DYNAMICS OF LATTICE DIFFERENTIAL EQUATIONS

John Mallet-Paret

Brown University

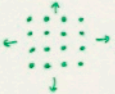
Lattice Differential Equations

Infinite (or high dimensional) system of ODE's with underlying spatial structure (lattice)

$$\dot{u}_\eta = F_\eta\left(\{u_{\eta+s}\}_{s\in\Lambda}\right) \qquad \eta\in\Lambda\subseteq\mathbb{R}^d$$

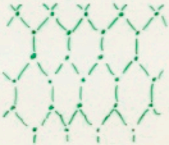
eg $\Lambda = \mathbb{Z}$

$\Lambda = \mathbb{Z}^d$



$\Lambda = \mathbb{Z}/m\mathbb{Z}$

$\Lambda \subseteq \mathbb{R}^2$ hexagonal lattice



$\Lambda \subseteq \mathbb{R}^3$ crystallographic lattices

Dynamical System Setting

$$\dot{u}_\eta = F_\eta\left(\{u_{\eta+s}\}_{s\in\Lambda}\right)$$

Regard $u = \{u_\eta\}_{\eta\in\Lambda}$ as an element $\boxed{u \in \ell^\infty(\Lambda)}$

and $F = \{F_\eta\}_{\eta\in\Lambda}$ as a map $\boxed{F : \ell^\infty(\Lambda) \to \ell^\infty(\Lambda)}$

$$\dot{u} = F(u)$$

Initial condition

$$u(0) = u_0 \in \ell^\infty(\Lambda)$$

Unique solution

$$u(t) \qquad t \in I$$

$$\dot{u}_i = \underbrace{\alpha}_{\substack{\uparrow \\ \text{coupling} \\ \text{constant}}} \overbrace{(u_{i+1} + u_{i-1} - 2u_i)}^{\text{discrete laplacian}} - \underbrace{f}_{\substack{\uparrow \\ \text{nonlinearity} \\ \text{(given)}}}(u_i) \qquad i \in \mathbb{Z}$$

Spatially discrete version of Chafee-Infante (Allen-Cahn) PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - f(u) \qquad \begin{array}{l} u = u(t,x) \\ x \in \mathbb{R} \end{array}$$

$\alpha > 0$ diffusion

$\alpha \to \infty$ PDE limit $\left(\alpha = \frac{1}{h^2} \quad h = \text{grid size}\right)$

$\alpha = 0$ decoupling

$$\left.\begin{array}{l} \alpha = O(1) \\ \alpha < 0 \quad \text{anti-diffusion} \end{array}\right] \text{new phenomena}$$

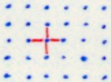
$$\dot{u}_i = \alpha(u_{i+1} + u_{i-1} - 2u_i) - f(u_i)$$

$$\dot{u}_i = \alpha(u_{i+1} - u_i) + \beta(u_{i-1} - u_i) - f(u_i)$$

$$\dot{u}_i = \sum_{r=-R}^{R} \alpha_r (u_{i+r} - u_i) - f(u_i)$$ long range $R < \infty$ or $R = \infty$

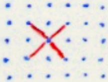
$$\dot{u}_i = \sum_{r=-R}^{R} \alpha_{i,r} (u_{i+r} - u_i) - f(u_i)$$ not translation invariant

$$\dot{u}_{ij} = \underbrace{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{ij}}_{\Delta u_{ij}} - f(u_{ij})$$



$(i,j) \in \mathbb{Z}^2$

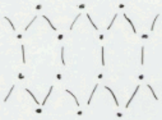
$$\dot{u}_{ij} = \underbrace{u_{i+1,j+1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i-1,j-1} - 4u_{ij}}_{\tilde{\Delta} u_{ij}} - f(u_{ij})$$

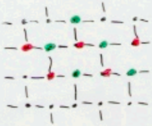


$$\dot{u}_{ij} = \alpha \Delta u_{ij} + \tilde{\alpha} \tilde{\Delta} u_{ij} - f(u_{ij})$$

$$\dot{u}_i = F(u_{i+1}, u_i, u_{i-1})$$ nonlinear coupling

etc...





$\mathbb{Z}^2$

same connections

$(i,j) \in \mathbb{Z}^2$ connects to

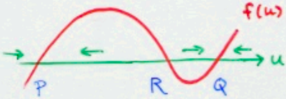
$(i \pm 1, j)$

$(i, j + \delta_{ij})$ $\qquad \delta_{ij} = (-1)^{i+j}$

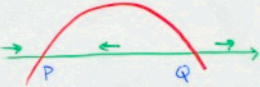
$$\dot{u}_{ij} = u_{i+1,j} + u_{i-1,j} + u_{i,j+\delta_{ij}} - 3u_{ij} - f(u_{ij})$$

Nonlinearity f

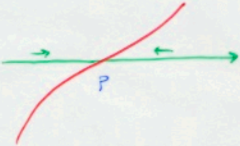
bistable



Fisher



monostable



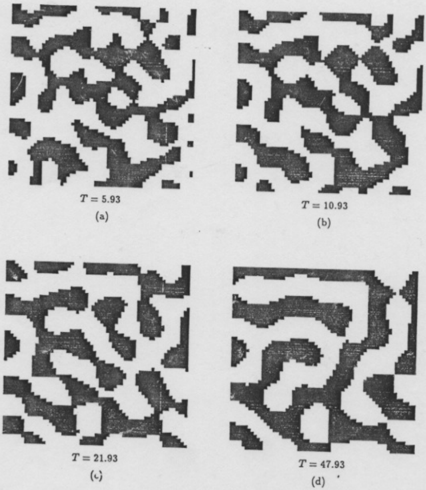


Figure 1. Neumann BCs ($\sigma = -2.5, \alpha_{10} = 0.5, \alpha_{11} = 0.25, c_0 = 0.0$)

We will usually consider only the case in which there is a double well potential; i.e. when $\sigma < -2$, but when either $\alpha_{10} < 0$ or $\alpha_{11} < 0$ we may consider $\sigma > -2$ since with a negative gradient energy coefficient the constant solution is not a minimizer. Since we typically obtain high amplitude solutions, "snapshots" of the time evolution are displayed using a white square to indicate a value greater than zero and a black square to indicate a value less than zero.

In Figure 1 we display "snapshots" of the time evolution in the range of parameters that corresponds to a well-posed PDE; i.e. the Cahn-Hilliard equation. Note that the solution undergoes coarsening like the continuous equation. Figure 1 shows what is known as spinodal decomposition.

$T = 0.96$

(a)

$T = 1.35$

(b)

$T = 2.76$

(c)

$T = 10.82$

(d)

Figure 2. Neumann BCs ($\sigma = -1.0, \alpha_{10} = -0.25, \alpha_{11} = 0.5, c_0 = 0.0$)

Figure 2 shows the ordering that occurs for $\alpha_{10} < 0$ and $\alpha_{11} > 0$. The bulk phases here are "checkerboard" patterns, one with even parity and one with odd parity. The bulk phases are separated by an interface.

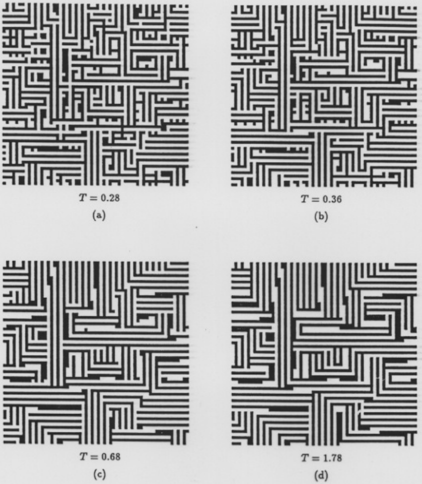
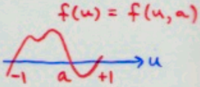


Figure 3. Neumann BCs ($\sigma = -4.0, \alpha_{10} = 2.0, \alpha_{11} = -1.0, c_0 = 0.0$)

We have $\alpha_{10} > 0$ and $\alpha_{11} < 0$ in Figure 3. For these parameter values the bulk phases are even and odd parity horizontal stripes and even and odd parity vertical stripes.

Bistable Reaction-Diffusion PDE

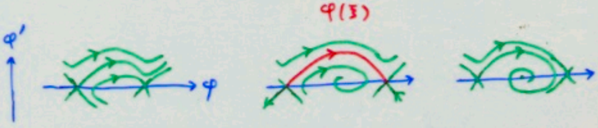
$$u_t = u_{xx} - f(u)$$



Traveling Wave $\underline{u(t,x) = \varphi(x-ct)}$

$$(*) \quad \begin{cases} -c\varphi'(\xi) = \varphi''(\xi) - f(\varphi(\xi)) \\ \varphi(\pm\infty) = \pm 1 \end{cases}$$

<u>THM</u> $\exists 1$ $c = c_*$ for which $(*)$ has a solution. The solution $\varphi(\xi)$ is unique and monotone. Also, $c = c_*(a)$ depends smoothly on a.

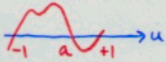


$c < c_*$ $\qquad$ $c = c_*$ $\qquad$ $c > c_*$

Bistable Reaction-Diffusion PDE

$$u_t = u_{xx} - f(u)$$

$f(u) = f(u,a)$



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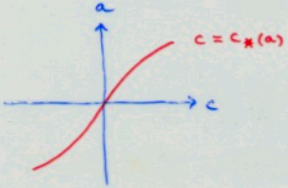


f monotone in a

$$\Downarrow$$

$c_*(a)$ monotone in a

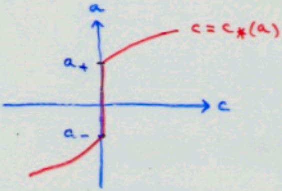
Lattice Differential Equation

$$\dot{u}_i = \alpha(u_{i+1} + u_{i-1} - 2u_i) - f(u_i)$$

Traveling wave $\underline{u_i(t) = \varphi(i - ct)}$

$$(\#) \quad \begin{cases} -c\varphi'(\xi) = \alpha\big(\varphi(\xi+1) + \varphi(\xi-1) - 2\varphi(\xi)\big) - f(\varphi(\xi)) \\ \varphi(\pm\infty) = \pm 1 \end{cases}$$

<u>THM</u> $\exists 1$ $c = c_*$ for which (#) has a monotone solution. If $c_* \neq 0$ the solution $\varphi(\xi)$ is unique and monotone. If $c_* = 0$ there exists a monotone solution, generically an even number of them.



<u>THM</u> Generically $\underline{a_- < a_+}$

The interval $[a_-, a_+]$ is called the <u>PINNING</u> <u>REGION</u>

$P=2$

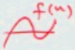
$$\dot{u}_i = \begin{cases} \alpha(u_{i+1}-u_i) + \beta(u_{i-1}-u_i) - f(u_i) & i=\text{even} \\ \tilde{\alpha}(u_{i+1}-u_i) + \tilde{\beta}(u_{i-1}-u_i) - f(u_i) & i=\text{odd} \end{cases}$$

$u_{2i}(t) = \varphi(i-ct)$

$u_{2i+1}(t) = \tilde{\varphi}(i-ct)$

wave speed $= 2c$

$$\Phi(\xi) = \begin{pmatrix} \varphi(\xi) \\ \tilde{\varphi}(\xi) \end{pmatrix}$$



$$-c\,\Phi'(\xi) = A_+\Phi(\xi+1) + A_0\Phi(\xi) + A_-\Phi(\xi-1) - f(\Phi(\xi))$$

$$\Phi(\pm\infty) = \begin{pmatrix} \pm 1 \\ \pm 1 \end{pmatrix}$$

$$A_+ = \begin{pmatrix} 0 & 0 \\ \tilde{\alpha} & 0 \end{pmatrix} \qquad A_0 = \begin{pmatrix} -\alpha-\beta & \alpha \\ \tilde{\beta} & -\tilde{\alpha}-\tilde{\beta} \end{pmatrix} \qquad A_- = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}$$

"shift" matrices nonnegative $A_\pm \geq 0$

off-diagonal elements of A_0 nonnegative

similar for $3 \leq P < \infty$

$\underline{P = 1}$ (translation invariant equation)

but: solution <u>not</u> translation invariant

$$\dot{u}_i = \alpha(u_{i+1} - u_i) + \beta(u_{i-1} - u_i) - f(u_i)$$

wave joins homogeneous equilib $u_i \equiv -1$

to "checkerboard" equilib $u_i = \begin{cases} p & i = \text{even} \\ q & i = \text{odd} \end{cases}$

$$\begin{cases} 0 = \alpha(q-p) + \beta(q-p) - f(p) \\ 0 = \alpha(p-q) + \beta(p-q) - f(q) \end{cases}$$

As before $u_{2i}(t) = \varphi(i-ct)$

$u_{2i+1}(t) = \tilde{\varphi}(i-ct)$

$\Phi(\xi) = \begin{pmatrix} \varphi(\xi) \\ \tilde{\varphi}(\xi) \end{pmatrix}$

$\Rightarrow$ same equation for $\Phi(\xi)$

but different boundary conditions

$$\Phi(-\infty) = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \qquad \Phi(+\infty) = \begin{pmatrix} p \\ q \end{pmatrix}$$

<u>THM</u> Consider the system

$$-c\Phi'(\xi) = \sum_{k=0}^{K} A_k \Phi(\xi + r_k) - f(\Phi(\xi))$$

diag of $f : \mathbb{R} \to \mathbb{R}$

$$\Phi(\pm\infty) = p_\pm$$

$$\Phi : \mathbb{R} \to \mathbb{R}^d$$

where $A_k \geqslant 0 \quad 1 \le k \le K \qquad \sum_{k=0}^{K} A_k$ irreducible

$A_0 \geqslant \lambda I$ some $\lambda \in \mathbb{R}$

$r_0 = 0$

$p_\pm$ are "stable" equilibria

$p_- \ll p_0 \ll p_+$

only equilibrium between, "unstable"

Then $\exists$ unique $c \in \mathbb{R}$ and $\exists$ monotone $\Phi(\xi)$.

Also, $c \neq 0 \Rightarrow \Phi$ unique.

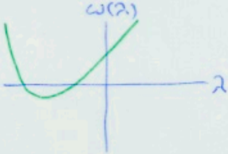
$$\Delta(\lambda) = \lambda c I + \sum_{k=0}^{K} A_k e^{\lambda r_k} - D$$

$\omega(\lambda) =$ largest eigenvalue of $\Delta(\lambda)$
(positive eigenvector)

$\lambda \longrightarrow \omega(\lambda)$ convex

"stable"

$\omega(0) < 0$



"unstable"

$\omega(0) > 0$

COR For the system

$$\dot{u}_i = \alpha_i(u_{i+1} - u_i) + \beta_i(u_{i-1} - u_i) - f(u_i)$$

$$\alpha_{i+P} = \alpha_i > 0$$

$$\beta_{i+P} = \beta_i > 0$$

with spatial period P, there exists $c \in \mathbb{R}$ and a traveling wave solution

$$u_{kP+j}(t) = \varphi_j(k - ct) \qquad 0 \le j < P$$

$$\varphi_j(\pm\infty) = \pm 1$$

which is strictly spatially monotone

$$u_i(t) < u_{i+1}(t) \qquad \forall i \in \mathbb{Z} \quad \forall t \in \mathbb{R}$$

If $c_* \neq 0$ then we have the orbit

$$\Gamma = \{ q(t) \mid t \in \mathbb{R} \} \cup \{-1, +1\} \subseteq \ell^\infty$$

$$q : \mathbb{R} \longrightarrow \ell^\infty \qquad q_i(t) = \varphi(i - c_* t)$$

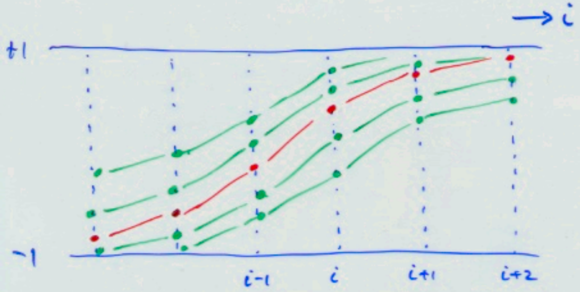
$$q(t+T) = S^{-1} q(t) \qquad T = c_*^{-1}$$

$\Gamma \subseteq [-1,1] \subseteq \ell^\infty$ is invariant,
totally ordered,
maximal

$\forall \xi \in [-1,1] \subseteq \mathbb{R} \quad \forall i \in \mathbb{Z} \quad \exists u \in \Gamma$ with $u_i = \xi$

If $c_* = 0$, what is the analog of Γ ?

<u>ANS</u> A "monotone complex" consisting of equilibria and their strong unstable manifolds.



Sample q(t) at same t

Sample q(t) at various t

A Generalized Traveling Wave for

$$\dot{u}_i = \alpha(u_{i+1} + u_{i-1} - 2u_i) - f(u_i)$$

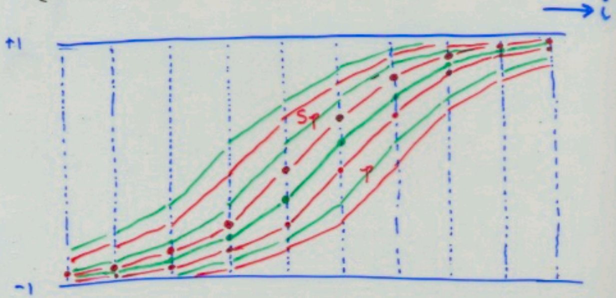
Assume $c_* = 0$

Assume for simplicity that every $u \in \mathcal{S}$ is hyperbolic, namely $\mathrm{spec}(D\mathcal{F}(u)) \cap i\mathbb{R} = \emptyset$

Fix $p \in \mathcal{S}$

Let $G \subseteq \{u \in \mathcal{S} \mid p \le u \le Sp\}$ be a maximal totally ordered set

(G must be a finite set.)



$q = \{q_i\}$ $\quad p < q < Sp$ $\quad p = \{p_i\}_{i \in \mathbb{Z}}$

G contains an odd number of elements, namely an even number modulo the equivalence $p \sim Sp$

$$G = \{ \underset{\underset{p}{\shortparallel}}{p^1} < q^1 < p^2 < q^2 < \cdots < p^N < q^N < \underset{\underset{Sp}{\shortparallel}}{p^{N+1}} \}$$

Elements of G alternate in stability

e.g. p^i = unstable $\qquad q^i$ = stable

Fill in gaps with strong unstable manifolds

$$\overset{p^1}{\bullet} \rightarrow \overset{q^1}{\bullet} \leftarrow \overset{p^2}{\bullet} \rightarrow \bullet \leftarrow \bullet \rightarrow \overset{q^N}{\bullet} \leftarrow \overset{p^{N+1} = Sp^1}{\bullet}$$

Then apply all powers of S to get

$$\Gamma = \bigcup_{n \in \mathbb{Z}} S^n \left(\bigcup_{i=1}^{N} W(p^i) \cup \{ q^i \} \right)$$

(*) $\Gamma \subseteq [-1,1] \subseteq \ell^\infty$ invariant

totally ordered

maximal

$\Rightarrow$ $\forall \xi \in [-1,1] \subseteq \mathbb{R}$, $\forall i \in \mathbb{Z}$, $\exists u \in \Gamma$ with $u_i = \xi$

Γ is closed in compact-open topology

$$\Gamma = \Gamma^{eq} \cup \underbrace{\Gamma^{up}}_{\dot u_i > 0 \ \forall i} \cup \underbrace{\Gamma^{dn}}_{\dot u_i < 0 \ \forall i}$$

(#) Also, $\Gamma \subseteq \mathcal{M}$

<u>DEF</u> A <u>generalized traveling wave</u> (GTW) is a set Γ satisfying (*). It is <u>regular</u> if also (#) holds.

Solution with $c \neq 0$ satisfies

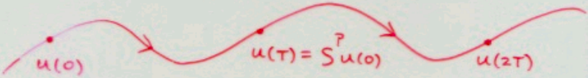
$$u(t+T) = S^p u(t) \qquad T = \frac{1}{c}$$

$S : \ell^\infty \longrightarrow \ell^\infty$

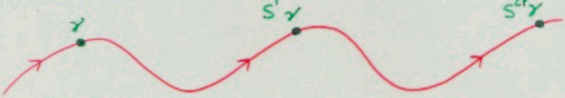
$\{u_i\} \longrightarrow \{u_{i+1}\}$

shift operator

In phase space ℓ^∞:

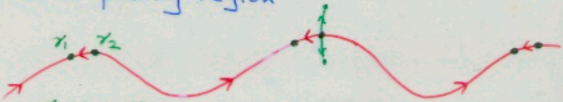


As $c \to 0$



$\gamma \in \ell^\infty$ equilibrium (saddle-node)

$c = 0$ in pinning region



$\gamma_1, \gamma_2 \in \ell^\infty$ hyperbolic equilibria

Techniques extend to more general system

$$\dot{u}_i = \alpha_i(u_{i+1} - u_i) + \beta_i(u_{i-1} - u_i) - f(u_i)$$

$$\alpha_{i+p} = \alpha_i > 0 \qquad \beta_{i+p} = \beta_i > 0$$

<u>THM</u> $\exists$ a regular GTW $\Gamma \subseteq \ell^\infty$.

<u>Non-periodic</u> coefficients α_i, β_i

Approximating system with coefficients

$$\begin{cases} \alpha_i^N = \alpha_i & |i| \le N \\ \alpha_{i+p}^N = \alpha_i^N & p = 2N+1 \end{cases} \quad + \text{ same with } \beta$$

$$\Rightarrow \Gamma^N \subseteq \ell^\infty \quad \text{RGTW}$$

Limit as $N \to \infty$

$$\Gamma^N \to \Gamma \quad \text{GTW}$$

But is Γ regular ?

THM Γ is regular if either

$$\alpha_i = \beta_i \quad \forall i$$

or else if

$$\underline{\alpha}(e^{\lambda}-1) + \bar{\beta}(e^{-\lambda}-1) + k > 0 \quad \forall \lambda \geq 0$$

and

$$\bar{\alpha}(e^{\lambda}-1) + \underline{\beta}(e^{-\lambda}-1) + k > 0 \quad \forall \lambda \leq 0$$

$$\underline{\alpha} = \liminf_{i \to \pm\infty} \alpha_i \qquad \underline{\beta} = \text{similar}$$

$$\bar{\alpha} = \limsup_{i \to \pm\infty} \alpha_i \qquad \bar{\beta} = \text{similar}$$

$$k = -f'(a) > 0$$