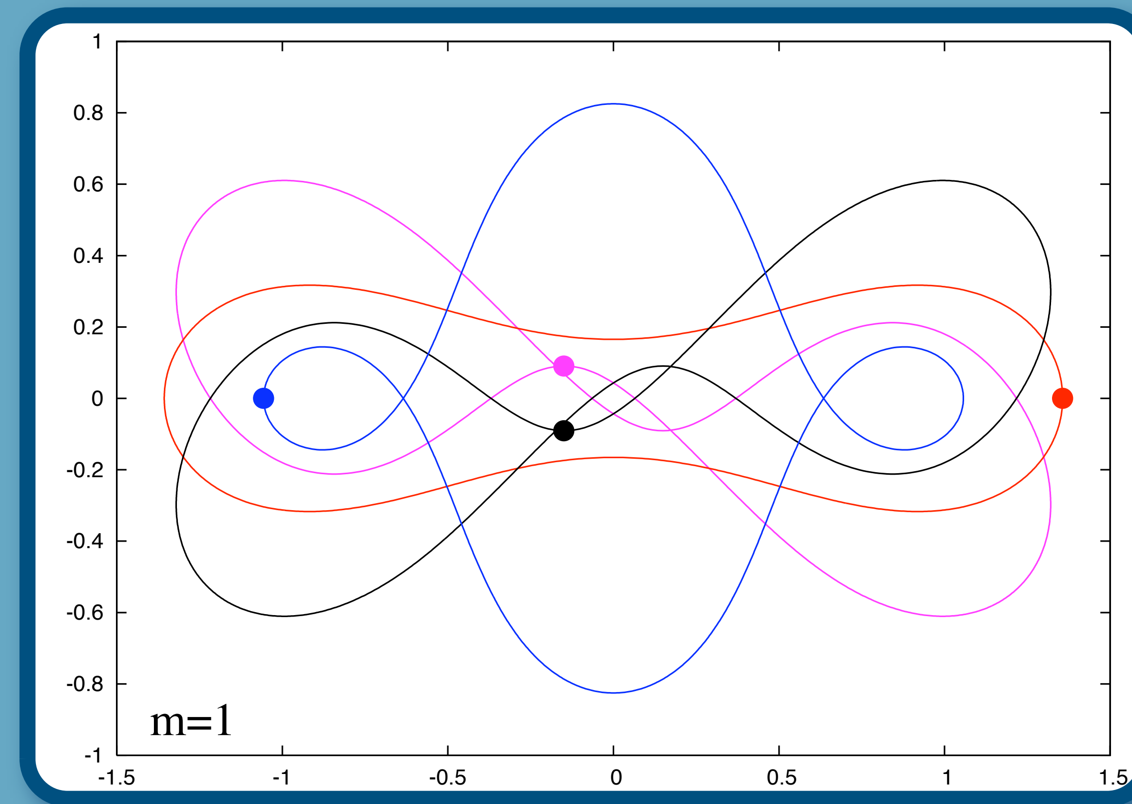


Continuation of the supereight

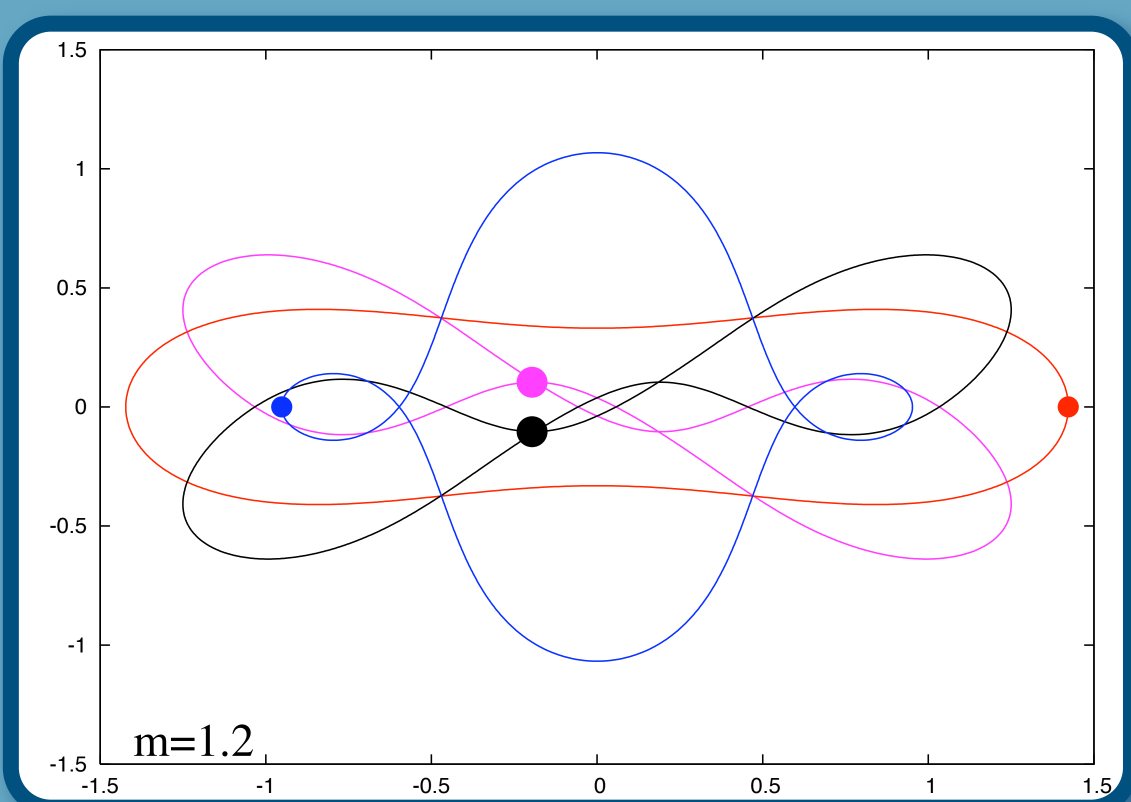
Along the foregoing branch one finds numerically non-normal behaviour and bifurcation at $m = 0.712412$. After switching branching (something AUTO can do very well) one can calculate a new branch of R_0-symmetric solutions, still using m as the continuation parameter.

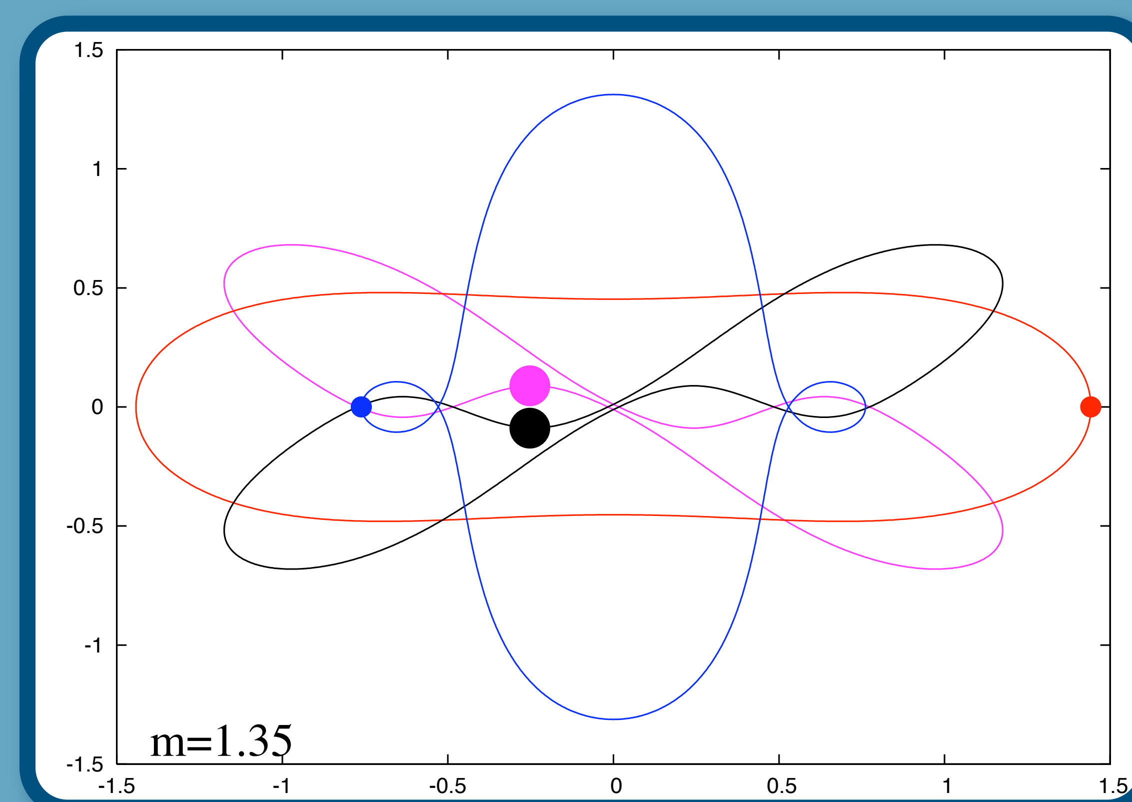

0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-1.5
-1
-0.5
0
0.5
1
1.5
m=0.8
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.5
-1
-0.5
0
0.5
1
1.5
m=1
1.5
1
0.5
0
-0.5
-1
-1.5
-1.5
-1
-0.5
0
0.5
1
1.5
m=1.2
1.5
1
0.5
0
-0.5
-1
-1.5
-1.5
-1
-0.5
0
0.5
1
1.5
m=1.35

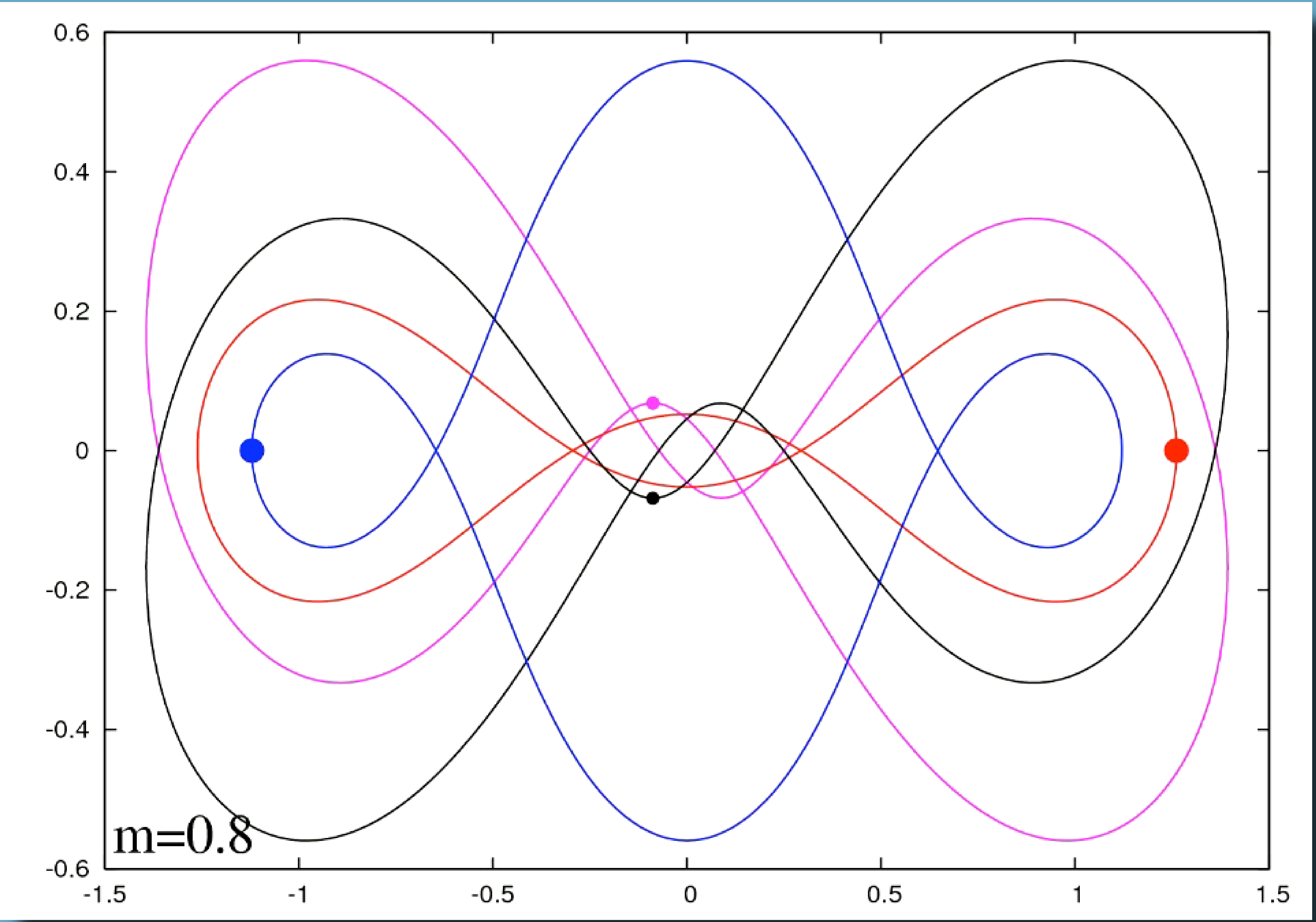


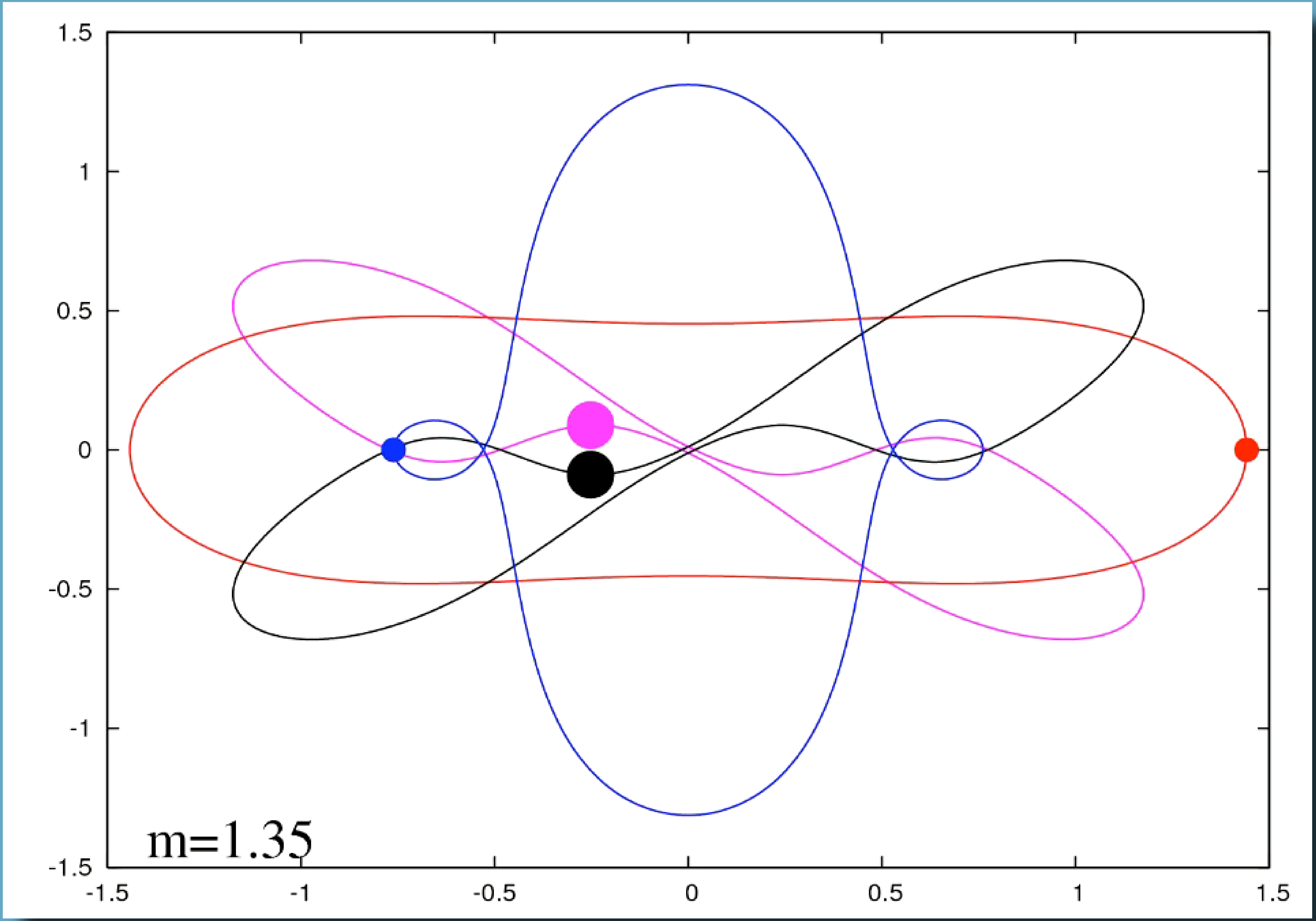


These are not choreographies at all!






0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-1.5
-1
-0.5
0
0.5
1
1.5
m=0.8



1.5
1
0.5
0
-0.5
-1
-1.5
-1.5
-1
-0.5
0
0.5
1
1.5
m=1.35

Continuation of the supereight

The supereight can also be considered as a $\tilde{R}_0$-symmetric solution with basic domain $[0, 4T_0]$ and with the reversor $\tilde{R}_0$ given by

$$\tilde{R}_0 := R \circ \Sigma_{2,4} \circ \Phi \circ \Psi_\pi.$$

Continuation of the supereight

The supereight can also be considered as a $\tilde{R}_0$-symmetric solution with basic domain $[0, 4T_0]$ and with the reversor $\tilde{R}_0$ given by

$$\tilde{R}_0 := R \circ \Sigma_{2,4} \circ \Phi \circ \Psi_\pi.$$

$\tilde{R}_0$ remains a reversor as long as $m_2 = m_4$.
So, in particular the case

$$m_1 = m_3 = m \quad \text{and} \quad m_2 = m_4 = 1$$

which we considered before, is allowed.

Continuation of the supereight

This time P_2 is the only first integral which is constant on $\mathrm{Fix}(\tilde{R}_0)$; a continuation, keeping the basic domain fixed, including a phase condition corresponding to P_2, and using m as the continuation parameter, gives a one-dimensional branch.

Continuation of the supereight

This time P_2 is the only first integral which is constant on $\mathrm{Fix}(\tilde{R}_0)$; a continuation, keeping the basic domain fixed, including a phase condition corresponding to P_2, and using m as the continuation parameter, gives a one-dimensional branch.

Using this $\tilde{R}_0$-symmetric continuation we obtain the same branch as the one we found before using a R_0-symmetric continuation. This means that the start and end configurations along this branch belong to

$$\mathrm{Fix}(R_0) \cap \mathrm{Fix}(\tilde{R}_0).$$

Continuation of the supereight

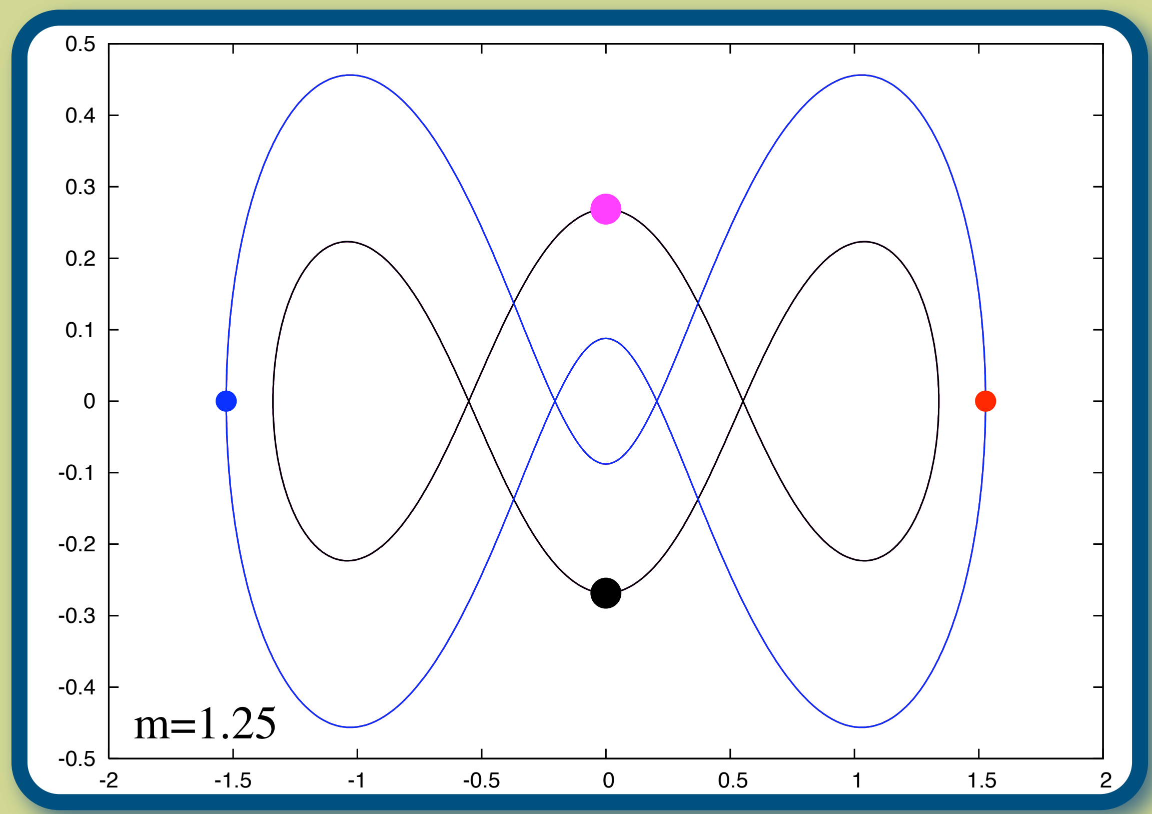
As a consequence the solutions along this branch are $\tilde{R}_0 R_0$-symmetric, with

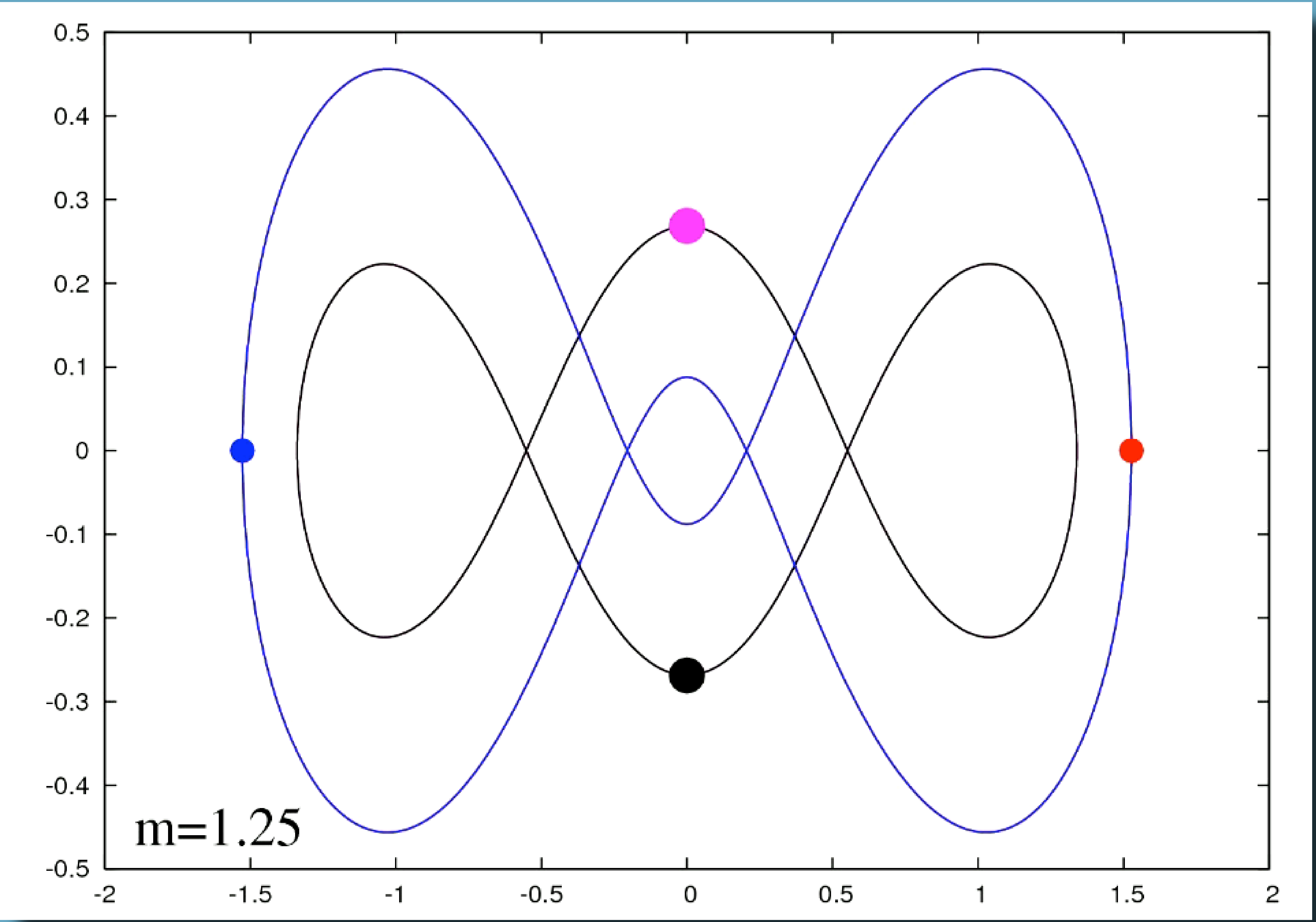
$$\tilde{R}_0 R_0 = -\Sigma_{1,3} \circ \Sigma_{2,4}.$$

Continuation of the supereight

As a consequence the solutions along this branch are $\tilde{R}_0 R_0$-symmetric, with

$$\tilde{R}_0 R_0 = -\Sigma_{1,3} \circ \Sigma_{2,4}.$$




m=1.25
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

Continuation of the supereight

However, the $\tilde{R}_0$-symmetric continuation gives different bifurcation points:

Continuation of the supereight

However, the $\tilde{R}_0$-symmetric continuation gives different bifurcation points:

$$m = 0.2534$$

$$m = 0.6853$$

$$m = 1.4037$$

$$m = 1.4592$$

$$m = 3.9458$$

Continuation of the supereight

However, the $\tilde{R}_0$-symmetric continuation gives different bifurcation points:

$$\left.\begin{array}{l} m = 0.2534 \\ m = 0.6853 \end{array}\right\rangle \text{connected}$$

$$m = 1.4037$$

$$\left.\begin{array}{l} m = 1.4592 \\ m = 3.9458 \end{array}\right\rangle \text{connected}$$

Continuation of the supereight

However, the $\tilde{R}_0$-symmetric continuation gives different bifurcation points:

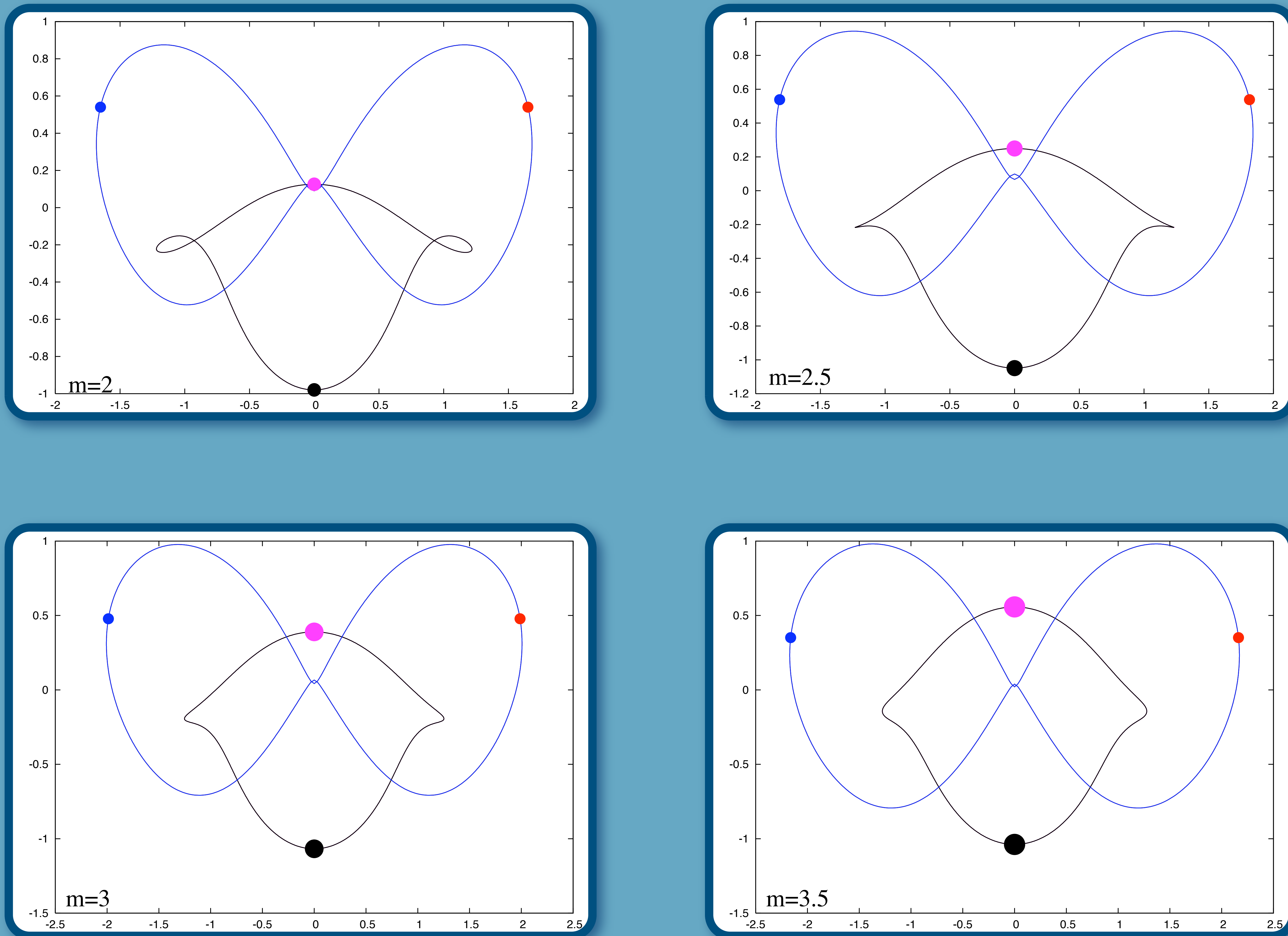
$$\left.\begin{array}{l} m = 0.2534 \\ m = 0.6853 \end{array}\right\rangle \text{connected}$$

$$m = 1.4037$$

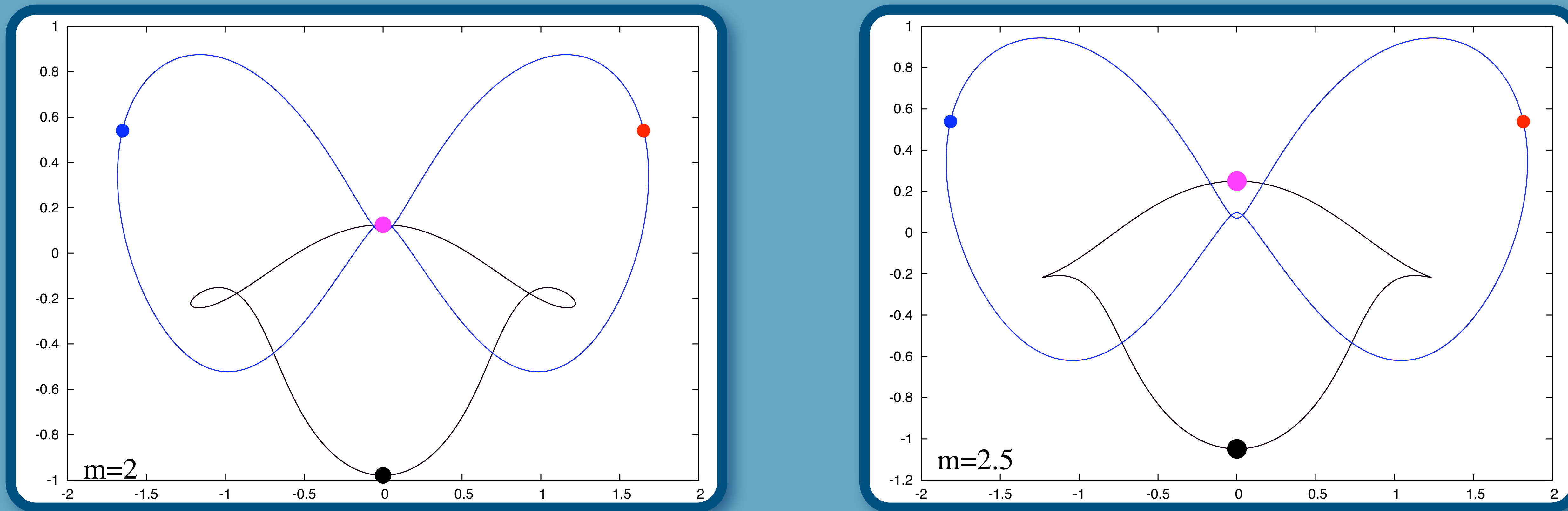
$$\left.\begin{array}{l} m = 1.4592 \\ m = 3.9458 \end{array}\right\rangle \text{connected}$$

The bifurcating branches contain $\tilde{R}_0$-symmetric solutions which are no longer R_0-symmetric.

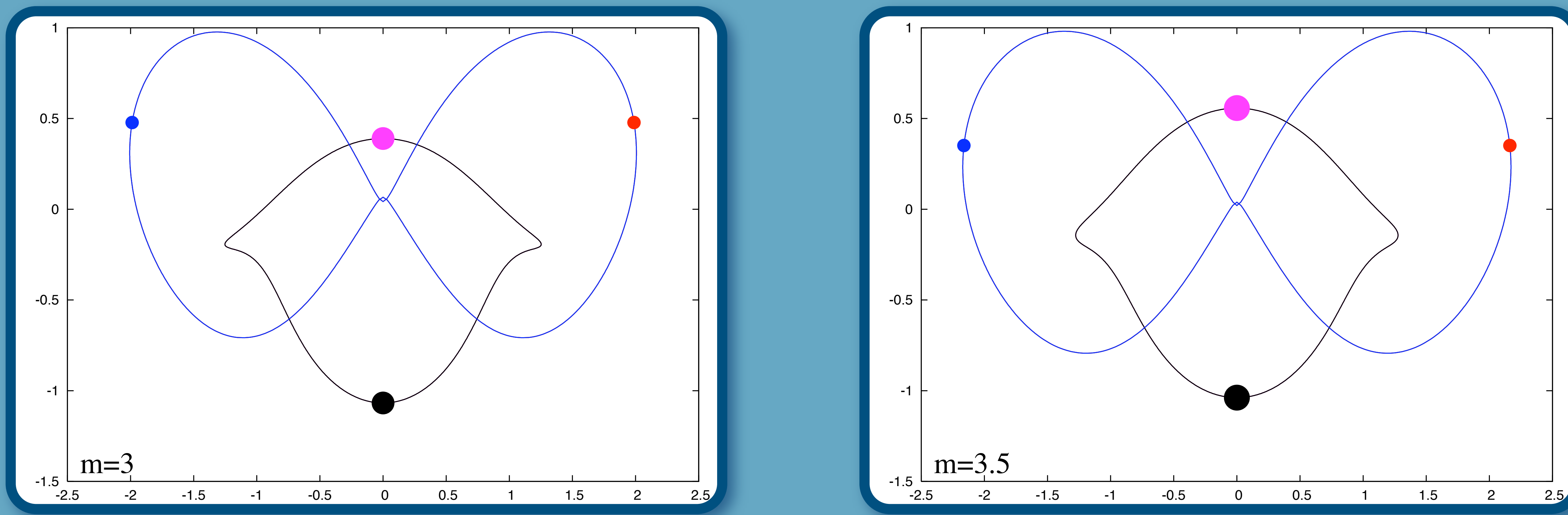
Along the branch from m=1.45 to m=3.95

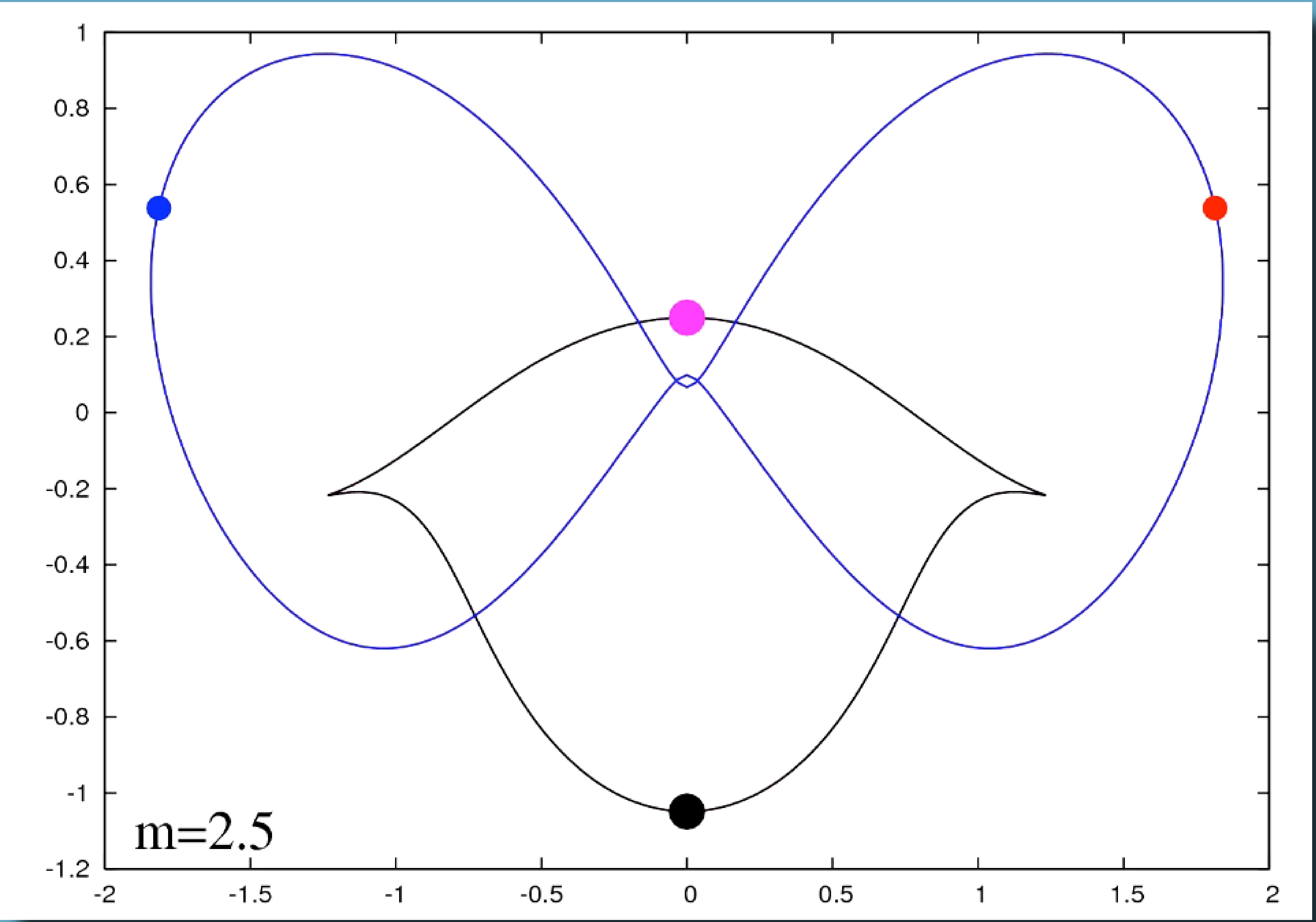


Along the branch from m=1.45 to m=3.95



These are again partial choreographies




1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
m=2.5

Continuation of the supereight

The supereight can also be continued as a **periodic orbit**, using the schemes of Lecture 1. When we use again the mass m of the 1st and 3rd body as continuation parameter we find the same branch as before, and no new bifurcations are detected.

Continuation of the supereight

Next we see how we can continue the supereight when we change just one of the masses. We take the following mass distribution:

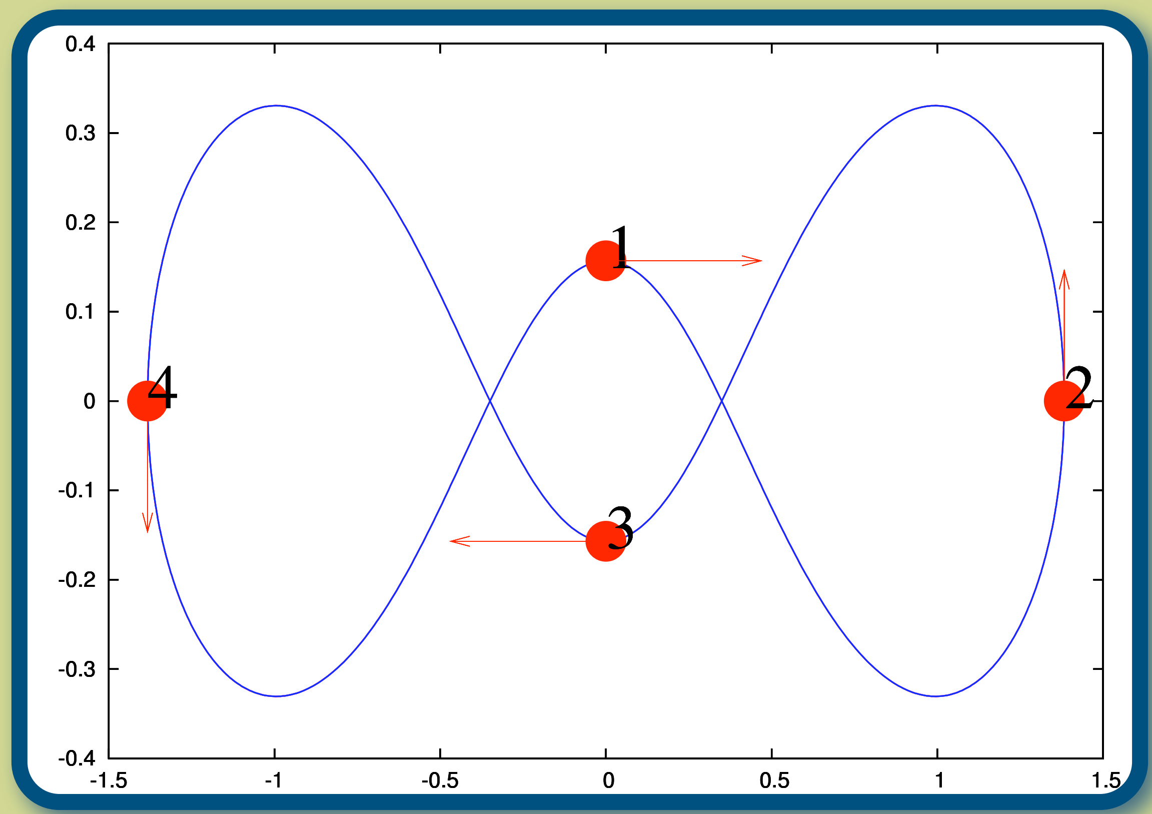
$$m_2 = \mu \quad \text{and} \quad m_1 = m_3 = m_4 = 1.$$

Continuation of the supereight

Next we see how we can continue the supereight when we change just one of the masses. We take the following mass distribution:

$$m_2 = \mu \quad \text{and} \quad m_1 = m_3 = m_4 = 1.$$

This mass configuration is compatible with the reversor R_0.



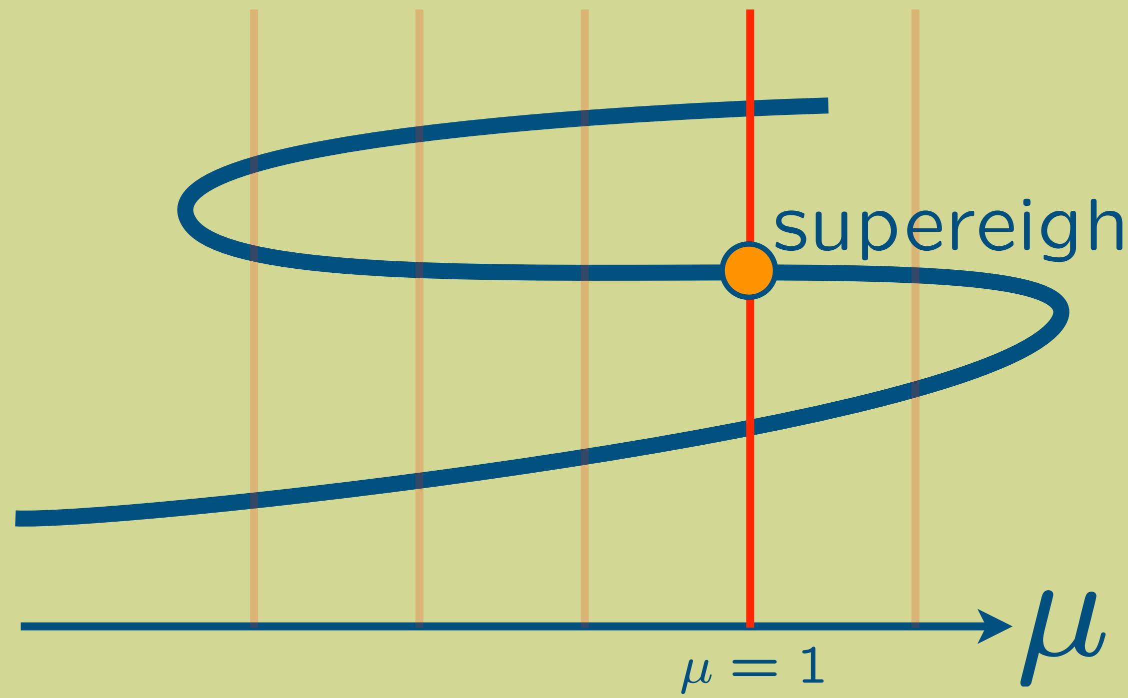
Continuation of the supereight

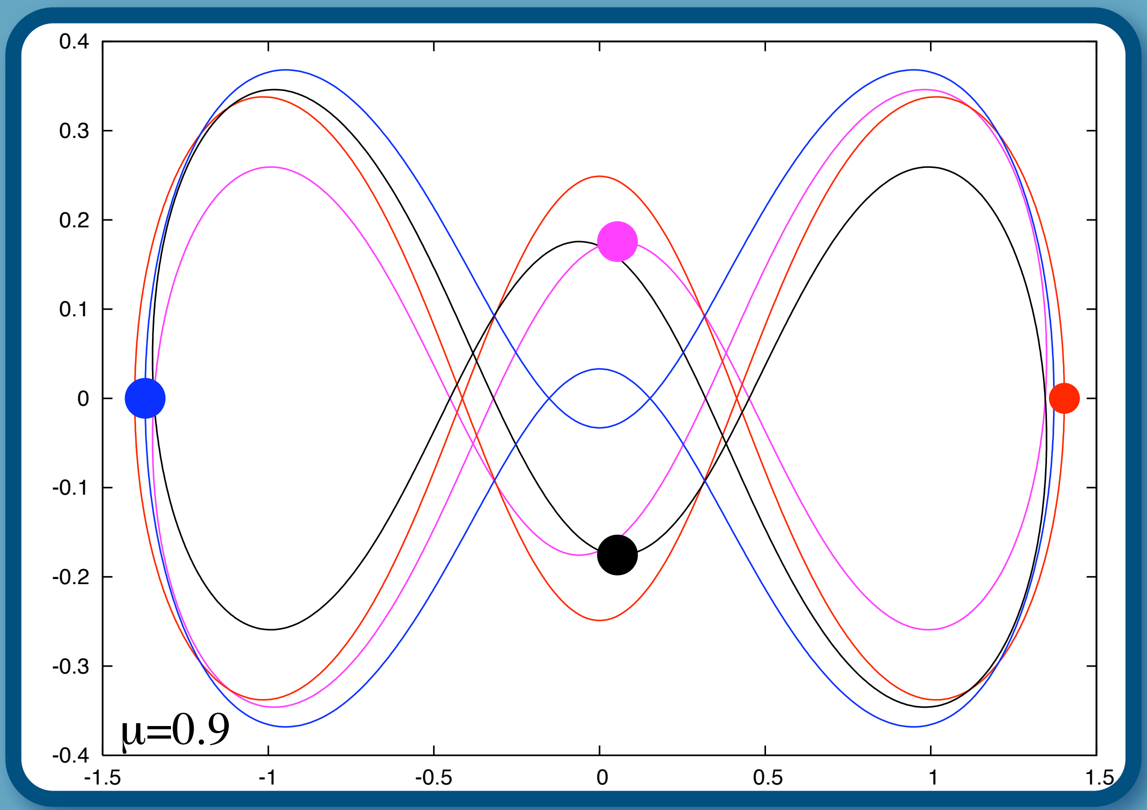
So we can start from the supereight to get a branch of R_0-symmetric solutions with basic domain $[0, 4T_0]$, and using μ as the continuation parameter.

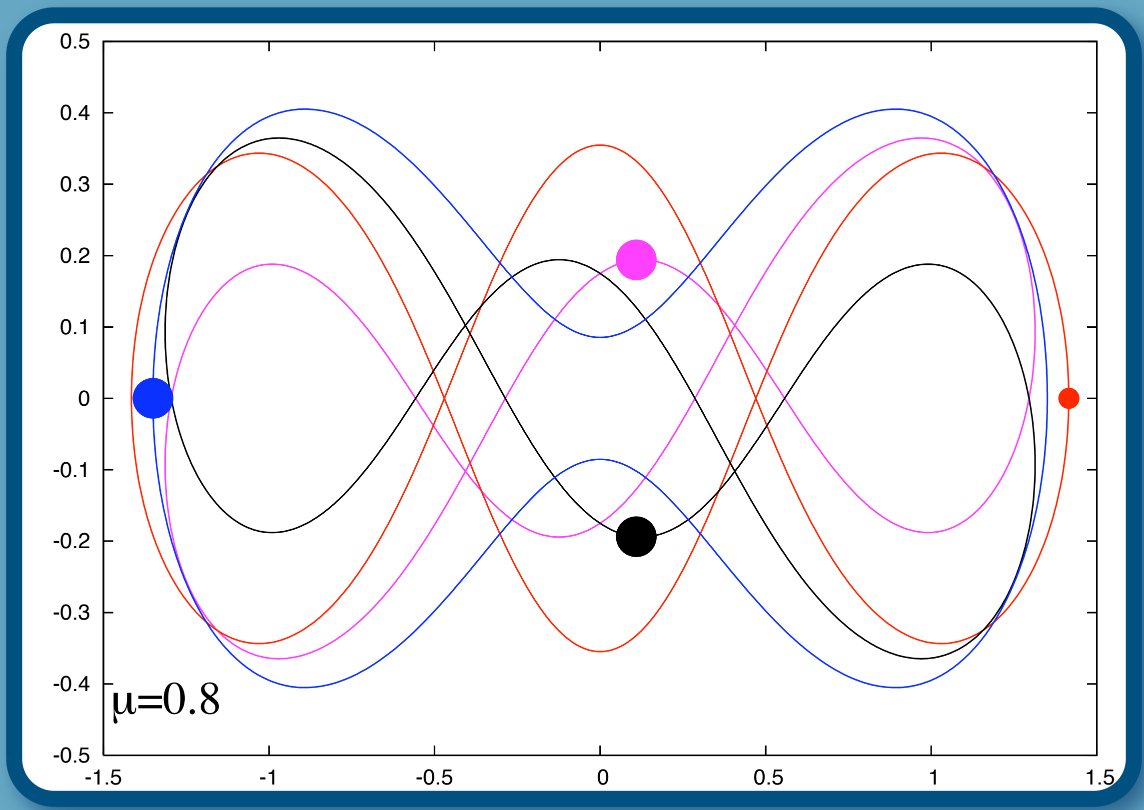
Continuation of the supereight

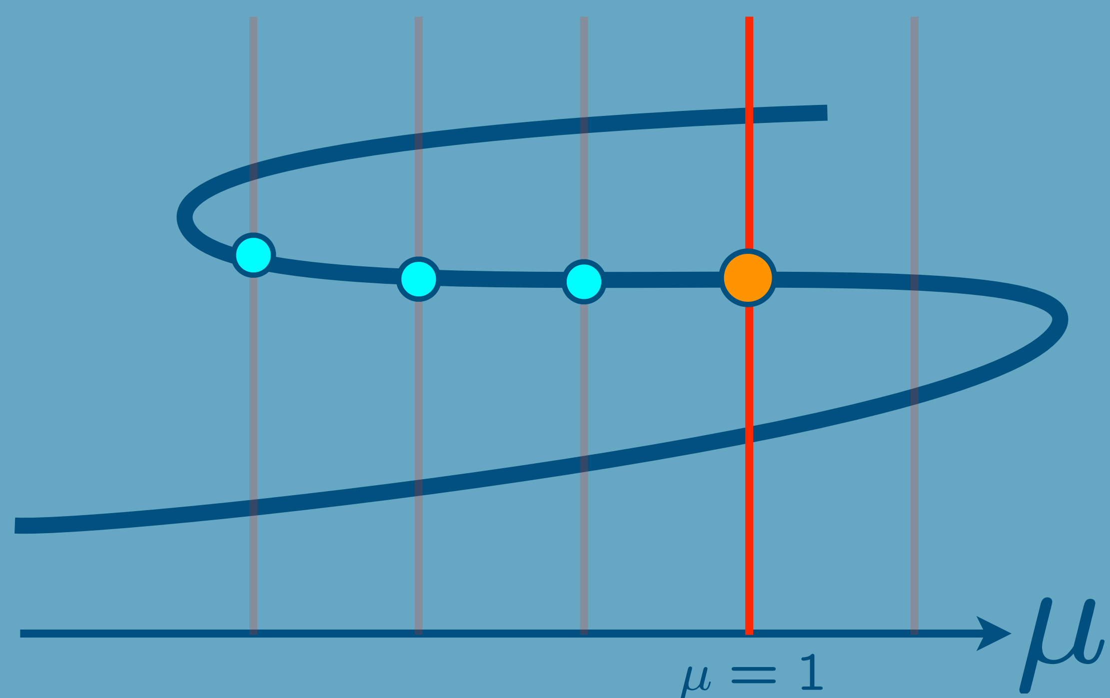
So we can start from the supereight to get a branch of R_0-symmetric solutions with basic domain $[0, 4T_0]$, and using μ as the continuation parameter.

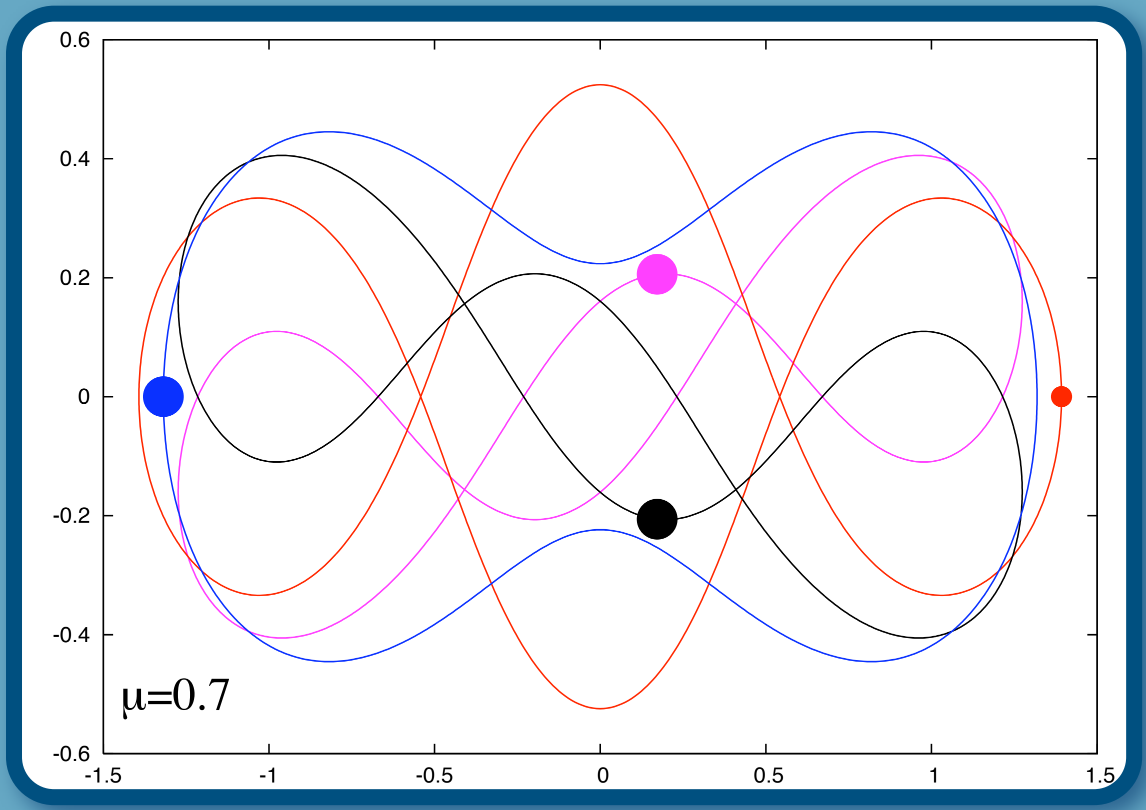
The resulting branch looks as follows:

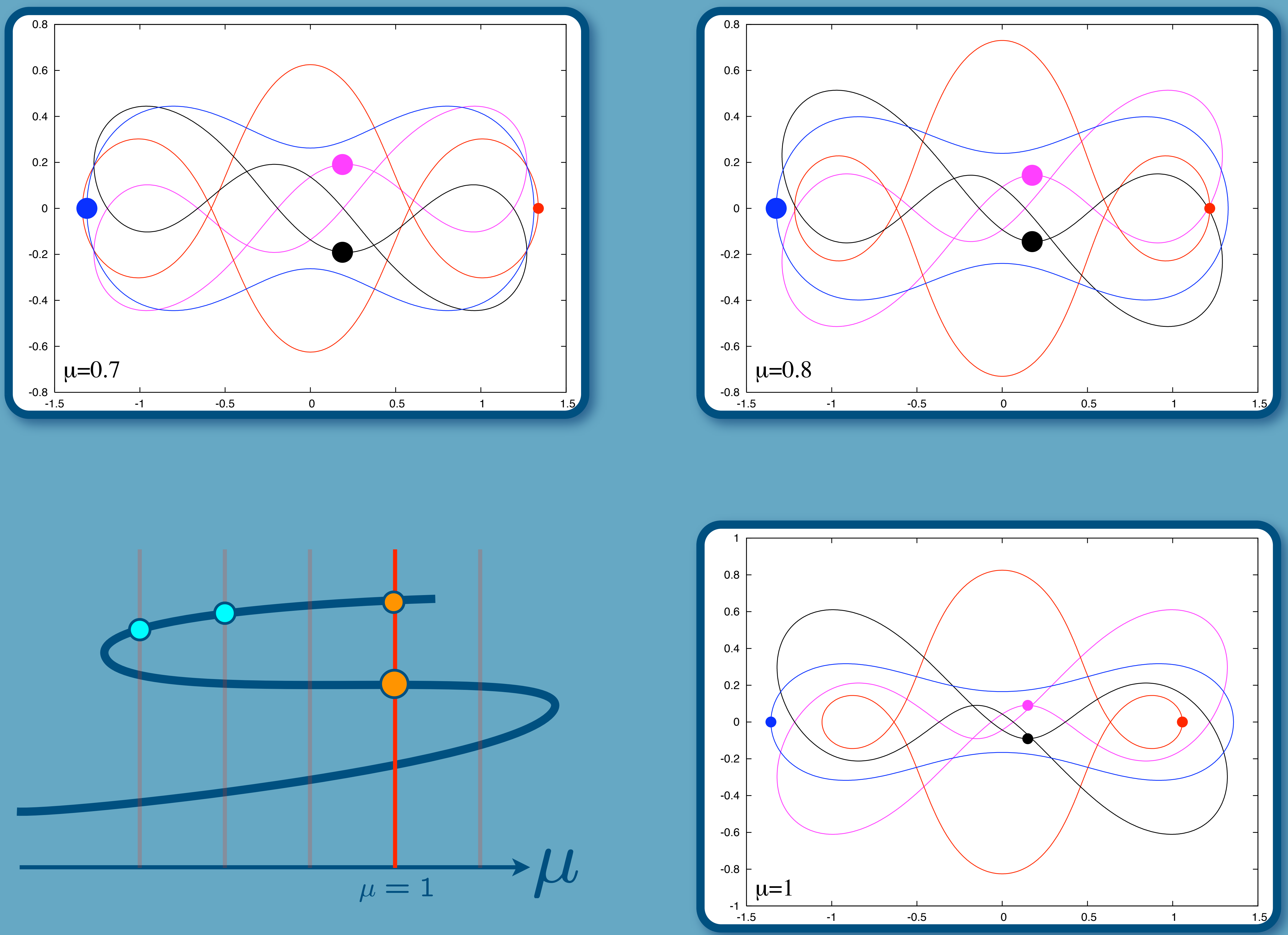


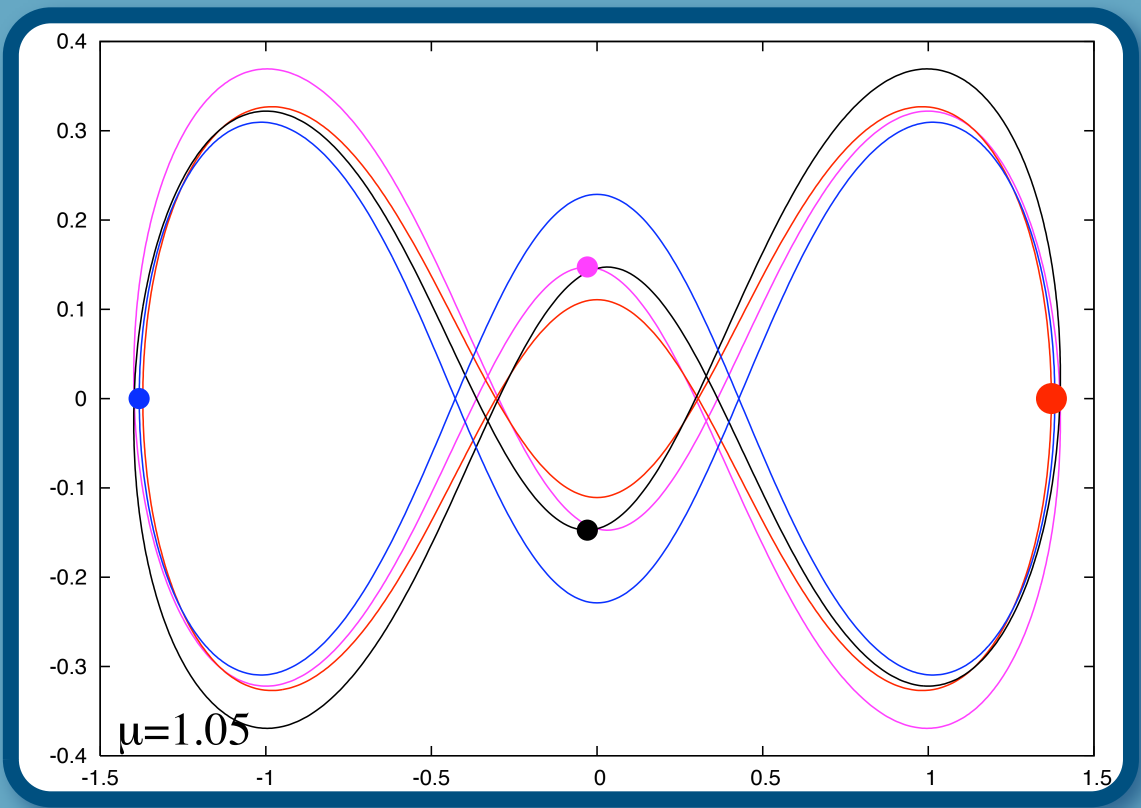

0.4 0.3 0.2 0.1 0 -0.1 -0.2 -0.3 -0.4
-1.5 -1 -0.5 0 0.5 1 1.5
μ=0.9

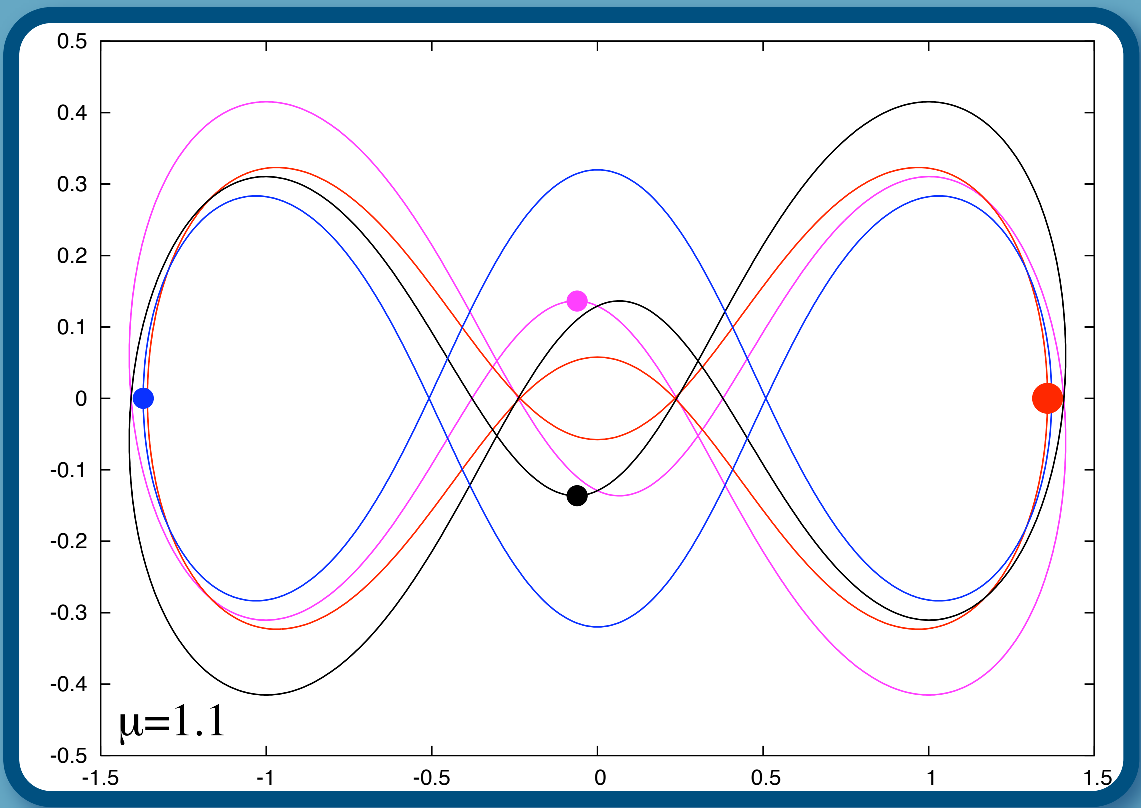

0.5 0.4 0.3 0.2 0.1 0 -0.1 -0.2 -0.3 -0.4 -0.5
-1.5 -1 -0.5 0 0.5 1 1.5
μ=0.8

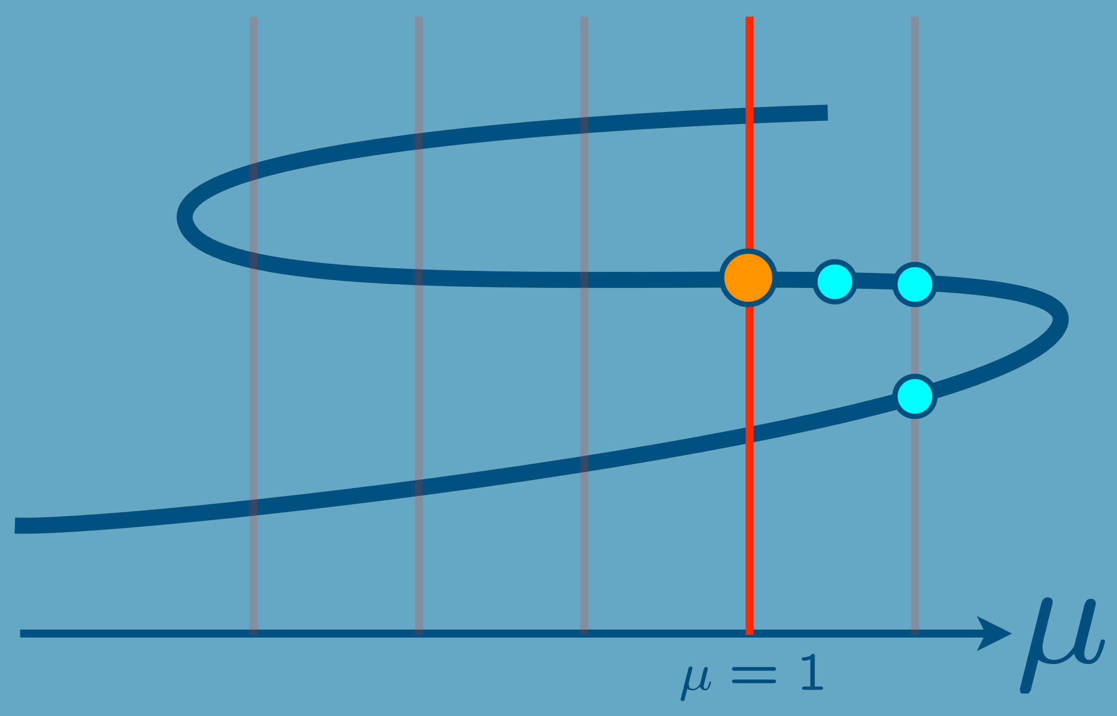

$\mu = 1$
μ

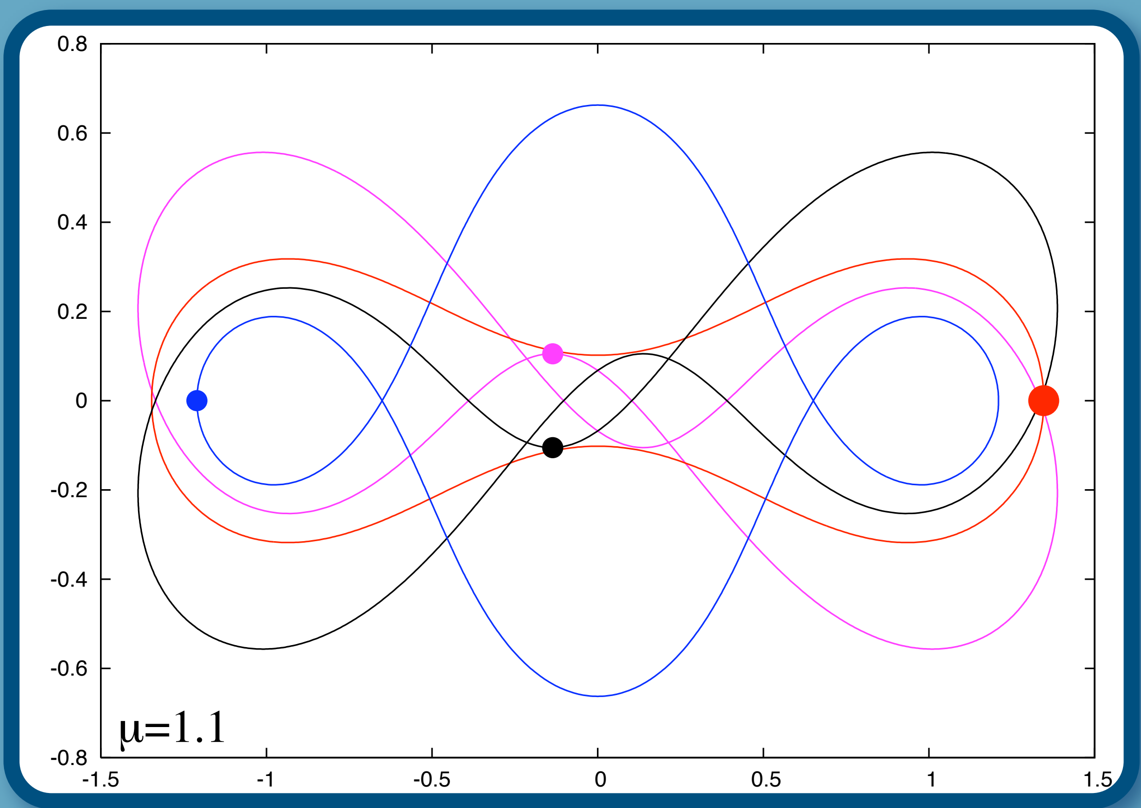

0.6 0.4 0.2 0 -0.2 -0.4 -0.6
-1.5 -1 -0.5 0 0.5 1 1.5
μ=0.7

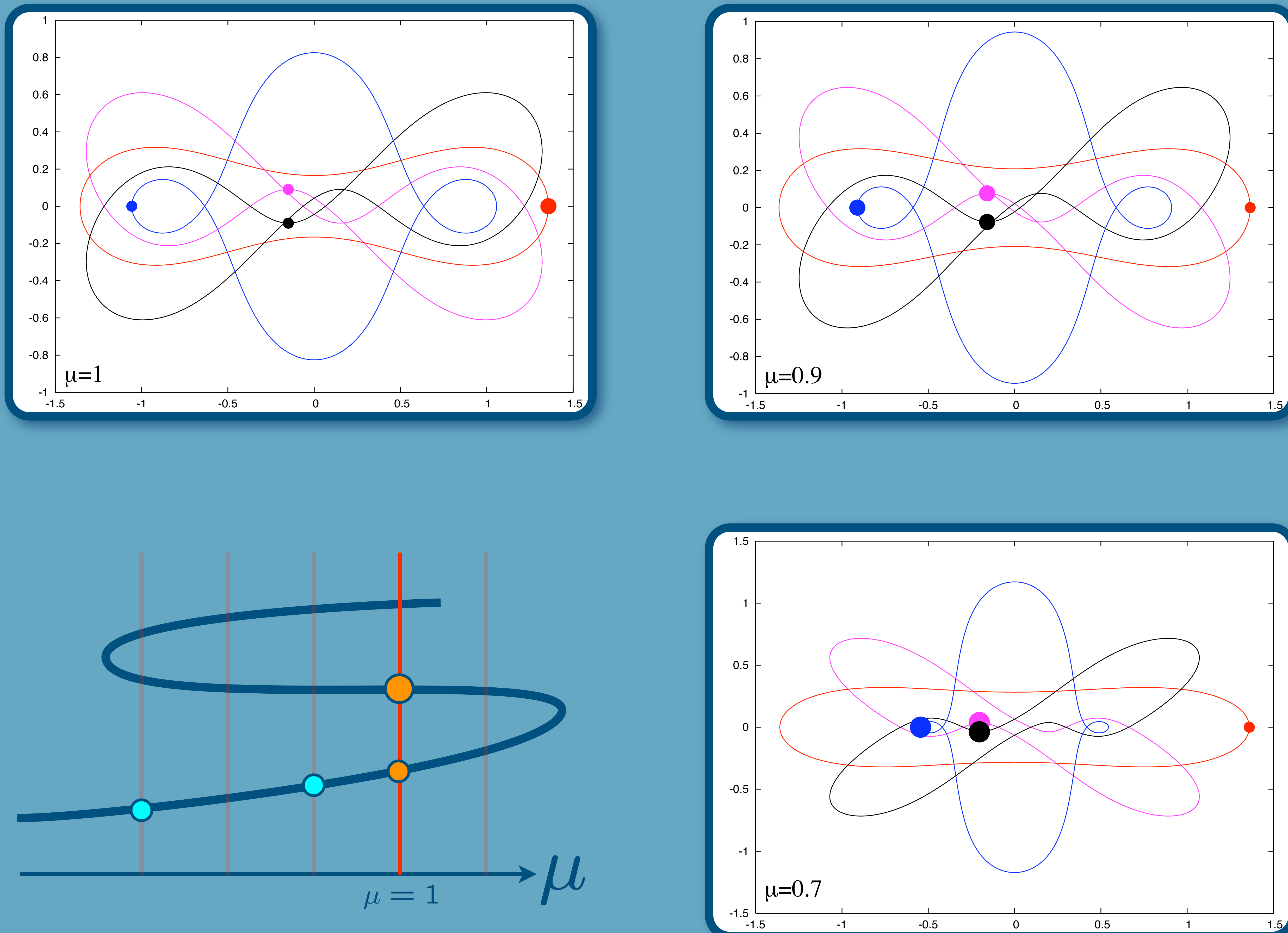

μ=0.7
μ=0.8
μ=1
$\mu = 1$
μ

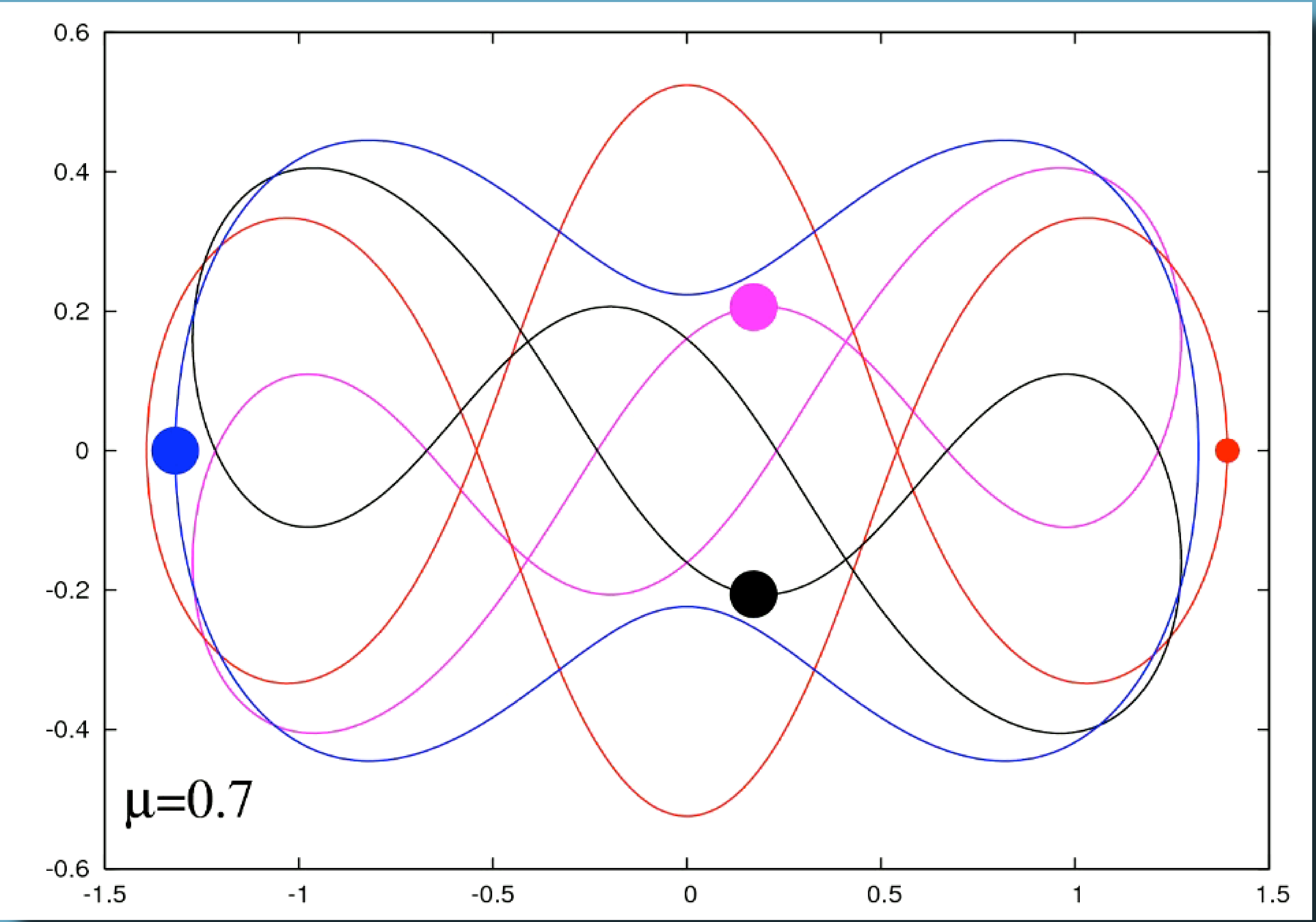

μ=1.05


μ=1.1

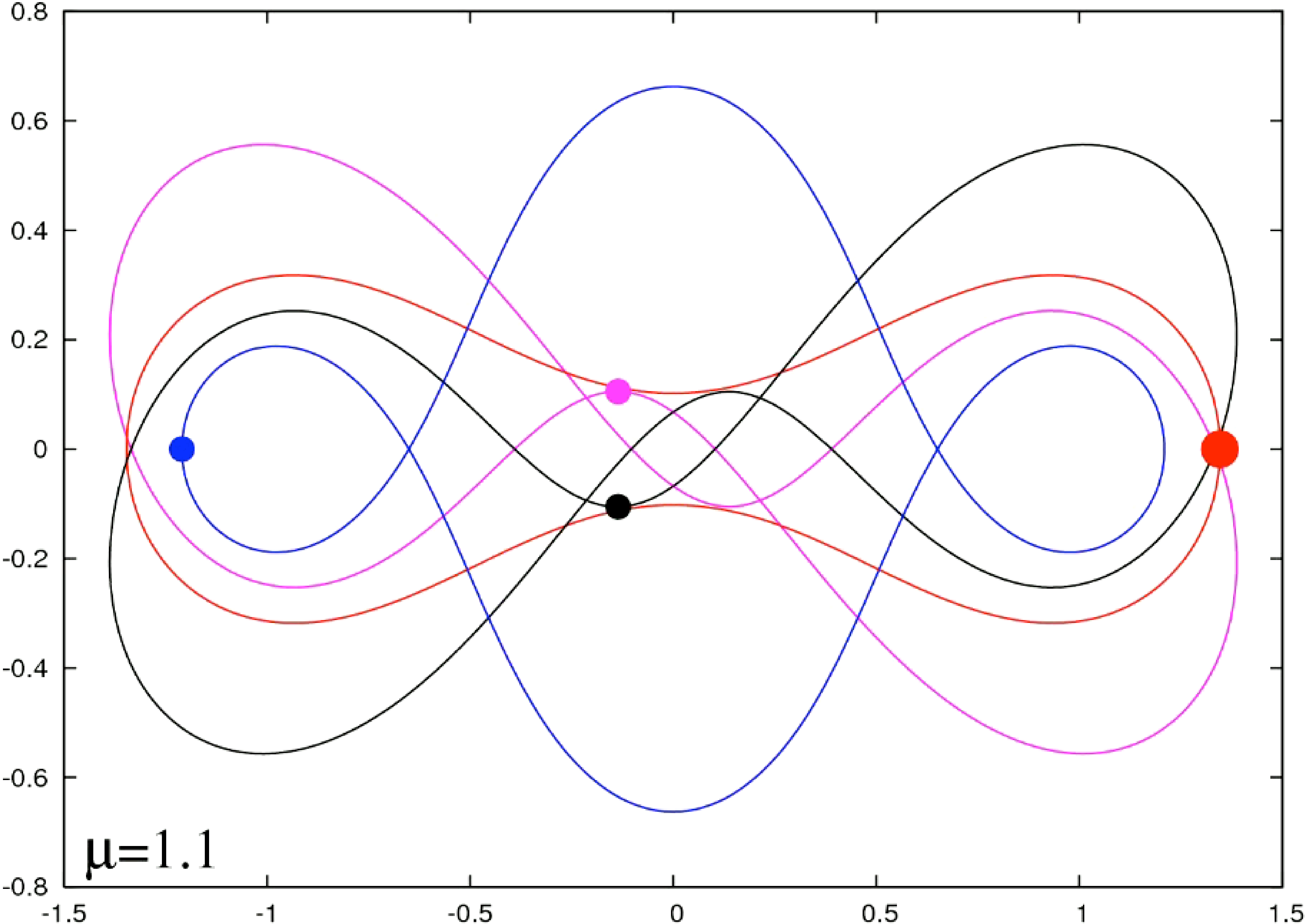

$\mu = 1$
μ

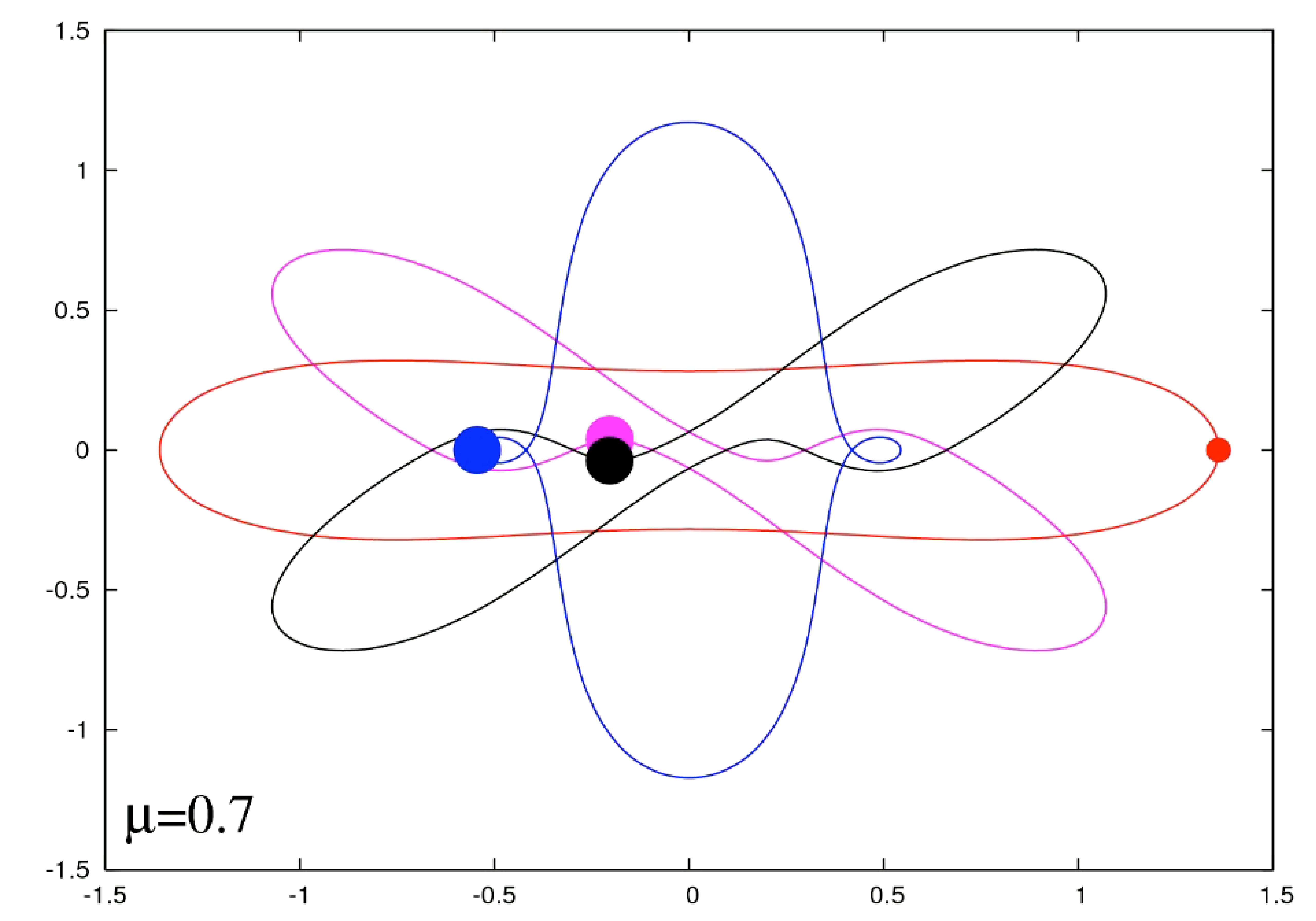

μ=1.1


μ=1
μ=0.9
μ=0.7
$\mu = 1$
μ



0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-1.5
-1
-0.5
0
0.5
1
1.5
μ=0.7

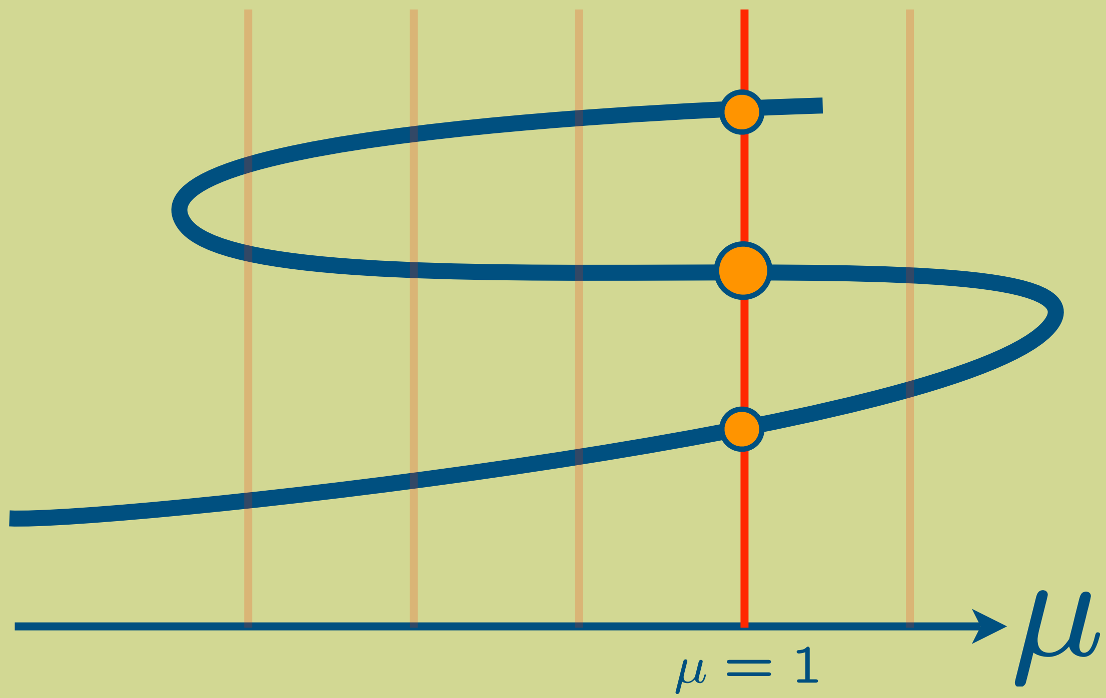

0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1.5
-1
-0.5
0
0.5
1
1.5
μ=1.1

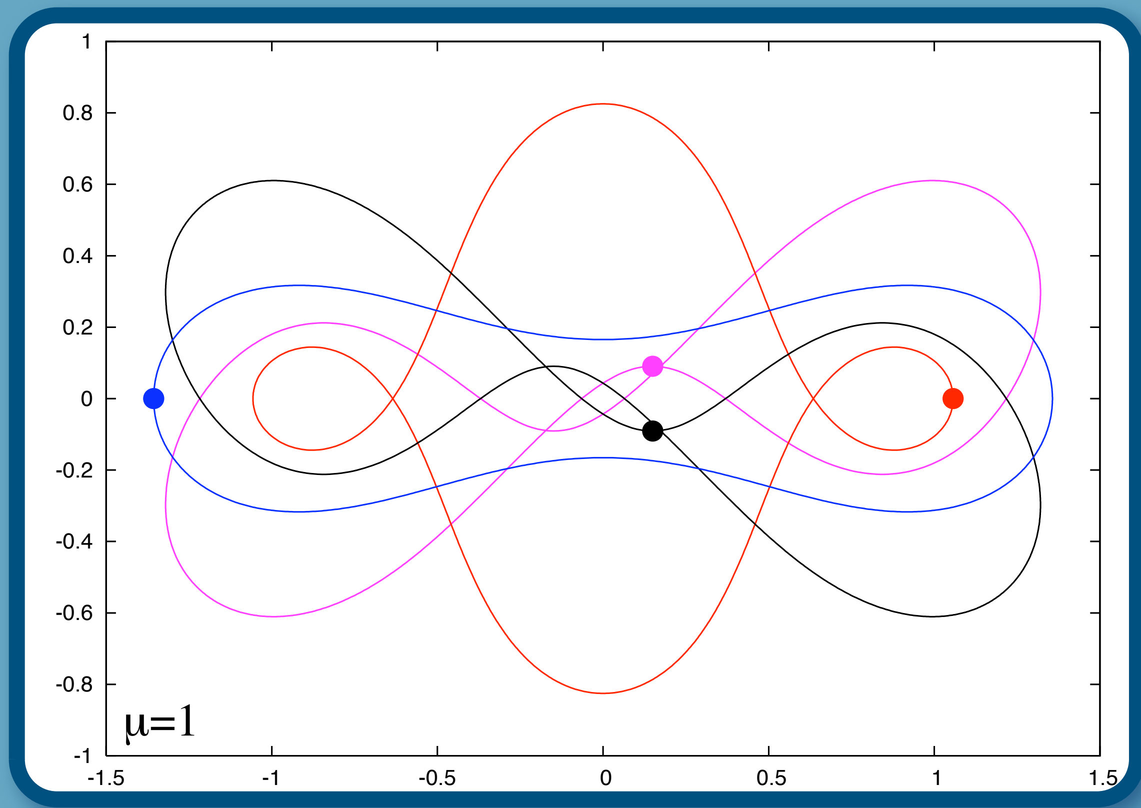


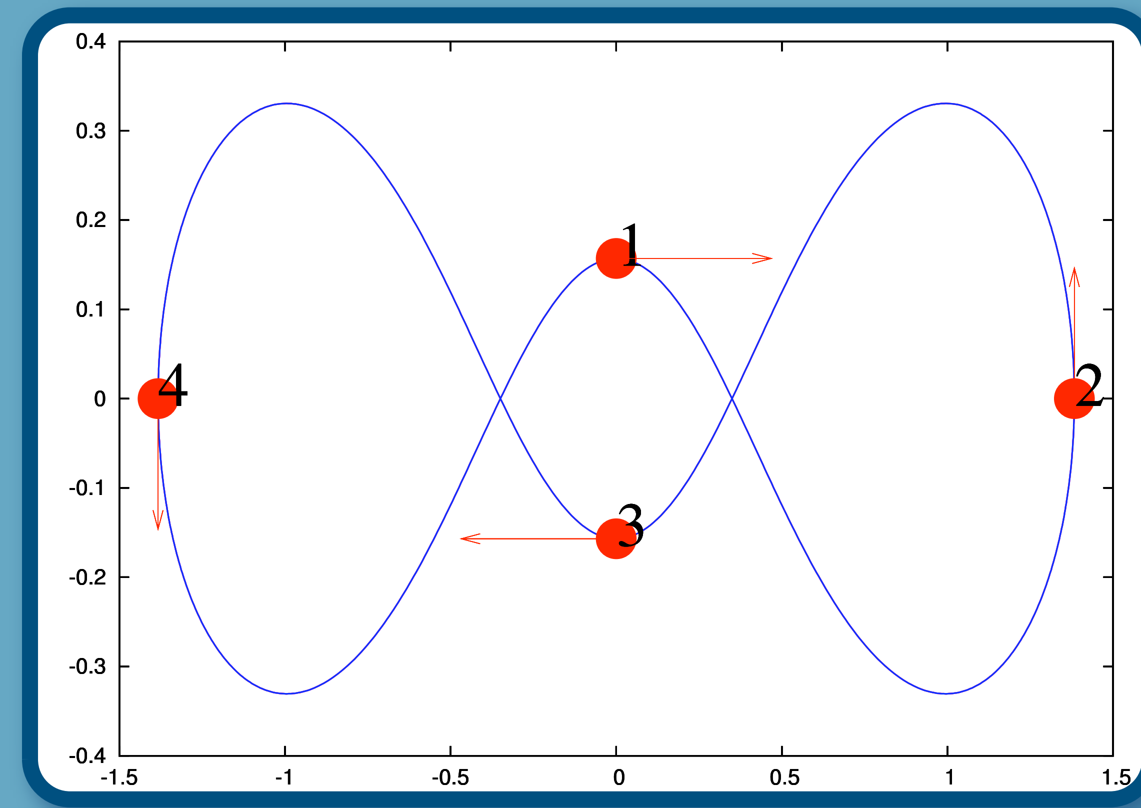
1.5
1
0.5
0
-0.5
-1
-1.5
-1.5
-1
-0.5
0
0.5
1
1.5
μ=0.7

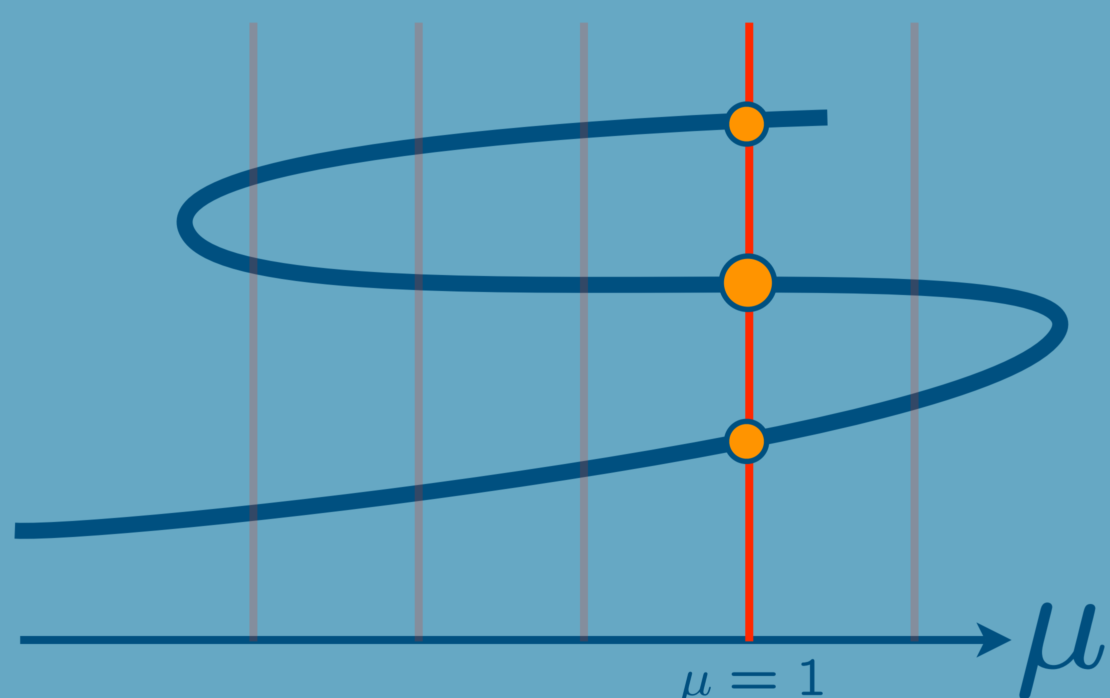
Continuation of the supereight

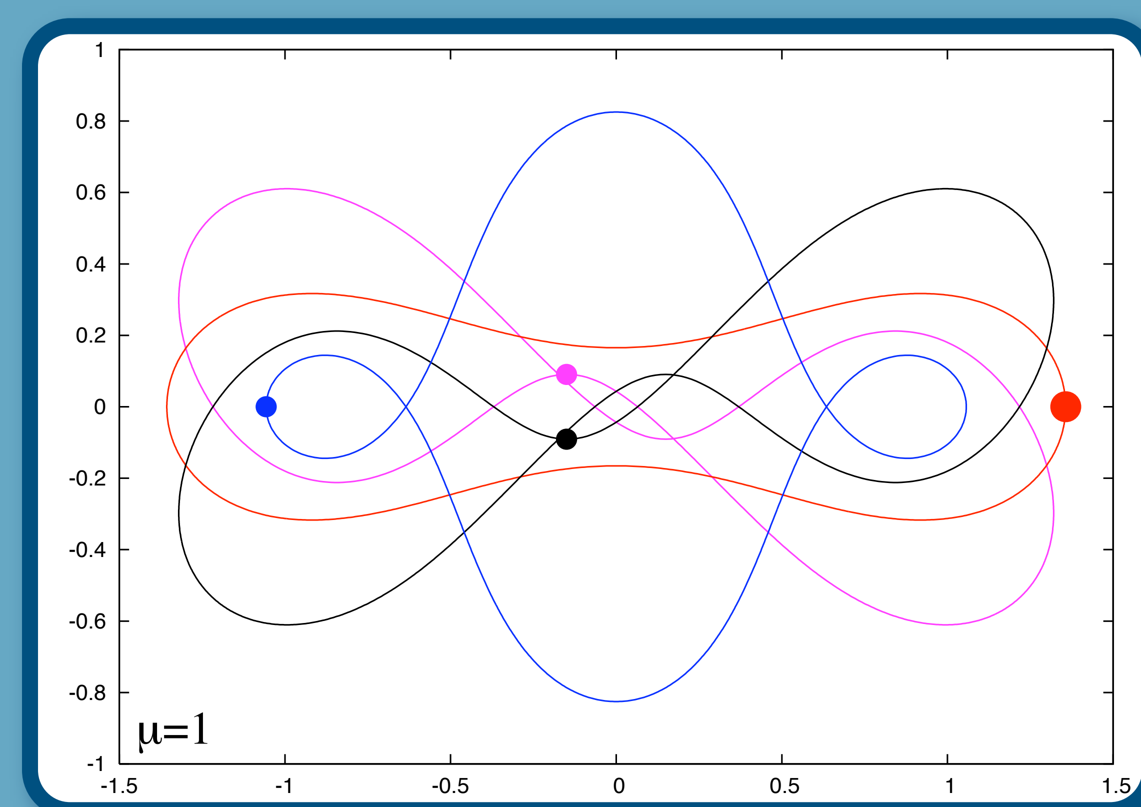
Observe that next to the supereight we have found two other (periodic) solutions with four equal masses ($\mu = 1$).

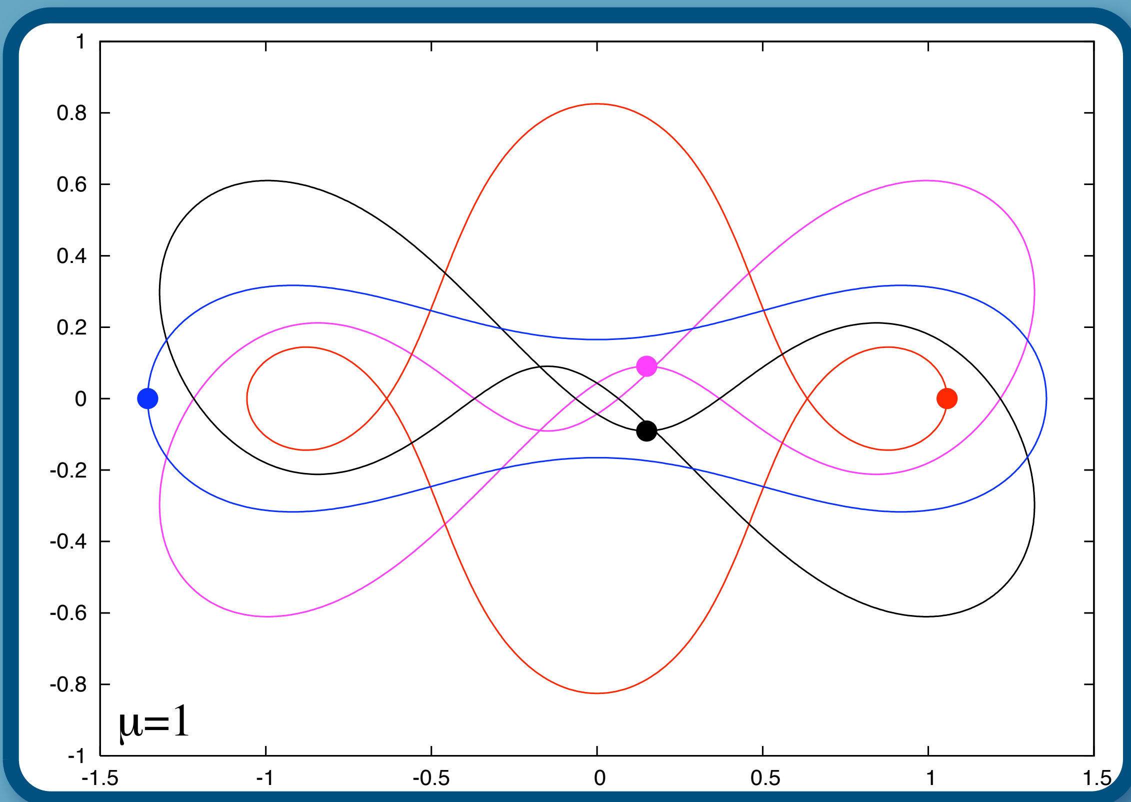



μ=1

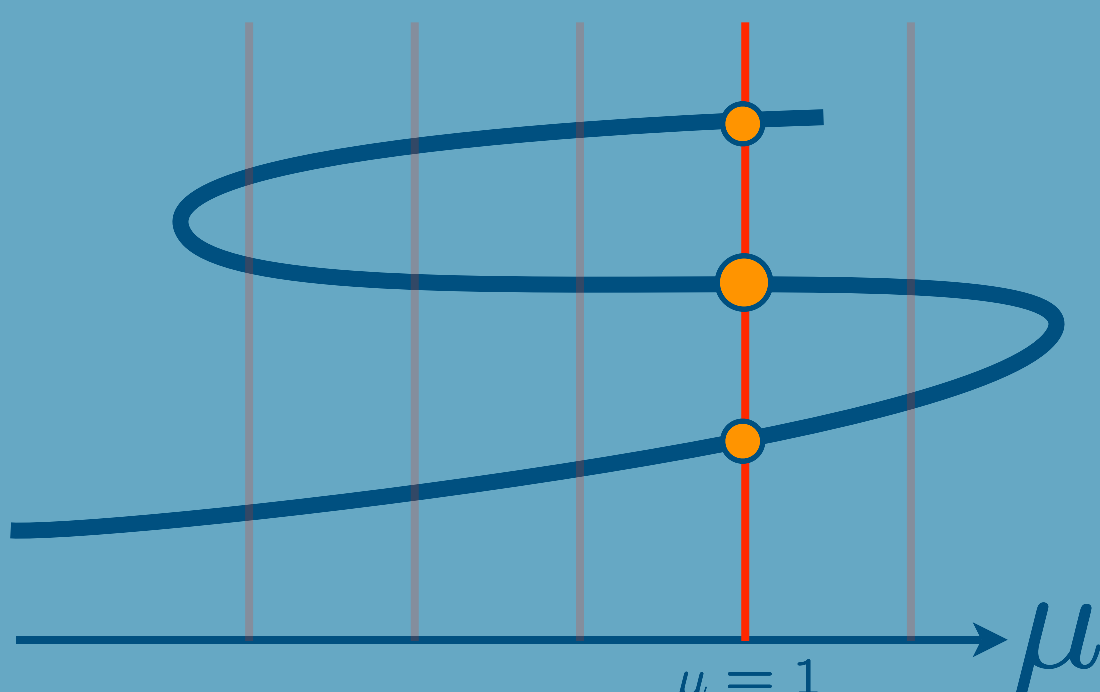

1
2
3
4

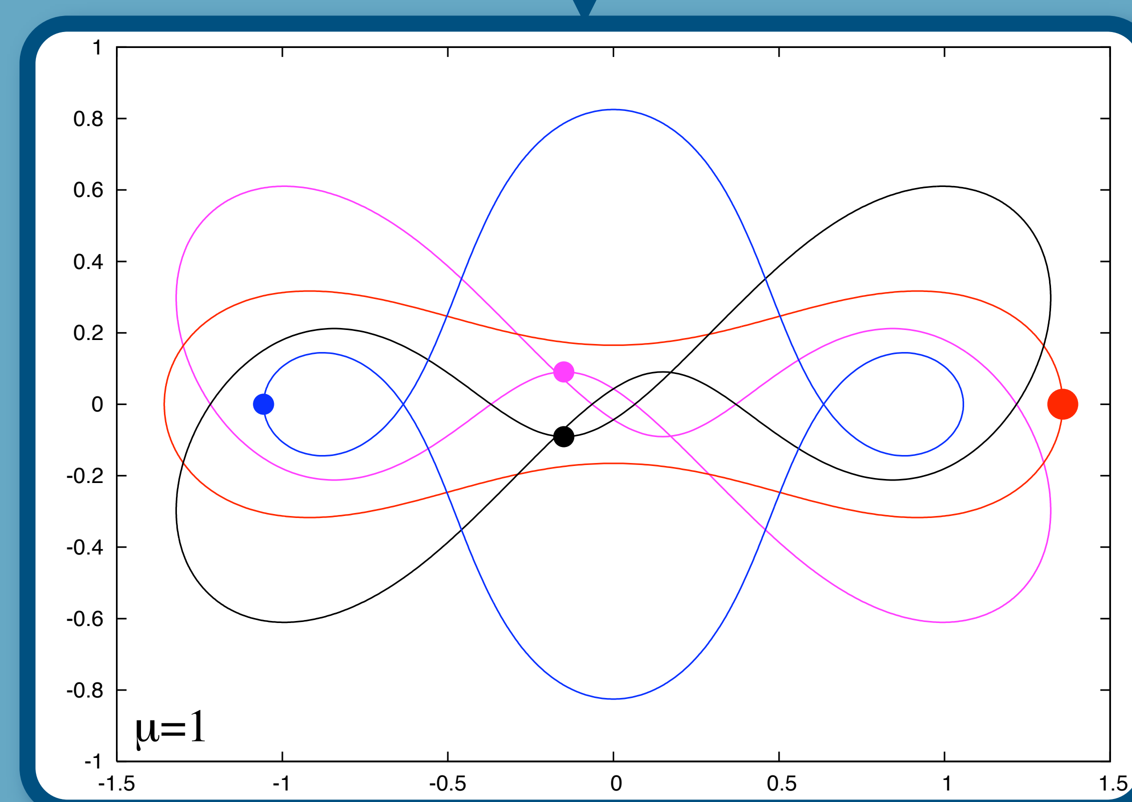

$\mu = 1$
μ

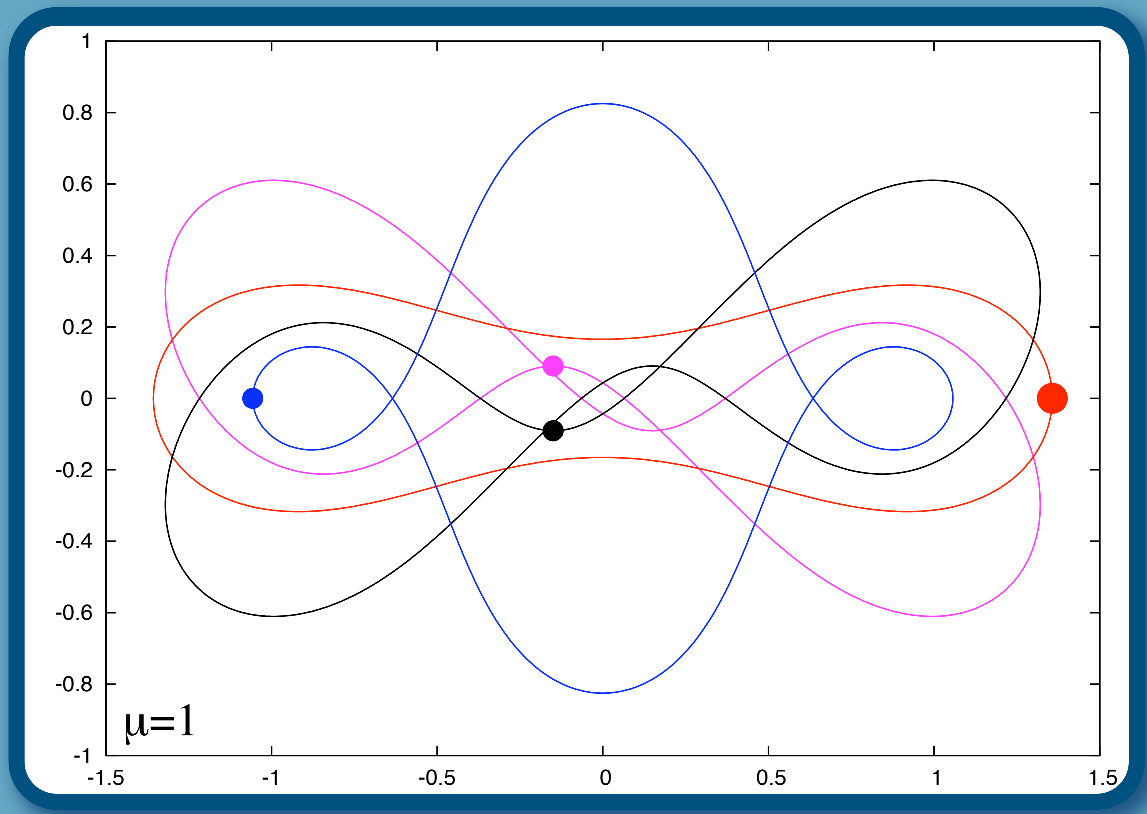

μ=1



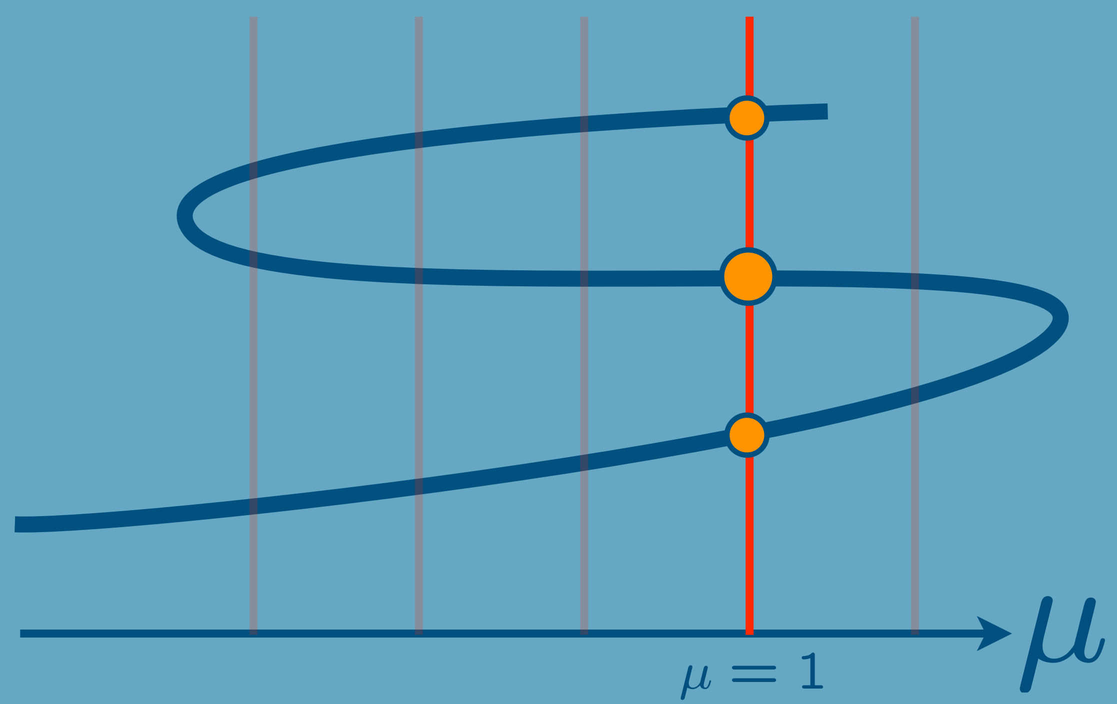
These two orbits are related by an exchange symmetry.

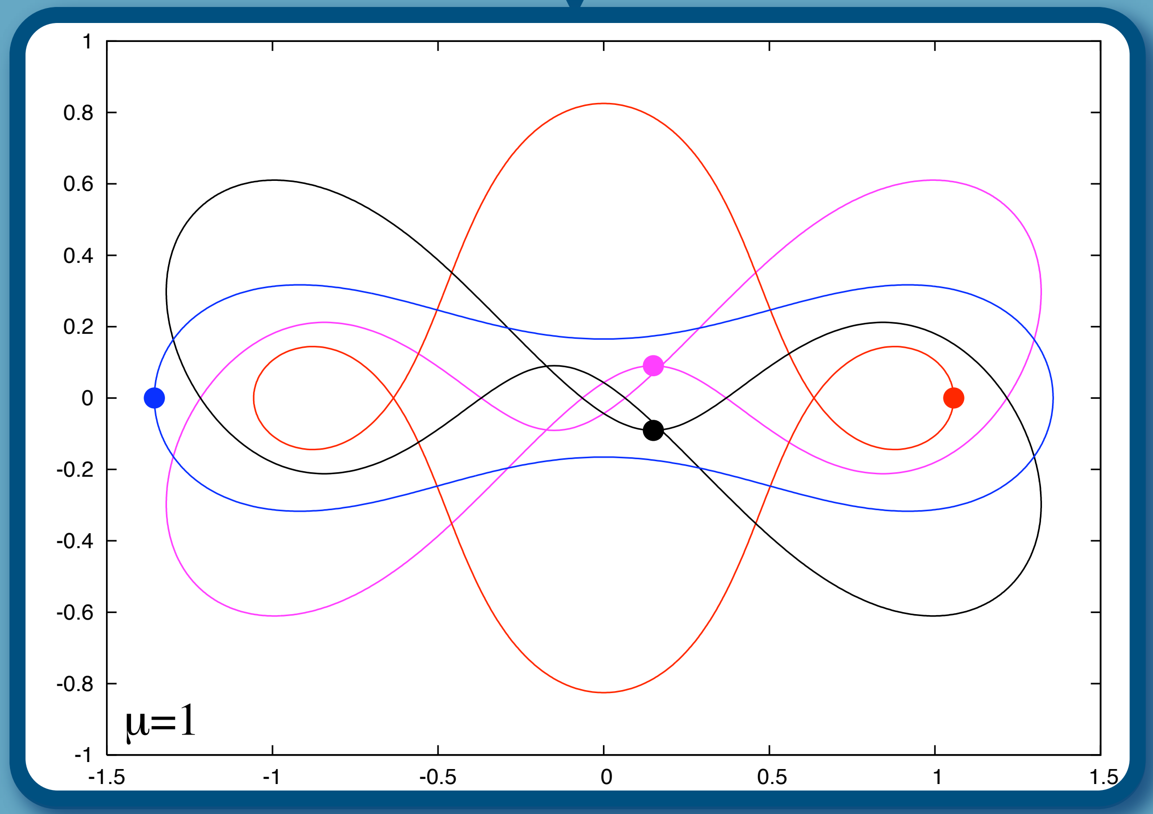


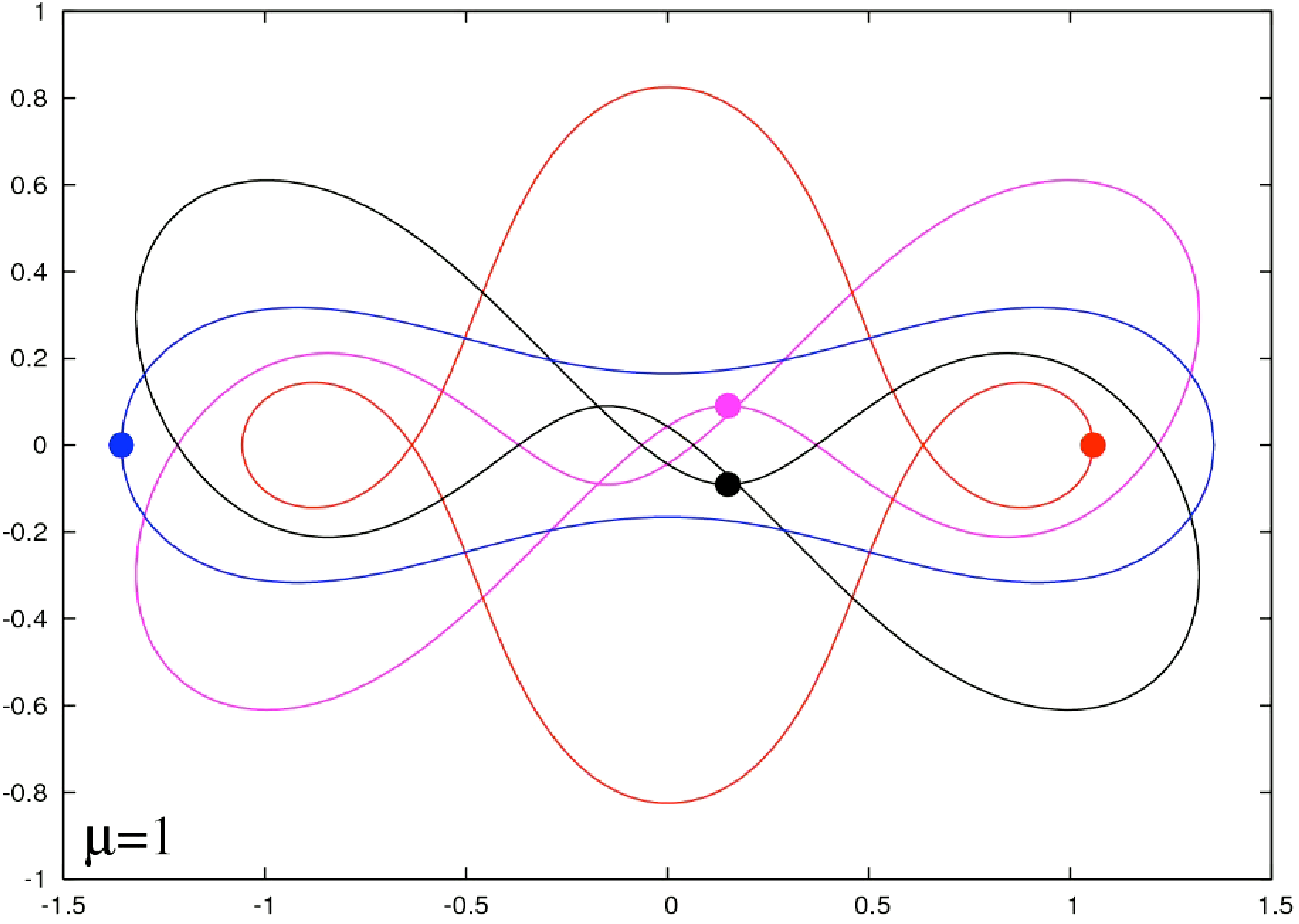


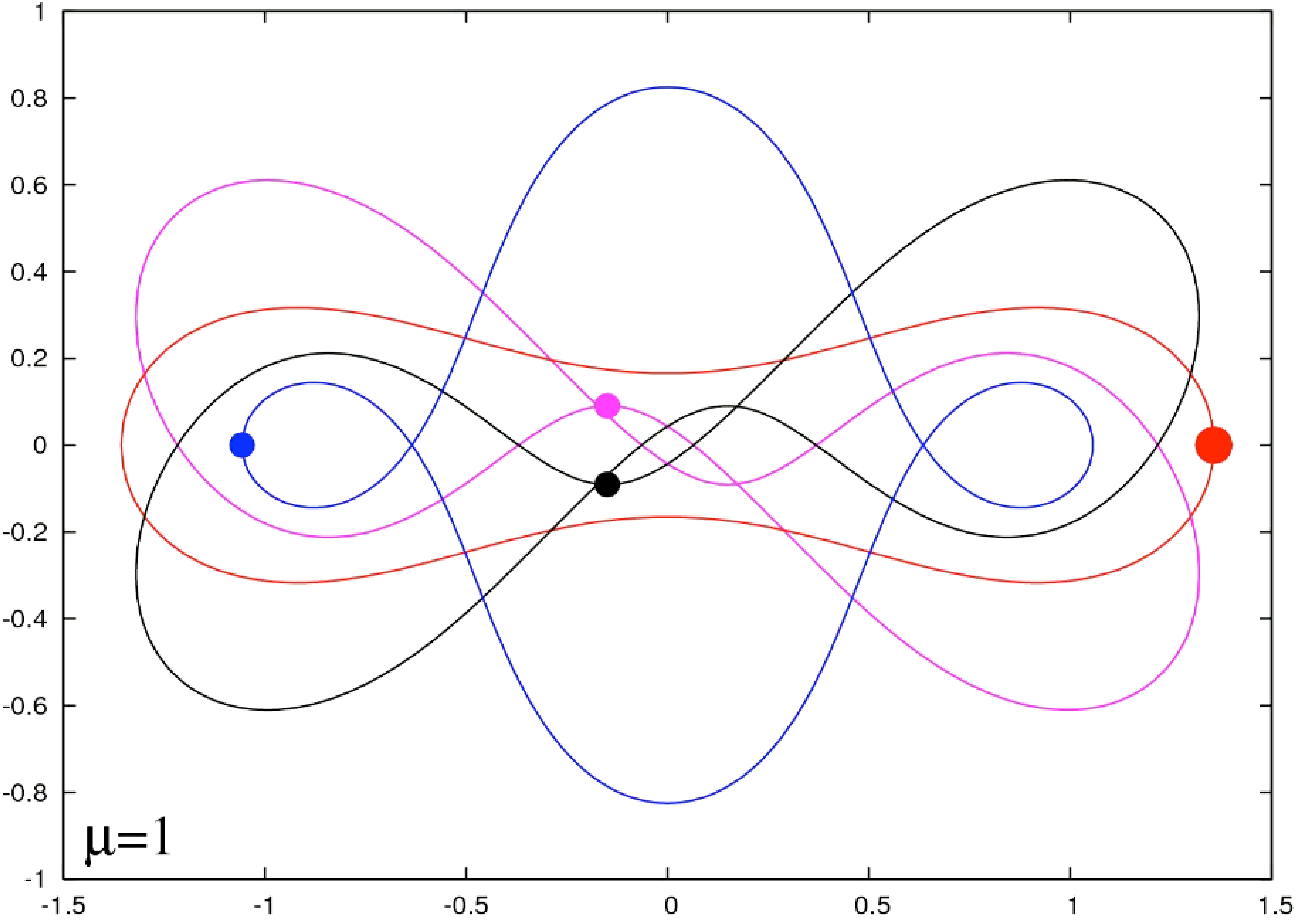


These two orbits are related by an exchange symmetry.






1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.5
-1
-0.5
0
0.5
1
1.5
μ=1



1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.5
-1
-0.5
0
0.5
1
1.5
μ=1

Continuation of the supereight

Finally we consider still a different mass distribution, namely

$$m_1 = m_4 = M \quad \text{and} \quad m_2 = m_3 = 1.$$

Continuation of the supereight

Finally we consider still a different mass distribution, namely

$$m_1 = m_4 = M \quad \text{and} \quad m_2 = m_3 = 1.$$

This is compatible with the reversor

$$\hat{R}_0 :=$$
$$R \circ \Sigma_{2,3} \circ \Sigma_{1,4} \circ \Phi \circ \Psi_\pi.$$

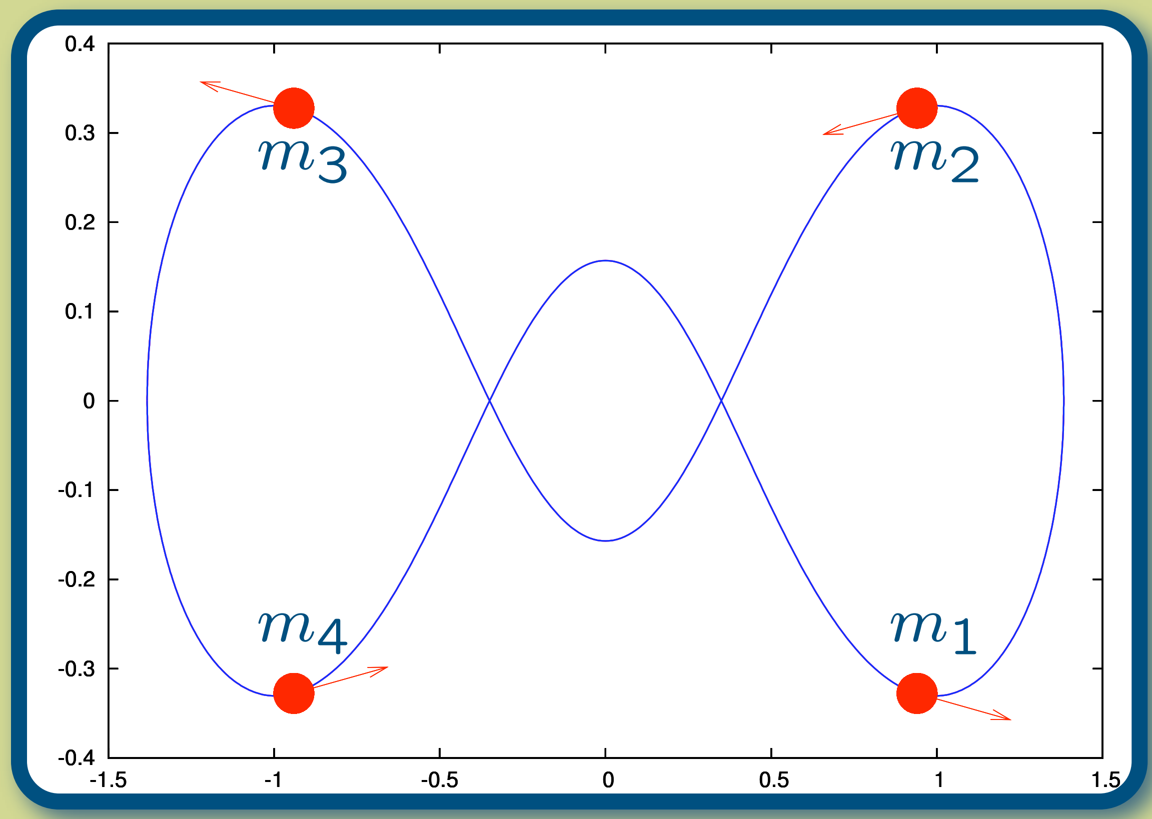
Continuation of the supereight

Finally we consider still a different mass distribution, namely

$$m_1 = m_4 = M \quad \text{and} \quad m_2 = m_3 = 1.$$

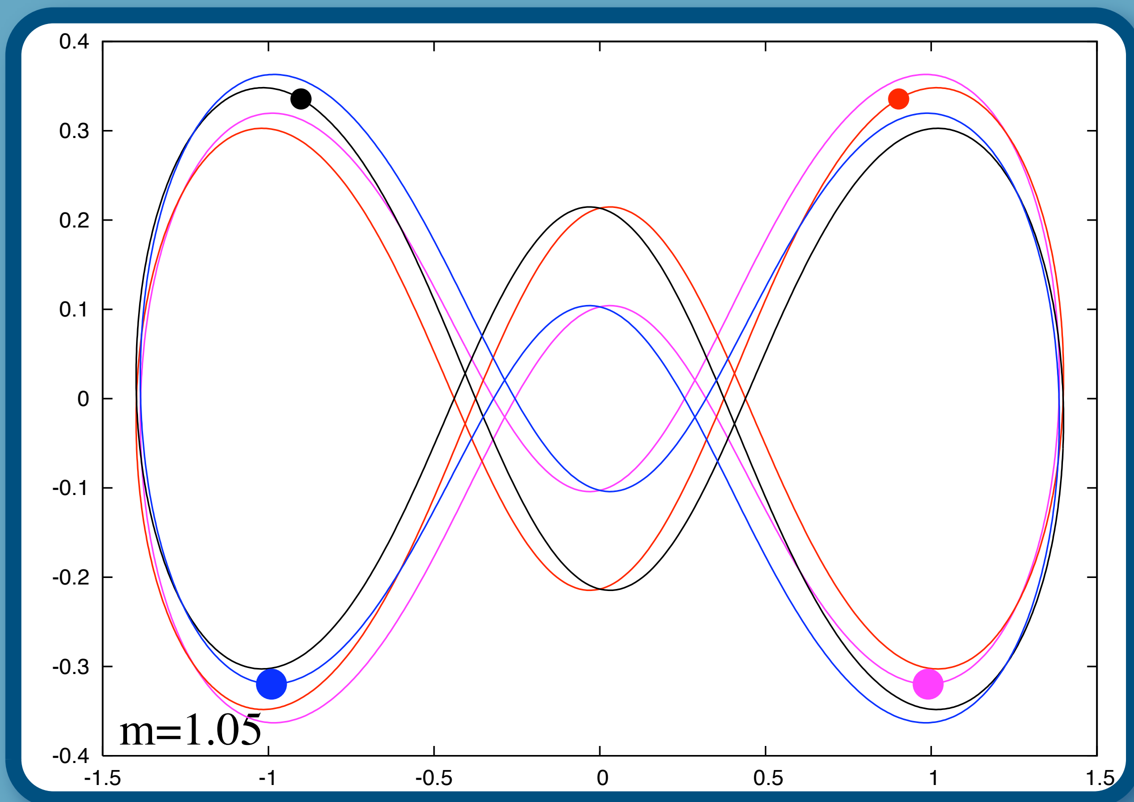
This is compatible with the reversor

$$\hat{R}_0 := R \circ \Sigma_{2,3} \circ \Sigma_{1,4} \circ \Phi \circ \Psi_\pi .$$

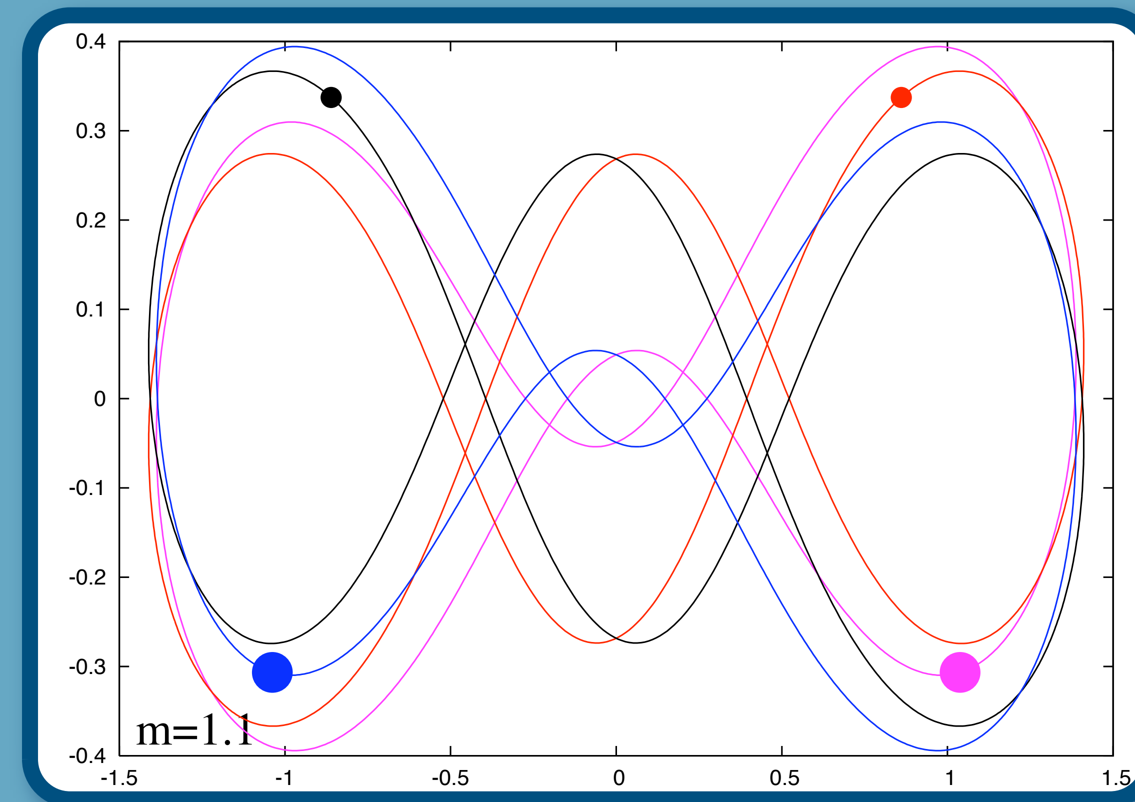


Continuation of the supereight

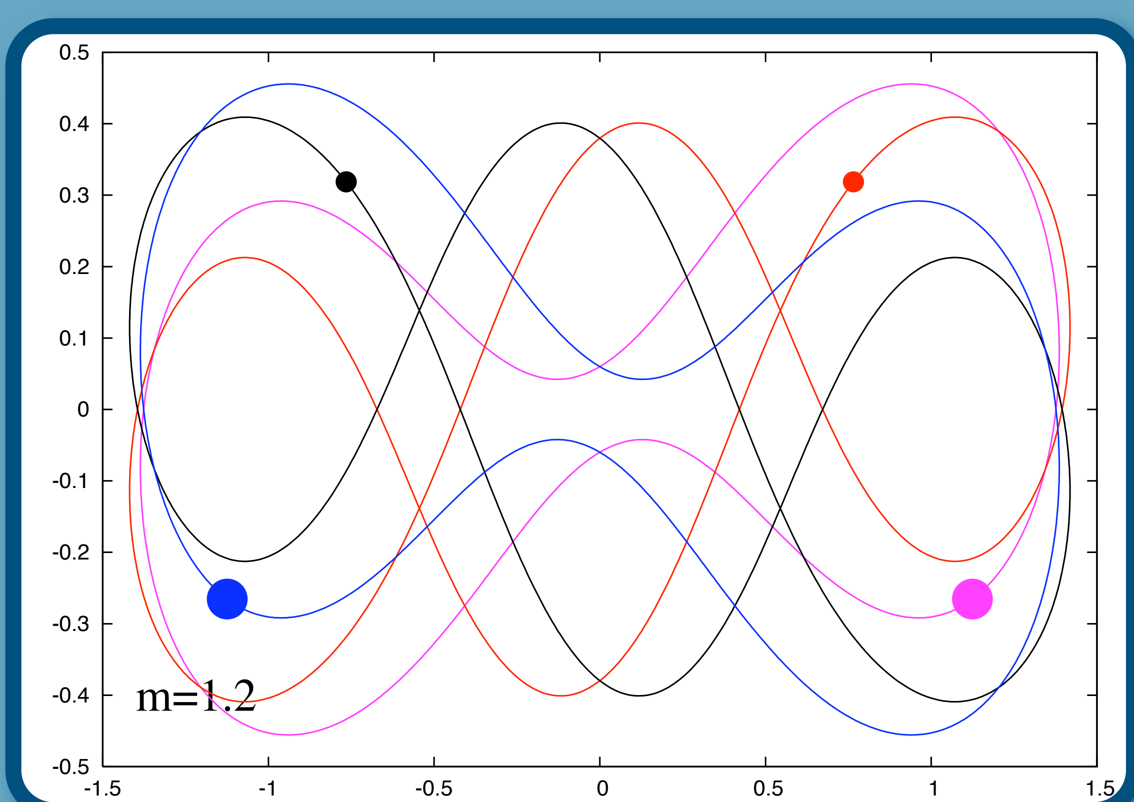
Continuing the supereight as a $\widehat{R}_0$-symmetric solution (with basic domain $[0, 4T_0]$) and using M as the continuation parameter we obtain a branch along which the solutions look as follows.



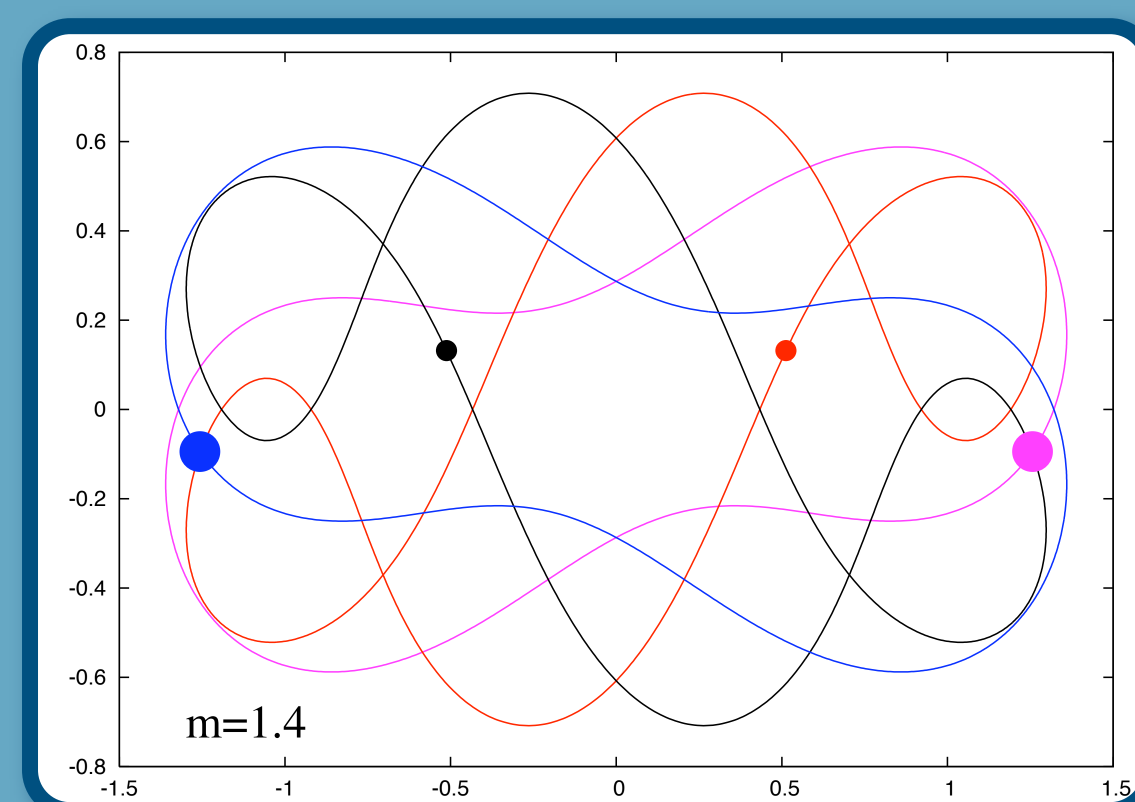
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-1.5
-1
-0.5
0
0.5
1
1.5
m=1.05



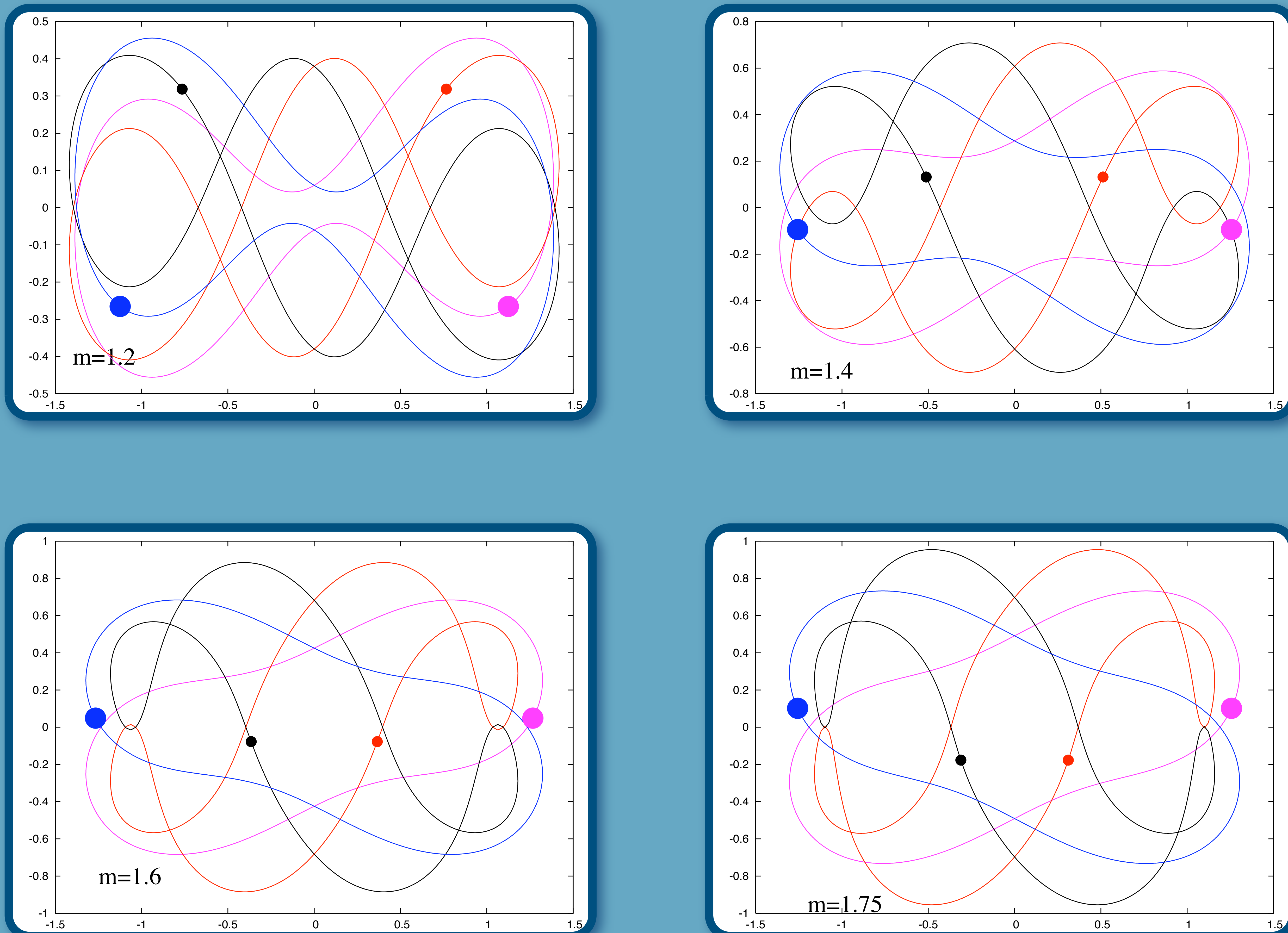
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-1.5
-1
-0.5
0
0.5
1
1.5
m=1.1

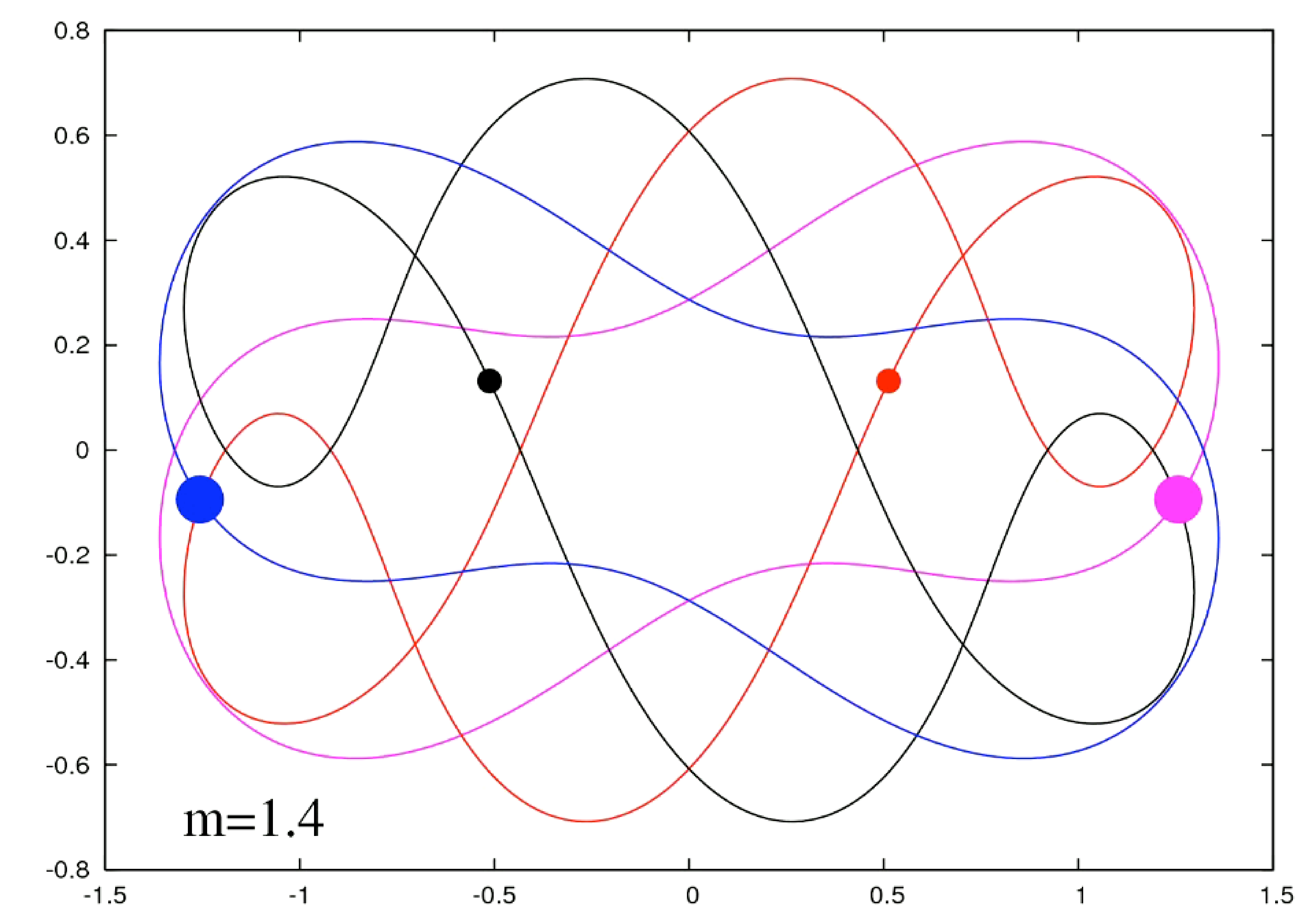


0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-1.5
-1
-0.5
0
0.5
1
1.5
m=1.2



0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1.5
-1
-0.5
0
0.5
1
1.5
m=1.4


m=1.2
m=1.4
m=1.6
m=1.75


0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1.5
-1
-0.5
0
0.5
1
1.5
m=1.4

Continuation of the supereight

One can also continue the supereight as a periodic orbit (forgetting the reversibility and using the techniques of lecture 1) under the mass configuration

$$m_1 = m_4 = M \quad \text{and} \quad m_2 = m_3 = 1,$$

and with M as the continuation parameter.

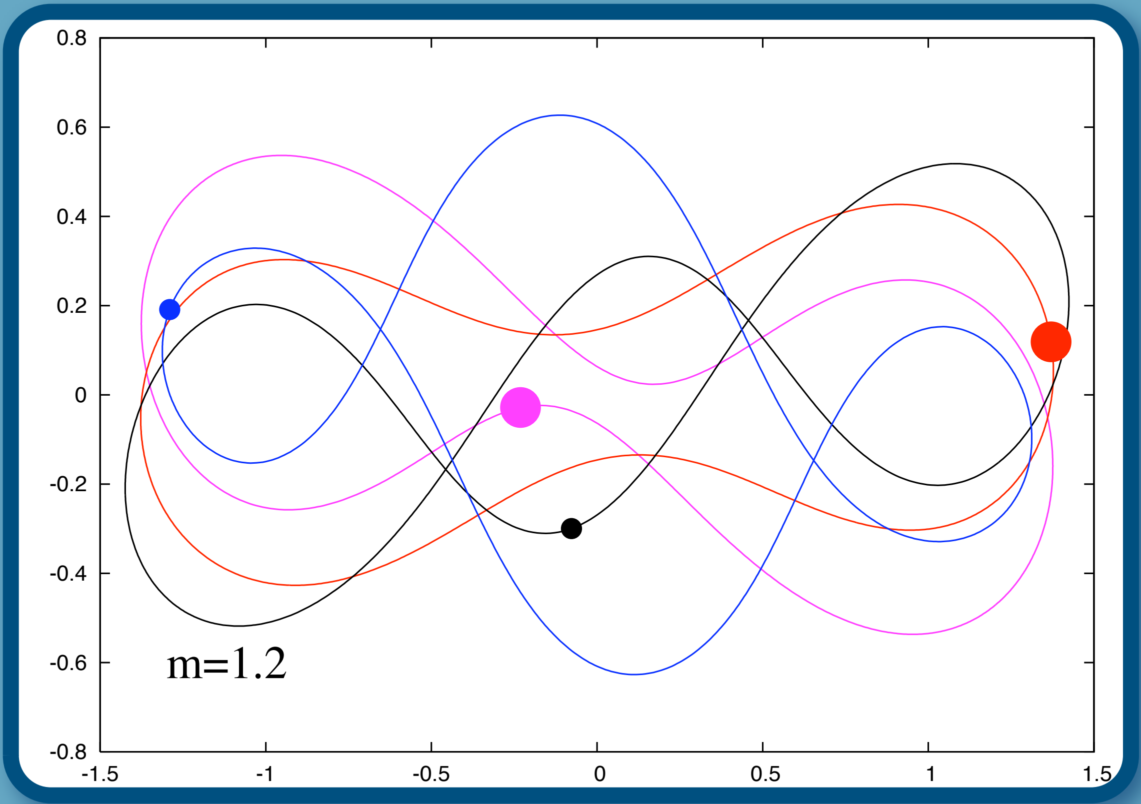
Continuation of the supereight

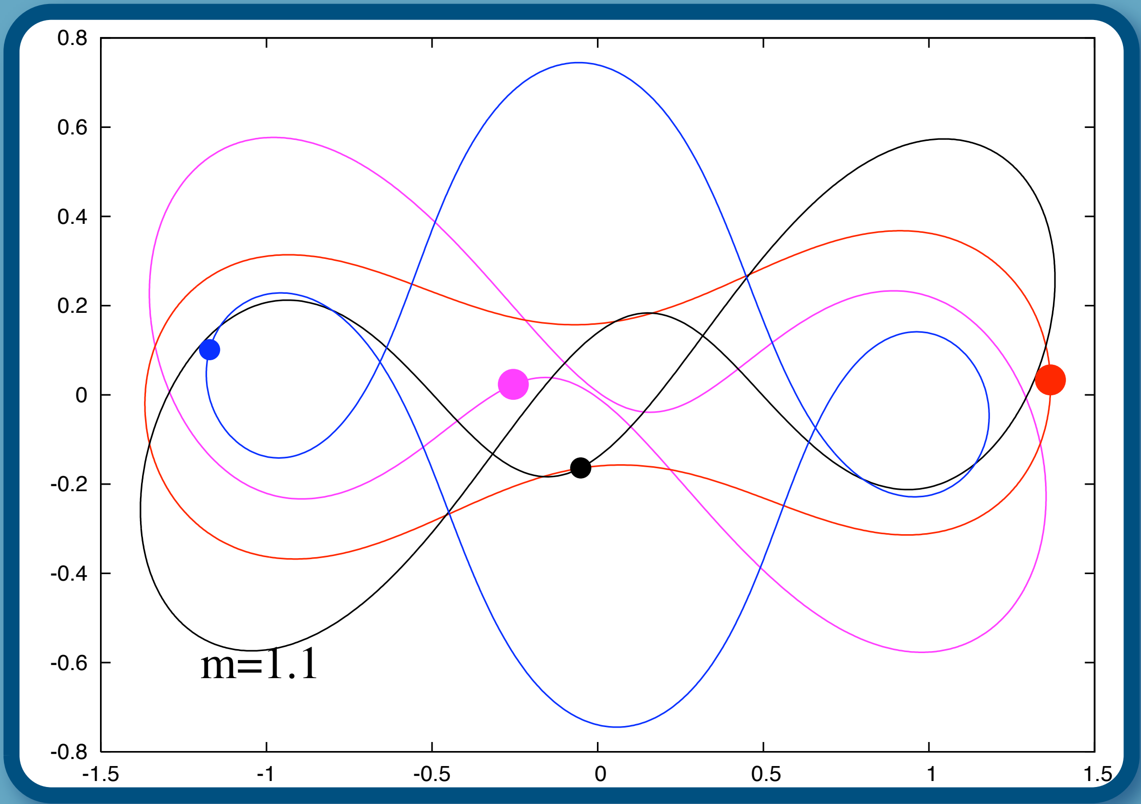
One can also continue the supereight as a periodic orbit (forgetting the reversibility and using the techniques of lecture 1) under the mass configuration

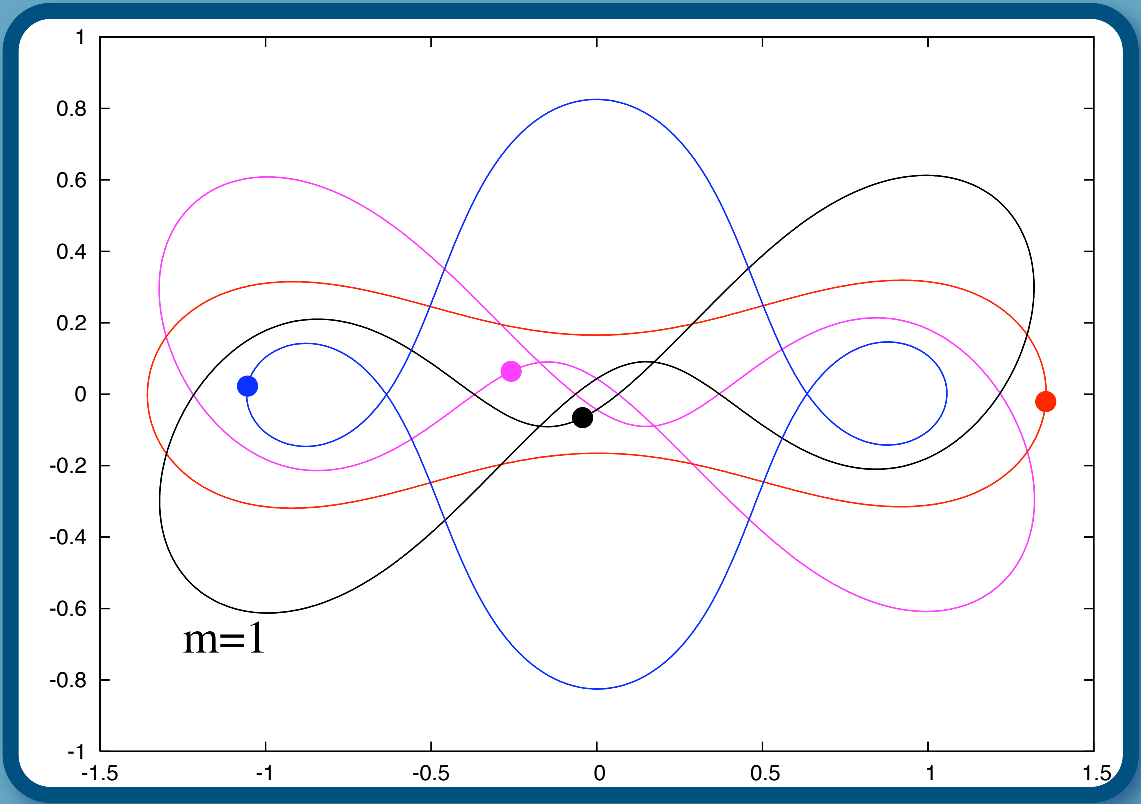
$$m_1 = m_4 = M \quad \text{and} \quad m_2 = m_3 = 1,$$

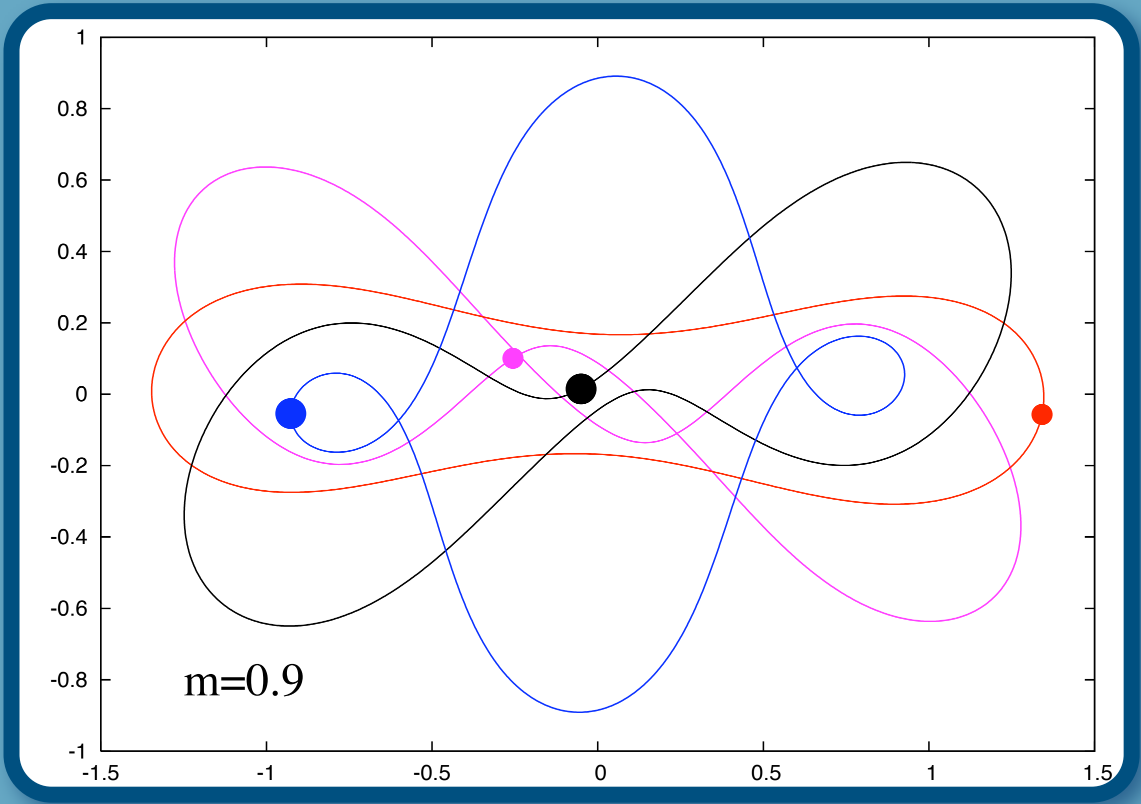
and with M as the continuation parameter.

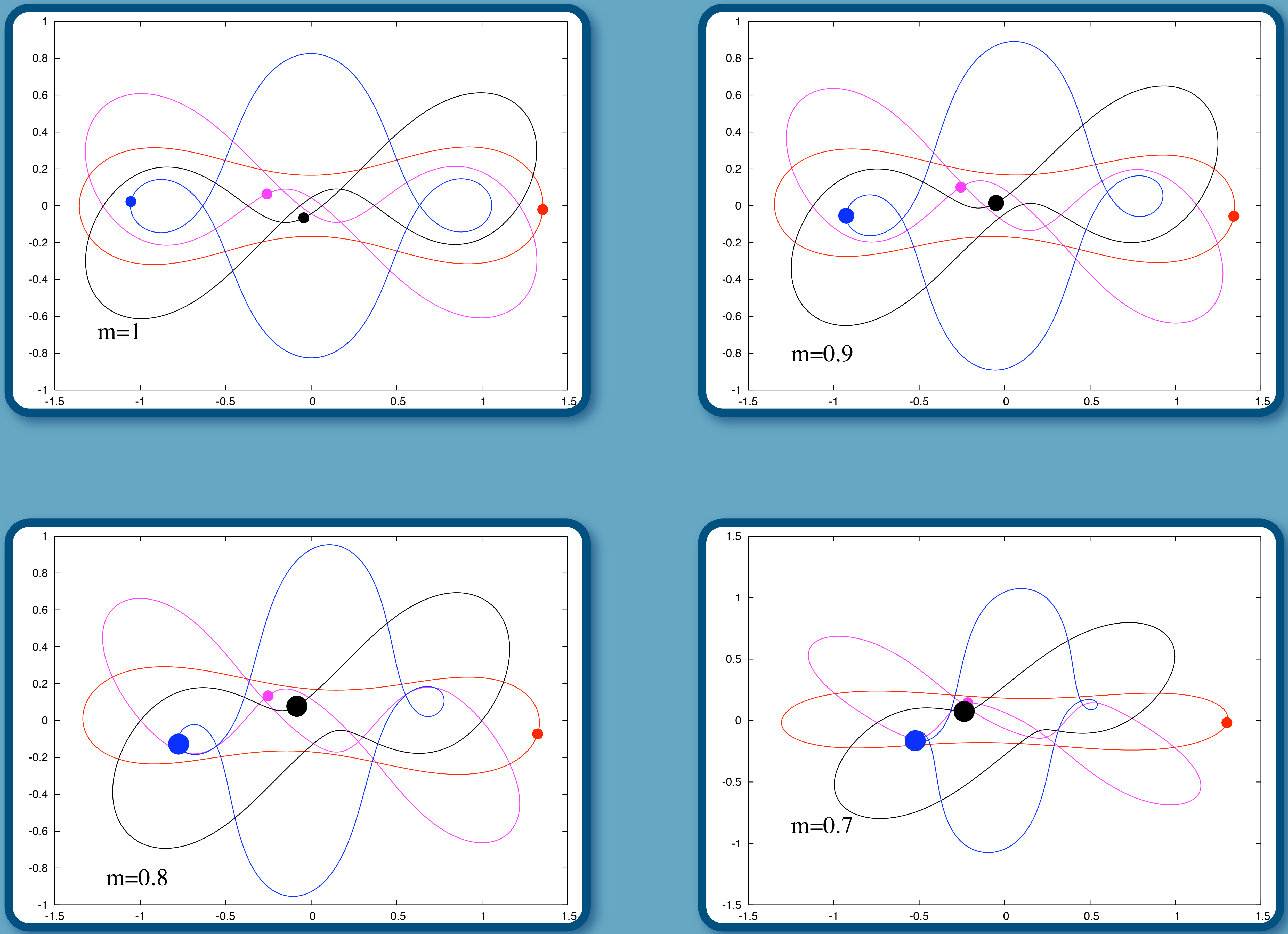
One obtains the same branch as when using the $\widehat{R}_0$-symmetry, only this time one detects a bifurcation at $M = 1.24871$. The bifurcating solutions are periodic but have no symmetry at all.

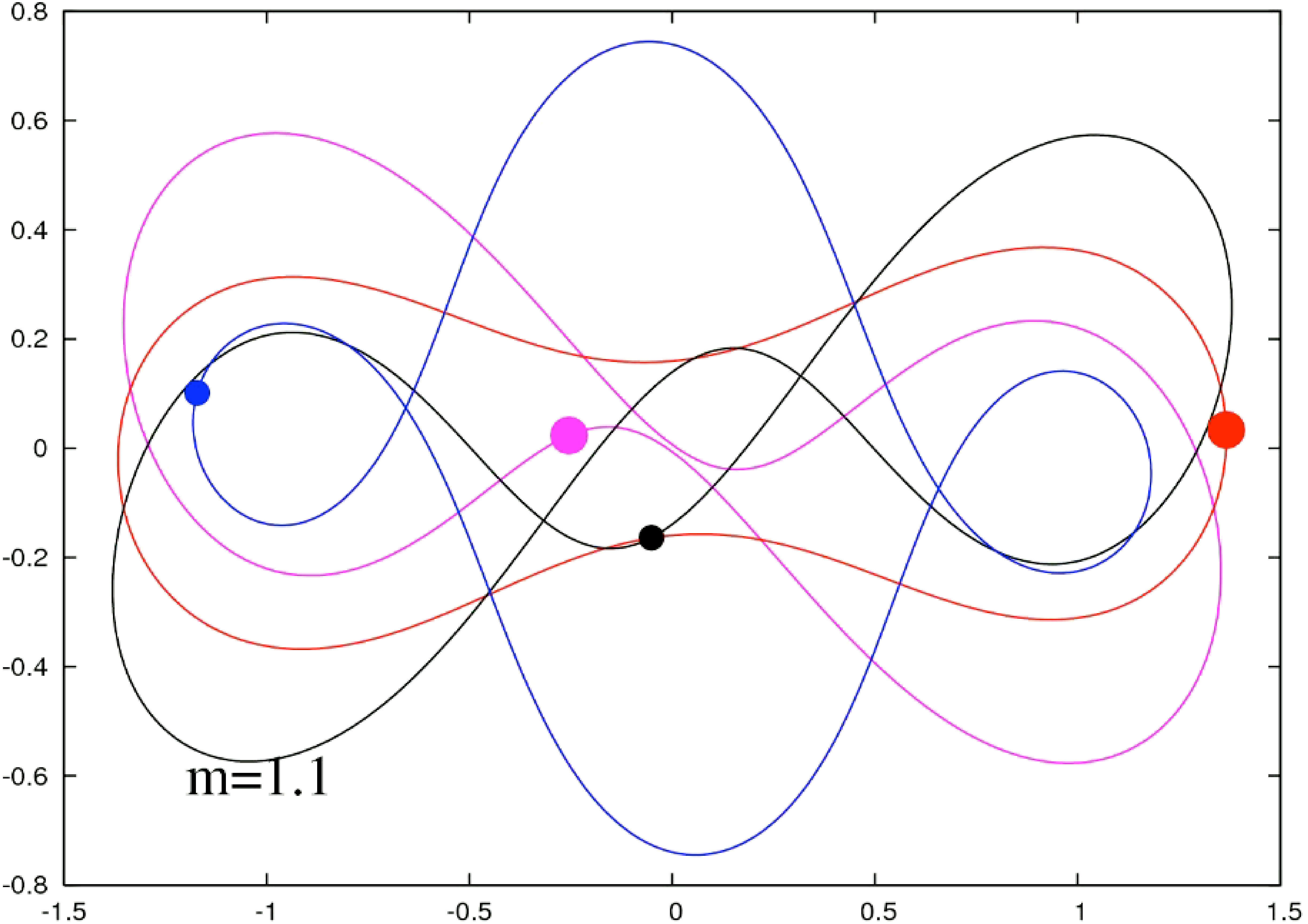

m=1.2
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1.5
-1
-0.5
0
0.5
1
1.5


m=1.1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1.5
-1
-0.5
0
0.5
1
1.5

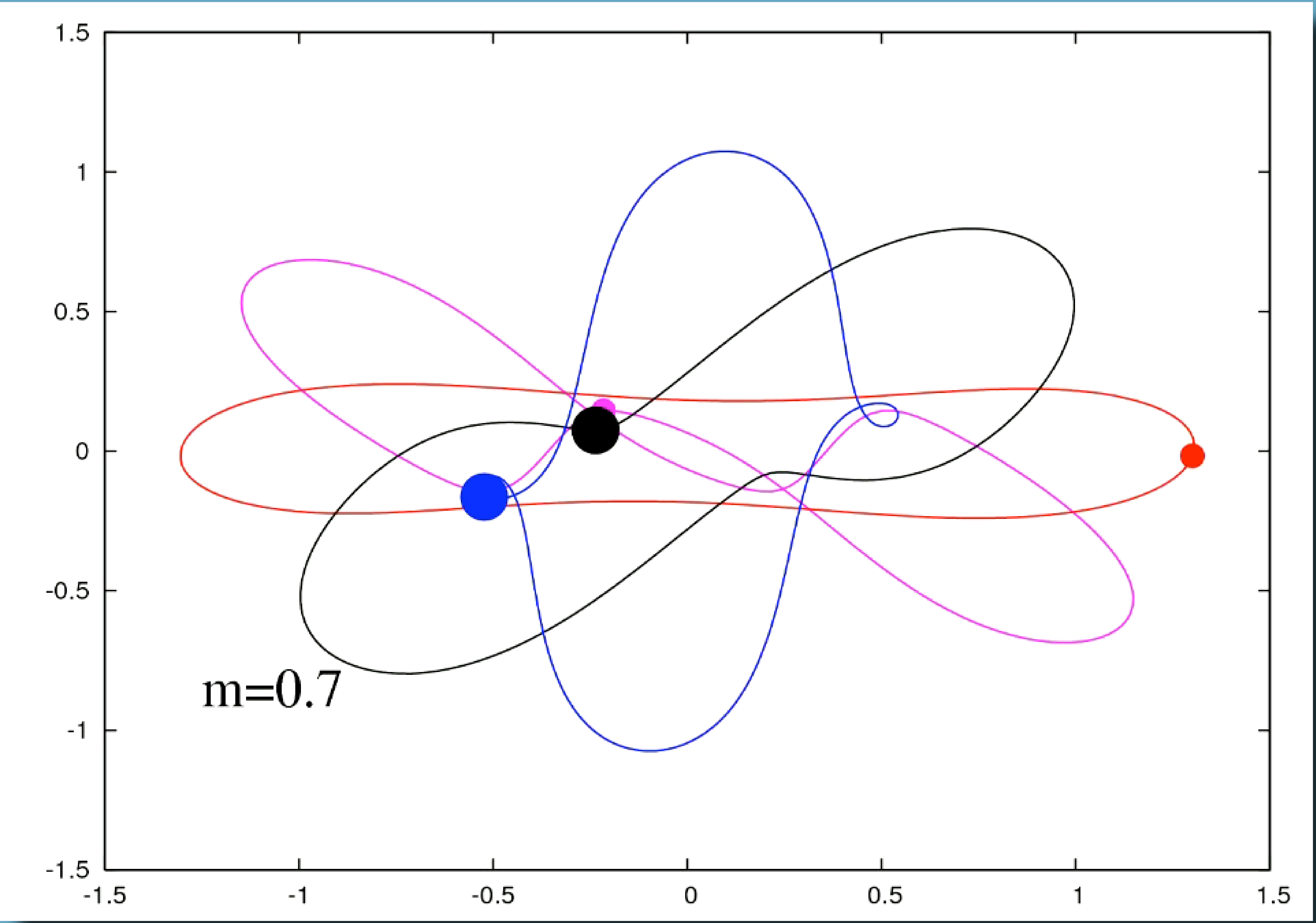

m=1
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.5
-1
-0.5
0
0.5
1
1.5


m=0.9
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.5
-1
-0.5
0
0.5
1
1.5

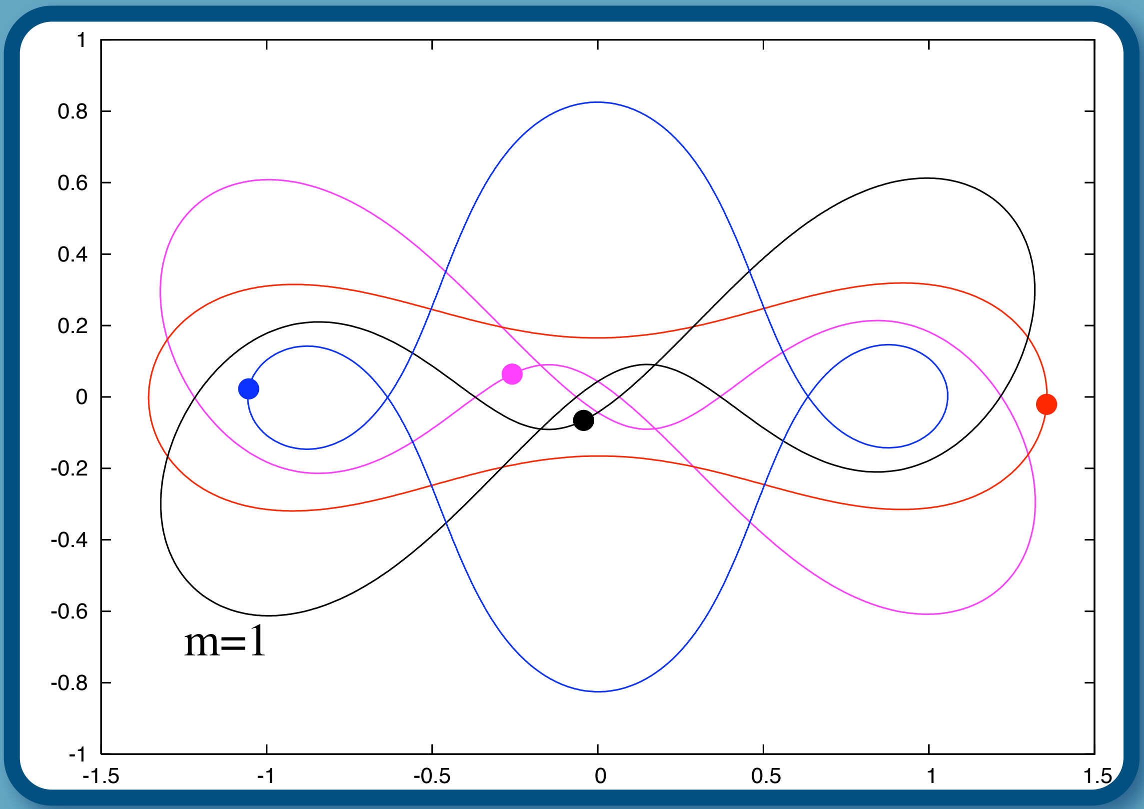

m=1
m=0.9
m=0.8
m=0.7



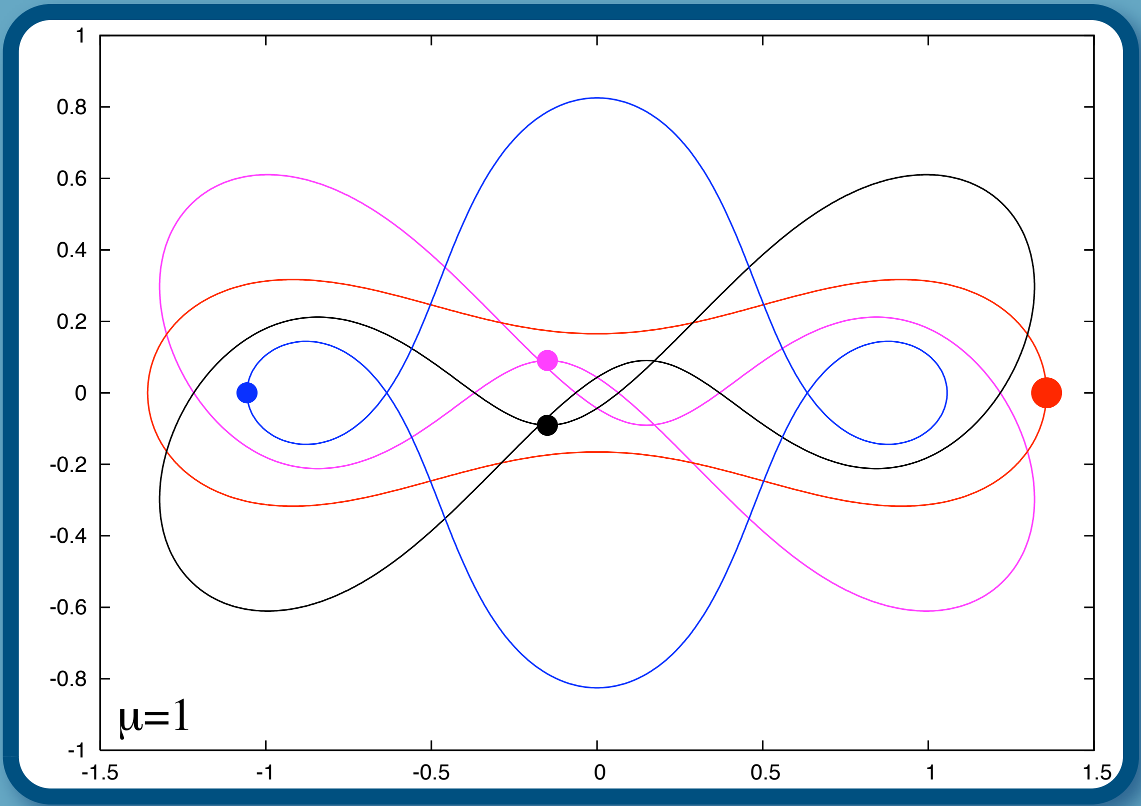
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1.5
-1
-0.5
0
0.5
1
1.5
m=1.1


1.5
1
0.5
0
-0.5
-1
-1.5
-1.5
-1
-0.5
0
0.5
1
1.5
m=0.7

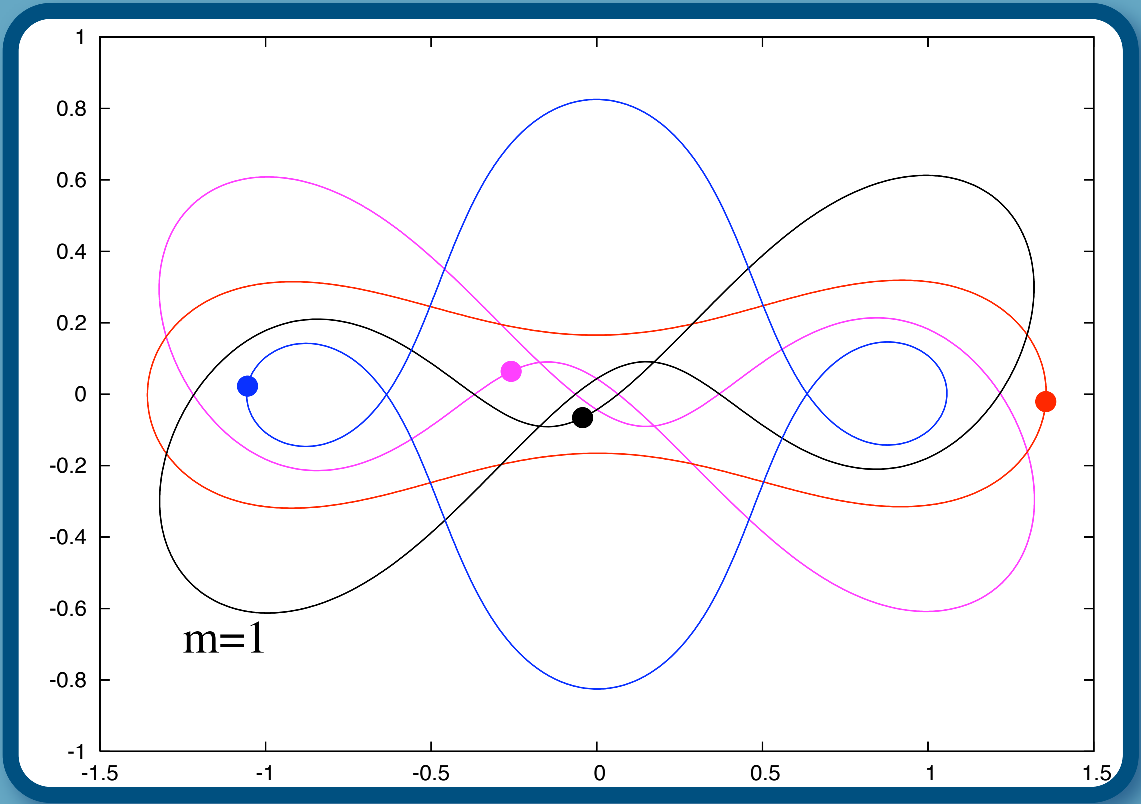
One more solution with 4 equal masses...

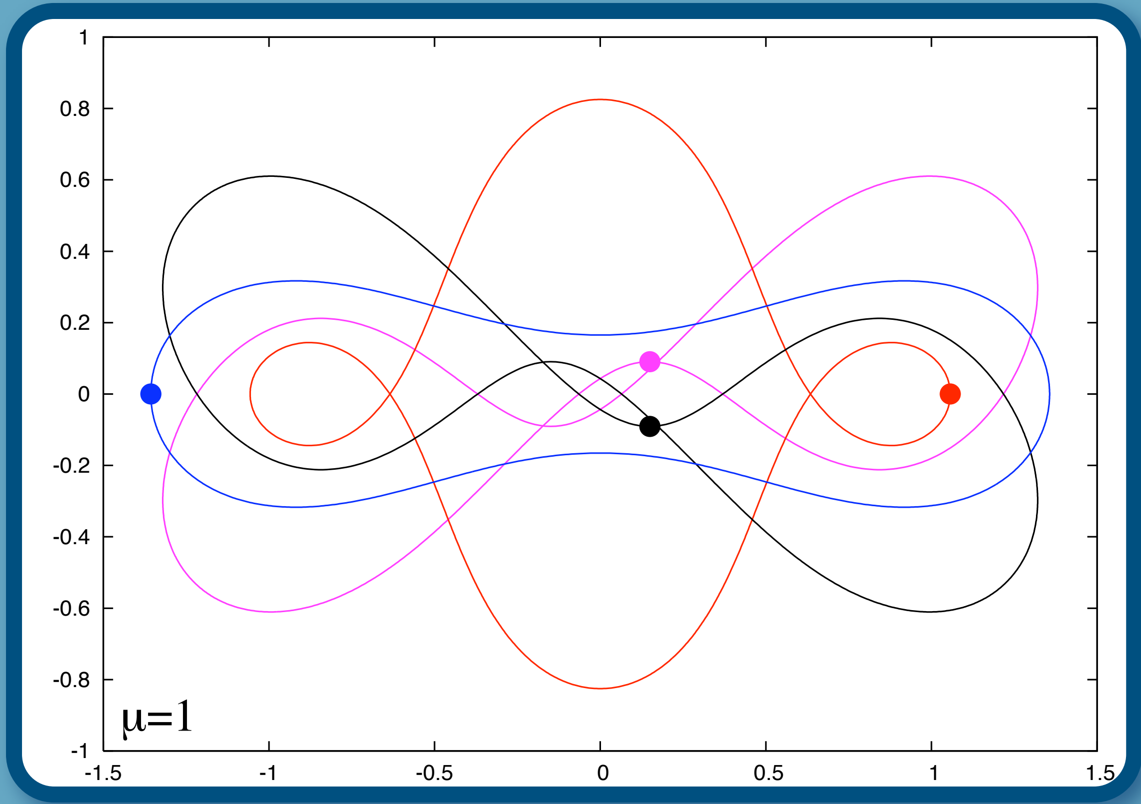


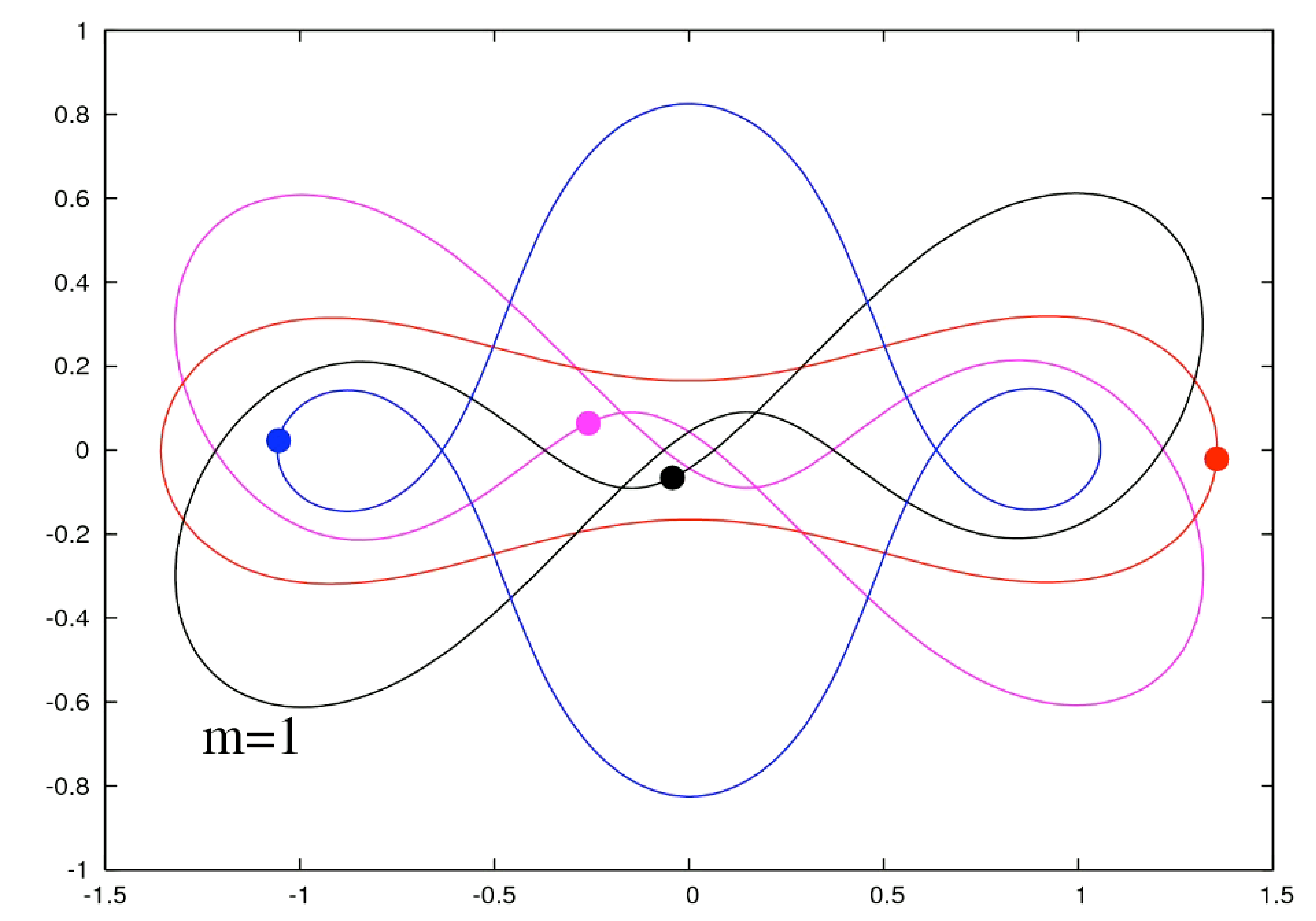
One more solution with 4 equal masses...



Up to an exchange of the bodies it is the same as the ones we found before.







1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.5
-1
-0.5
0
0.5
1
1.5
m=1

Thank You

UNIVERSITEIT GENT