SINGULARLY PERTURBED STATE-DEPENDENT DELAY EQUATIONS

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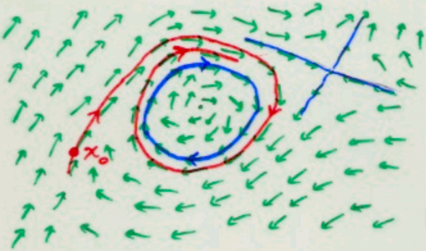
$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$$

$f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$x = x(t)$ solution



$$\begin{cases} \dot{x}(t) = f(x(t-r)) \\ x(t) = \varphi(t) \quad t \in [-r, 0] \end{cases}$$

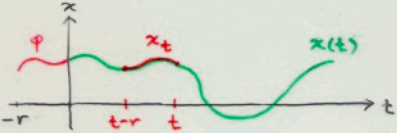
$f: \mathbb{R}^n \to \mathbb{R}^n$

$r \geq 0$ time delay (given)

$\varphi \in C([-r,0], \mathbb{R}^n)$ given initial condition

$x(t)$ unique solution for $t \geq 0$

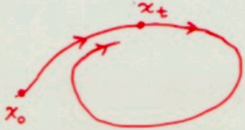
(solution may not exist for $t < 0$)



$$x_t(s) = x(t+s) \qquad s \in [-r, 0]$$

$$x_t \in C([-r,0], \mathbb{R}^n) = X \quad \text{phase space}$$

Dynamical System (Semiflow) in X



$$\begin{cases} \dot{x}(t) = f(x(t-r)) \\ x(t) = \varphi(t) \quad t \in [-r, 0] \end{cases}$$

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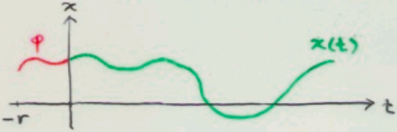


More generally $\begin{cases} \dot{x}(t) = f(x(t), x(t-r_1), \dots, x(t-r_m)) \\ x(t) = \varphi(t) \quad t \in [-r, 0] \end{cases}$

$0 \leq r_i \leq r$

Distributed delays e.g. $\int_{-r}^{0} g(x(t+s))\, ds$

Variable delays e.g. $r(t)$, $r(x(t))$, etc.

Picard (1908 - Int. Congress)

Volterra (Elasticity)

$\gtrsim$ 1940's – Control theory

$\gtrsim$ 1960's – Other applications, particularly life sciences (biology, physiology, demography)

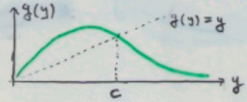
- Beginning of qualitative/geometric theory using ideas and tools from dynamical systems

- Nonlinear problems

Due to the infinite-dimensionality of the phase space $\mathbf{X}$, adapting the tools from finite-dimensional dynamical systems (ODE's) often presents significant challenges.

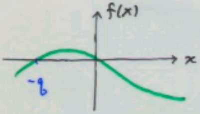
Mackey – Glass Equation

$$\dot{y}(t) = -y(t) + g(y(t-r))$$



Let $y = c + x$ to get

$$\dot{x}(t) = -x(t) + f(x(t-r))$$



Note the range of negative feedback

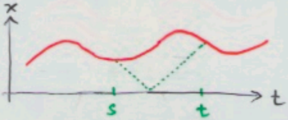
$$x f(x) < 0 \qquad \begin{matrix} x > -q \\ x \neq 0 \end{matrix}$$

Negative feedback tends to induce oscillations in the solutions.

Control of Satellite

$x(t)$ = distance from earth (scalar)

speed of signal = 1



$$t - s = x(t) + x(s)$$

$$\left.\begin{aligned} \ddot{x}(t) &= F(x(s)) \\ x(s) + s &= x(t) - t \end{aligned}\right\}$$

State-dependent, implicitly determined delay $r = t - s$

Linear Theory

$$\dot{x}(t) = \sum_{i=1}^{m} A_i x(t - r_i)$$

$$\det\left(\lambda I - \sum_{i=1}^{m} A_i e^{-\lambda r_i}\right) = 0 \iff \exists \; x(t) = e^{\lambda t} v$$

Finitely many roots with $\operatorname{Re} \lambda > 0$

Nonlinear Perturbations

$$\dot{x}(t) = \sum_{i=1}^{m} A_i x(t - r_i) + g(x(t - r_1), \dots, x(t - r_m))$$

$$g(0) = 0 \quad g'(0) = 0$$

Invariant manifolds (local) if $r_i =$ constant

Stable, center, unstable manifolds

Global Results via Topological Methods

Single delay + Negative feedback

Periodic oscillations / Morse decompositions

Monotone feedback $\Rightarrow$ Poincaré-Bendixson

Global Results via Analytical Methods

Singular perturbations (two time scales)

$$\varepsilon \dot{x}(t) = -x(t) + f(x(t-r))$$

constant delay $\quad r \equiv 1$

state-dependent delay $\quad r = r(x(t))$

$$\varepsilon \dot{x}(t) = g(x(t-r_1), x(t-r_2), \ldots, x(t-r_N))$$

multiple delays $\quad r_i \geq 0$

(constant vs state dependent)

singular perturbation

<u>very few</u> global results for case of <u>multiple</u> <u>delays</u>

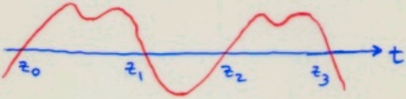
$$\varepsilon \dot{x}(t) = -x(t) + f(x(t-1))$$

① existence of periodic solutions
TOOL: degree theory in cones

$x f(x) < 0 \quad x \neq 0$

$f'(0) = -k < -1 \qquad 0 < \varepsilon < \varepsilon_* = \varepsilon_*(k)$

$f(x)$ bounded for $x < 0$



period $= z_{i+2} - z_i$

$z_{i+1} - z_i > 1$

② Similar approach yields slowly oscillating periodic solutions for

$$\dot{x}(t) = -x(t) + f(x(t-r_1), \dots, x(t-r_N)),$$

$r_i = r_i(x(t))$, provided $r_1(0) = r_2(0) = \dots = r_N(0) > 0$

$$\varepsilon \dot{x}(t) = -x(t) + f(x(t-1))$$

① existence of periodic solutions

<u>TOOL</u>: degree theory in cones

$x f(x) < 0 \quad x \neq 0$

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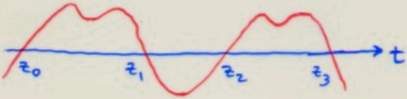


period $= z_{i+2} - z_i$

$z_{i+1} - z_i > 1$

③ Poincaré-Bendixson / Morse-Smale results

<u>TOOL</u>: oscillation (lap) number

$f'(x) < 0 \quad \forall x \in \mathbb{R}$ or

$f'(x) > 0 \quad \forall x \in \mathbb{R}$

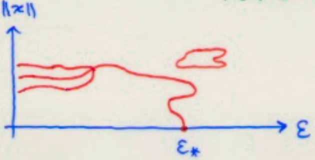
	single delay	multiple delays
constant delay(s)	very much known	less hopeful
state-dependent delay(s)	much is known	hopeful

$$\varepsilon \dot{x}(t) = -x(t) + f(x(t-1))$$

$x f(x) < 0 \quad x \neq 0$

$f(x)$ bounded $\quad x \leq 0$

$f'(0) < -1$



global branch of slowly oscillating periodic solutions

Thm $\quad \|x\|_{L^\infty} \not\to 0$ as $\varepsilon \to 0$

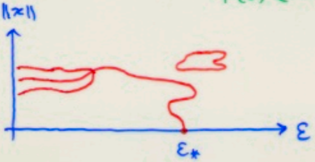
Thm $\quad$ period of $x(\cdot) = 2 + O(\varepsilon)$

$$\varepsilon \dot{x}(t) = -x(t) + f(x(t-1))$$

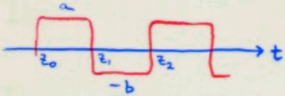
$x f(x) < 0 \quad x \neq 0$

$f(x)$ bounded $\quad x \leq 0$

$f'(0) < -1$



global branch of slowly oscillating periodic solutions



$z_{i+1} - z_i = 1 + O(\varepsilon)$

$f : [-B, A] \circlearrowleft$

$-b, a \in [-B, A]$

$f(a) = -b \quad f(-b) = a$

$f^n(x) \xrightarrow[n\to\infty]{} \{a, -b\} \quad \forall x \neq 0$

May not be unique for $\varepsilon \ll 1$.

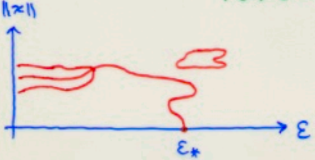
May not be stable for $\varepsilon \ll 1$.

$$\varepsilon \dot{x}(t) = -x(t) + f(x(t-1))$$

$x f(x) < 0 \quad x \neq 0$

$f(x)$ bounded $\quad x \leq 0$

$f'(0) < -1$



global branch of slowly oscillating periodic solutions

$$\varepsilon \dot{x}(t) = -x(t) + f(x(t-r))$$

global branch as above $\Leftarrow \begin{cases} f = \text{as above} \\ r(x) \geq 0 \quad r(0) = 1 \end{cases}$

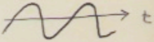
<u>Thm</u> $\|x\|_{L^\infty} \not\to 0$ as $\varepsilon \to 0$

provided $r'(0) \neq 0$ or $r''(0) > 0$.

$\|x\| \not\to 0$ not known if $r'(0) = 0$ and $r''(0) < 0$.

Q Take a sequence $\varepsilon^n \to 0$ and a sequence $x^n(t)$ of SOPS's. What is the limiting shape of the graphs of these solutions?

THM If $f'(x) < 0 \ \forall x$ then each $x^n(t)$ has exactly one maximum point and one minimum point per period.



THM If either $r'(0) \neq 0$ or else if $r'(0) = 0$ and $r''(0) > 0$ then

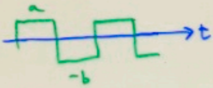
$$\|x^n\| \not\to 0$$

$\underline{Q}$ Take a sequence $\varepsilon^n \to 0$ and a sequence $x^n(t)$ of SOPS's. What is the limiting shape of the graphs of these solutions?

$$\Gamma^n = \{ (t, \xi) \in \mathbb{R}^2 \mid \xi = x^n(t) \text{ for some } t \}$$
$$= \text{graph of the solution}$$

$$\Gamma^n \longrightarrow \Omega \subseteq \mathbb{R}^2 \quad \text{as } n \to \infty$$

constant delay $r \equiv 1$



period 2 square wave

$f(a) = -b$

$f(-b) = a$

Q Take a sequence $\varepsilon^n \to 0$ and a sequence $x^n(t)$ of SOPS's. What is the limiting shape of the graphs of these solutions?

$$\Gamma^n = \{ (t,\xi) \in \mathbb{R}^2 \mid \xi = x^n(t) \text{ for some } t \}$$

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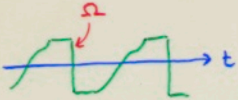
state-dependent delay $r = r(x(t))$



THM The ascending and descending parts of Ω are described by solutions of a max-plus eigenproblem.

Q Take a sequence $\varepsilon^n \to 0$ and a sequence $x^n(t)$ of SOPS's. What is the <u>limiting shape</u> of the graphs of these solutions?

$$\Gamma^n = \{ (t, \xi) \in \mathbb{R}^2 \mid \xi = x^n(t) \text{ for some } t \}$$

$$= \text{graph of the solution}$$

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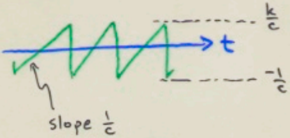
affine case $f(x) = -kx \quad r(x) = 1 + cx$

$\Rightarrow$ sawtooth

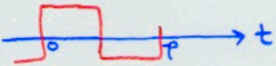


period $k+1$

Stability and uniqueness governed by

Layer Equations

$y(\tau) = x(-\varepsilon\tau)$
$z(\tau) = x(1+\varepsilon\varrho-\varepsilon\tau)$



$\Downarrow$

$p = 2(1+\varepsilon\varrho) = \text{period}$

$\varrho = \varrho(\varepsilon)$

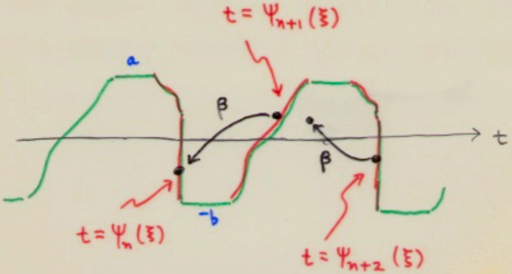
$$y'(\tau) = y(\tau) - f(z(\tau-\varrho))$$

$$z'(\tau) = z(\tau) - f(y(\tau-\varrho))$$

Boundary conditions in the limit $\varepsilon \to 0$

$$(y(-\infty), z(-\infty)) = (a, -b)$$

$$(y(\infty), z(\infty)) = (-b, a)$$



$$\Psi_{n+1}(\xi) = \max_{-b \le s \le \xi} \left(r(s) + \Psi_n(f^{-1}(s)) \right)$$

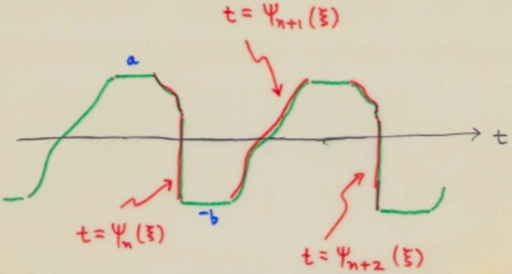
$$\Psi_{n+2}(\xi) = \max_{\xi \le s \le a} \left(r(s) + \Psi_{n+1}(f^{-1}(s)) \right)$$

$$\Psi_{n+2}(\xi) = \Psi_n(\xi) + p \qquad p = \text{limiting period}$$

Backdating map $(t, \xi) \xrightarrow[\beta]{} (t - r(\xi),\ f^{-1}(\xi))$

$$\Psi_{n+1}(\xi) - r(\xi) = \Psi_n(f^{-1}(\xi))$$

$$\Psi_{n+2}(\xi) - r(\xi) > \Psi_{n+1}(f^{-1}(\xi))$$



$$\Psi_{n+1}(\xi) = \max_{-b \le s \le \xi} \left(r(s) + \Psi_n(f^{-1}(s)) \right)$$

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$$\Psi_{n+2}(\xi) = \Psi_n(\xi) + p \qquad p = \text{limiting period}$$

$$\Psi_n(\xi) + p = \max_{\xi \le s \le a} \left(q(\xi, s) + \Psi_n(f^{-2}(s)) \right)$$

$$q(\xi, s) = \left(\max_{\xi \le \sigma \le s} r(\sigma) \right) + r(f^{-1}(s))$$

Special case $r'(x) \geqslant 0$

$$(*) \quad \psi(\xi) + p = \max_{\xi \le s \le a} \left(h(s) + \psi(f^{-2}(s)) \right)$$

$$h(s) = r(s) + r(f^{-1}(s))$$

<u>THM</u> $\quad p = \max_{0 \le s \le a} h(s)$

<u>BASIS THEOREM</u> Assume the set

$$Z = \{ s \in [0,a] \mid h(s) = p \text{ and } h(\sigma) < p \text{ in } (s, f^2(s)] \}$$

is finite, and $0 \notin Z$. Then there is a finite basis $\{\psi^i(\xi)\}_{i=1}^{d}$ for solutions of $(*)$

$$\psi(\xi) = (\psi^1(\xi) + c^1) \vee (\psi^2(\xi) + c^2) \vee \cdots \vee (\psi^d(\xi) + c^d)$$

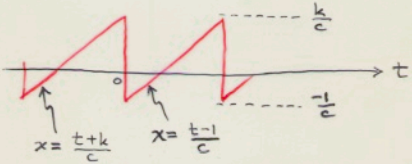
$d = \text{card}(Z) \qquad c^i \in [-\infty, \infty)$

$$\varepsilon \dot{x}(t) = -x(t) - k\,x(t-r)$$

$$r = r(x(t)) = 1 + c\,x(t)$$

$$c > 0 \qquad k > 1$$

<u>Thm</u> The limiting slowly oscillating periodic solution has the following shape as $\varepsilon \to 0$:



<u>Thm</u> $p = k + 1 + \alpha\varepsilon|\log\varepsilon| + O(\varepsilon)$ as $\varepsilon \to 0$ for the period of this solution

<u>Thm</u> $\mu = O(\varepsilon)$ as $\varepsilon \to 0$ for the nontrivial characteristic (Floquet) multipliers (superstability)

<u>Thm</u> Solution is unique for each $\varepsilon \ll 1$.

$$\varepsilon \dot{x}(t) = -x(t) - k\,x(t-r)$$

$$r = r(x(t)) = 1 + c\,x(t)$$

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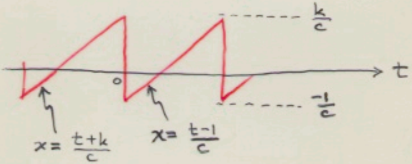

Layer equation

$$y(\tau) = x(\varepsilon\tau)$$

$$y'(\tau) = -y(\tau) - k\,\underbrace{x(\varepsilon\tau - 1 - c\,y(\tau))}_{\approx\ \frac{k-1-c\,y(\tau)}{c}}$$

$$y'(\tau) = (k-1)\left(y(\tau) - \frac{k}{c}\right)$$

is an ODE!

$$\varepsilon \dot{x}(t) = -x(t) - k\,x(t-r)$$

$$r = r(x(t)) = 1 + c\,x(t)$$

$$c > 0 \qquad k > 1$$

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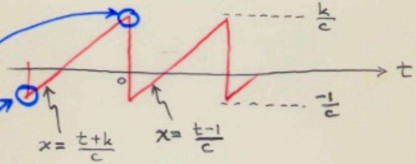


fast-slow system near hyperbolic point on slow manifold

$$\dot{\xi} = a(\xi, \eta)$$
$$\varepsilon^2 \dot{\eta} = b(\xi, \eta)$$

fast-slow system near non-hyperbolic point on slow manifold (like van der Pol)

Asymptotics of the slowly oscillating periodic solution

$-\varepsilon T_1 \le t \le \varepsilon T_2$

$x(t) = \varphi\left(\frac{t}{\varepsilon}\right)$ $\qquad \varphi(\tau) \approx \frac{k}{c}\left(1 - e^{(k-1)\tau}\right)$

$\varphi'(\tau) \le -K < 0$ valid up to $\varphi(T_2) = \frac{-1}{c} + O(\varepsilon)$

$\varepsilon T_3 \le t \le k+1-\varepsilon T_4$

$x(t) = \frac{t-1}{c} + \varepsilon\sigma(t) + \varepsilon e^{-\eta(\varepsilon^{-2}(t-\varepsilon T_3))}$

$\sigma(t), \dot{\sigma}(t) = O(1)$ $\qquad \eta(0) = O(1)$

$0 < K_1 \le \eta'(\tau) \le K_2$

$k+1-\varepsilon T_4 \le t \le k+1-\varepsilon T_5$

$x(t) = \dfrac{k + \varepsilon(\tau - T_3) - \varepsilon^2\alpha(\tau)}{c}$ $\qquad t = k+1+\varepsilon\tau$

$k+1-\varepsilon T_5 \le t \le k+1+\dfrac{\varepsilon|\log\varepsilon|}{k-1} + \varepsilon T_6$

$x(t) = \frac{k}{c} - \varepsilon\,\zeta(\tau)\,e^{(k-1)\tau}$ $\qquad t = k+1+\varepsilon\tau$

In the interval $\varepsilon T_3 \le t \le k+1-\varepsilon T_4$ the delay time $t-1-cx(t)$ belongs to the interval $[-\varepsilon T_1, \varepsilon T_2]$

$$\varepsilon \dot{x}(t) = -x(t) - kx(t-1-cx(t))$$

$$= -x(t) - k\varphi\left(\frac{t-1-cx(t)}{\varepsilon}\right)$$

Write $x(t) = \frac{t-1}{c} + \varepsilon \gamma(t)$ to give

$$\varepsilon^2 \dot{\gamma} = F(t, \gamma, \varepsilon)$$

$$= -\left(\frac{t-1+\varepsilon}{c}\right) - \varepsilon\gamma - k\varphi(-c\gamma)$$

In the interval $\varepsilon T_3 \le t \le k+1-\varepsilon T_4$ the delay time $t-1-cx(t)$ belongs to the interval $[-\varepsilon T_1, \varepsilon T_2]$

$$\varepsilon \dot{x}(t) = -x(t) - kx(t-1-cx(t))$$

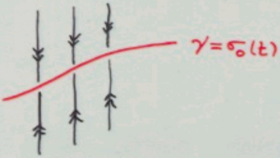
$$= -x(t) - k\varphi\left(\frac{t-1-cx(t)}{\varepsilon}\right)$$

Write $x(t) = \frac{t-1}{c} + \varepsilon\gamma(t)$ to give

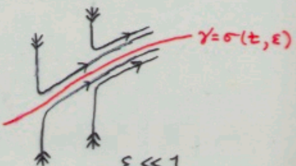
$$\varepsilon^2 \dot{\gamma} = F(t,\gamma,\varepsilon) \qquad F(t,\sigma_0(t),0) \equiv 0$$

$$\begin{cases} \gamma' = F(t,\gamma,\varepsilon) \\ t' = \varepsilon^2 \end{cases}$$

At $\varepsilon = 0$ $\exists$ normally hyperbolic invariant manifold which perturbs smoothly for $\varepsilon \neq 0$.



$\varepsilon = 0$

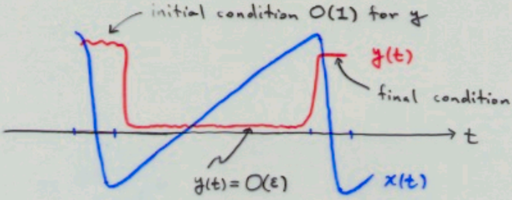


$\varepsilon \ll 1$

Linearization

$$\varepsilon \dot{y}(t) = a(t)\, y(t) - k\, y(t-1-c\,x(t))$$

$$a(t) = -1 + ck\,\dot{x}(t-1-c\,x(t))$$



Monodromy operator for y-equation

$$M : C[-q, 0] \longrightarrow C[-q, 0]$$

ψ = initial condition for y

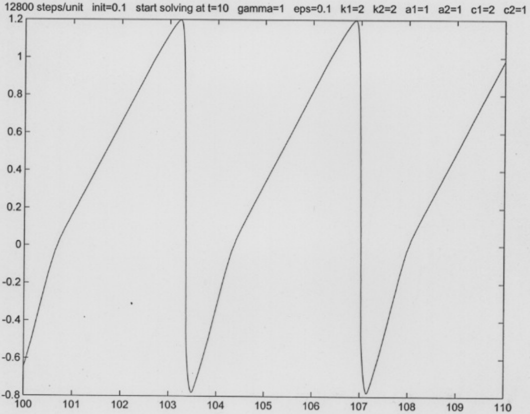
$M\psi$ = final condition (i.e. period map)

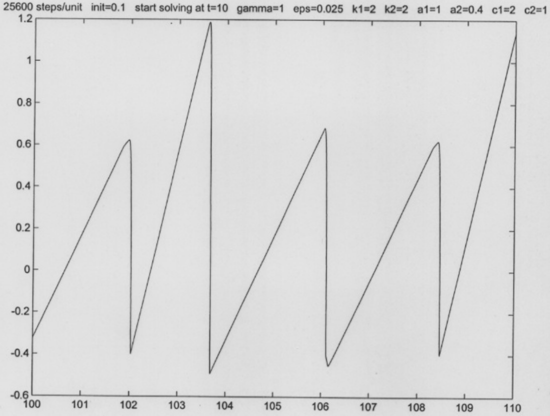
THM $M = M_1 + M_\varepsilon$ $\quad$ M_1 = rank 1

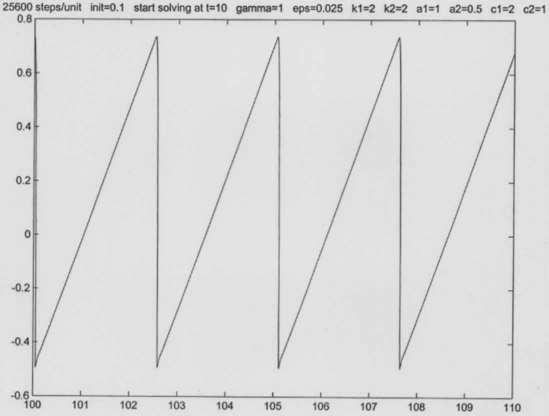
$$\|M_\varepsilon\| = O(\varepsilon)$$

Thus nontrivial characteristic multipliers $\mu = O(\varepsilon)$.

Superstability.


12800 steps/unit init=0.1 start solving at t=10 gamma=1 eps=0.1 k1=2 k2=2 a1=1 a2=1 c1=2 c2=1
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
100
101
102
103
104
105
106
107
108
109
110


25600 steps/unit init=0.1 start solving at t=10 gamma=1 eps=0.025 k1=2 k2=2 a1=1 a2=0.4 c1=2 c2=1
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
100
101
102
103
104
105
106
107
108
109
110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0.6
0.4
0.2
0
-0.2
-0.4
-0.6
100
101
102
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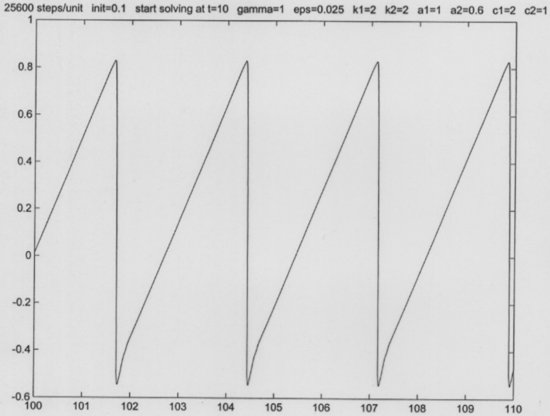
105
106
107
108
109
110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1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
100
101
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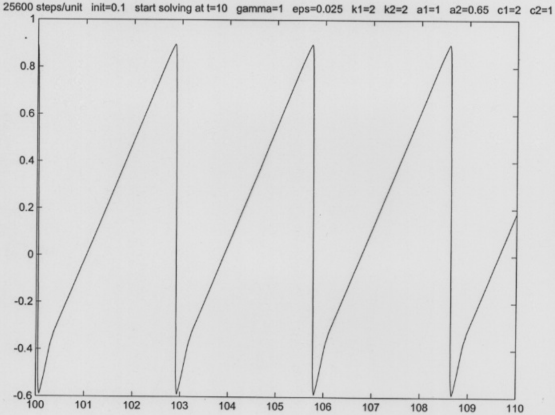
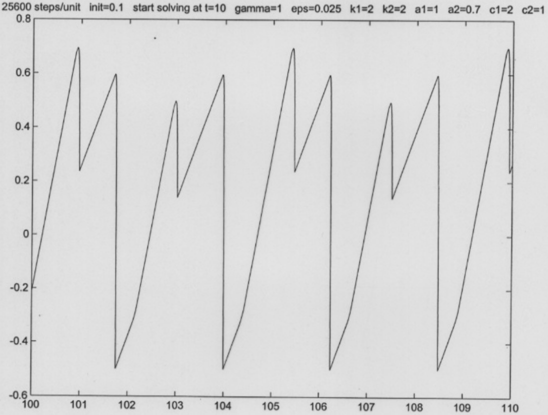
106
107
108
109
110


25600 steps/unit init=0.1 start solving at t=10 gamma=1 eps=0.025 k1=2 k2=2 a1=1 a2=0.65 c1=2 c2=1
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
100
101
102
103
104
105
106
107
108
109
110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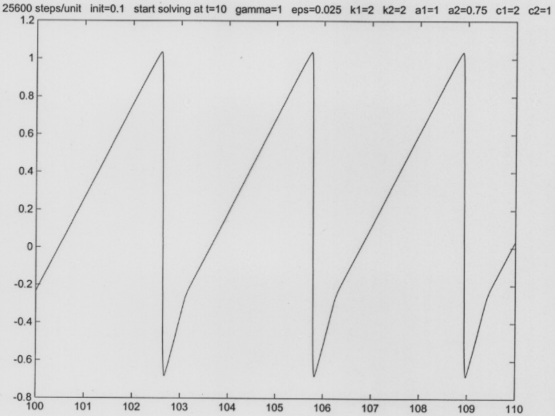
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
100
101
102
103
104
105
106
107
108
109
110


25600 steps/unit init=0.1 start solving at t=10 gamma=1 eps=0.025 k1=2 k2=2 a1=1 a2=0.75 c1=2 c2=1
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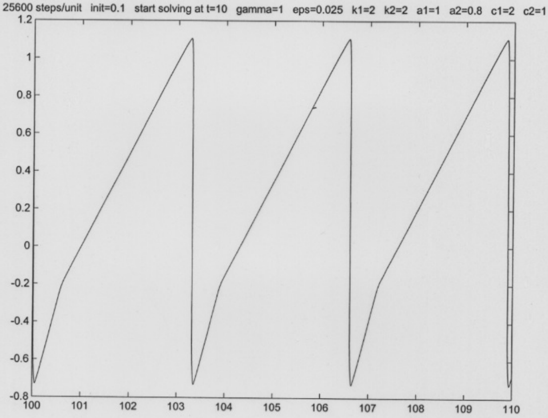
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
100
101
102
103
104
105
106
107
108
109
110



25600 steps/unit init=0.1 start solving at t=10 gamma=1 eps=0.025 k1=2 k2=2 a1=1 a2=0.8 c1=2 c2=1
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
100
101
102
103
104
105
106
107
108
109
110