Continuation of periodic orbits in conservative and Hamiltonian systems.

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Abstract

We introduce and justify a computational scheme for the continuation of periodic orbits in systems with one or more first integrals, and in particular in Hamiltonian systems having several independent symmetries. Our method is based on a generalization of the concept of a normal periodic orbit as introduced by Sepulchre and MacKay [21]. We illustrate the continuation method on some integrable Hamiltonian systems with two degrees of freedom and briefly discuss some further applications.

1 Introduction

The topic of this paper is the continuation, both theoretically and computationally, of periodic orbits in conservative systems (i.e. systems having a number of first integrals), and in particular in Hamiltonian systems having some independent constants of motion. It is well known that with respect to periodic orbits and their continuation conservative and Hamiltonian systems behave quite differently from general dissipative systems — we briefly explain how.

In dissipative systems periodic orbits are generically isolated, and therefore an external parameter is required in order to be able to continue such periodic orbits. Computationally the problem of finding a periodic orbit is formulated as a boundary value problem with the period as an additional parameter. In order to avoid phase shifts along the same orbit and ensure uniqueness one has to introduce an appropriate phase condition. So in the dissipative case the continuation problem presents itself as a boundary value problem for the initial point, subject to a phase condition, and depending on two parameters, the period and the external parameter. We refer to section 2 for more details.

This scheme no longer works for Hamiltonian systems or, more generally, for systems having a first integral. (Also time-reversible systems form an exceptional class, but we will discuss those systems in a later paper). In conservative systems periodic orbits typically belong to one-parameter families, parametrized by the value of the first integral (the energy in the Hamiltonian case). This "internal parameter" is not explicitly available, at least not directly, and this causes the general continuation scheme to fail. Additional complications arise for Hamiltonian systems having several independent constants of motion (symmetries): here periodic orbits belong to families having the dimension of the number of independent integrals, including the energy, and further "phase conditions" are required in order to uniquely identify members of such family.

In the literature one can find some basic continuation results for periodic orbits of Hamiltonian systems, such as for example the "cylinder theorem"of [17]. The aim of this paper is to present an approach which not only allows to prove a number of theoretical results but which can also be implemented directly for the numerical calculation of branches of periodic orbits. We will concentrate on Hamiltonian systems having several symmetries (i.e. first integrals). Our starting point will be a generalization of some continuation results of Sepulchre and MacKay; in their paper [21] these authors discuss the continuation of periodic orbits in systems having a first integral. They introduce the concept of a *normal* periodic orbit and show that such normal periodic orbits belong to one-parameter families of normal

periodic orbits. The key idea of their approach is to embed the conservative equation in a one-parameter family of dissipative systems by adding a small gradient perturbation term to the vector field in such a way that a periodic orbit can only exist when the perturbation is zero. Under appropriate conditions one can then invoke the implicit function theorem to obtain a continuation result for periodic orbits of the extended system, which by the basic property of the perturbation means that in fact one obtains a branch of periodic orbits for the unperturbed conservative system. This idea of adding a dissipative term which is later forced to be zero is not new; it is for example classically used to prove the Lyapunov Center Theorem as a special case of a vertical Hopf bifurcation (see e.g. [23]). Numerically the idea has been used in, for example, the thesis of Zufiría [28], the paper [1] by Aronson *et al*, and in several other papers.

In this paper we extend this idea (in a rather obvious way) to the case when there are k independent first integrals, with $k \geq 1$; a continuation result similar to the one in [21] then leads to a k-parameter family of periodic orbits of the given conservative system. We also give conditions under which this family can be parametrized by the values of the first integrals. For numerical computations one prefers to have one-parameter families, so for $k \geq 2$ one has to impose additional conditions (here called "phase conditions") in order to make the continuation result suitable for numerical implementation; we show how this can be done when the system under consideration is Hamiltonian. We also discuss in detail the application of our results to some simple (integrable) Hamiltonian system with two degrees of freedom and two constants of motion; we explore in particular the geometrical meaning of the phase conditions. Other applications will be reported elsewhere.

In section 2 we start with some mathematical preliminaries on continuation and on how these can be used for the continuation of periodic orbits of autonomous systems. In section 3 we generalize the results of Sepulchre and MacKay to conservative systems with several first integrals, in section 5 we specialize to Hamiltonian systems and show how appropriate phase conditions make the continuation scheme directly suitable for numerical implementation. In section 6 we consider two particular cases, namely the continuation of relative equilibria and the case of integrable Hamiltonian systems with two degrees of freedom, where we explain our continuation results in terms of action-angle variables. In sections 7 and 8 we give two detailed applications, and we conclude in section 9 with a survey of other applications and a discussion of possible extensions of our approach.

2 Some preliminaries

The basic tool to obtain (local) continuation results is of course the implicit function theorem. Stated very briefly, if $G_0 : \mathbb{R}^m \to \mathbb{R}^n$ (with $m > n$) is a smooth mapping, and if $x_0 \in \mathbb{R}^m$ is a solution of

$$G_0(x) = 0 \tag{2.1}$$

such that $DG_0(x_0) \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$ is surjective (i.e. G_0 is a *submersion* at x_0), then near x_0 the solution set of (2.1) is a smooth $(m-n)$-dimensional submanifold of $\mathbb{R}^m$. More precisely, if U and V are complementary subspaces of $\mathbb{R}^m$ such that $\dim(V) = n$ and $V \cap \mathrm{Ker}\,(DG_0(x_0)) = \{0\}$, then the solution set of (2.1) has near x_0 the form

$$\{x^*(u) = x_0 + u + v^*(u) \mid u \in U\},$$

where $v^* : U \to V$ is a smooth mapping with $v^*(0) = 0$. Moreover this result is persistent under small perturbations: if $G : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n$, $(x, \varepsilon) \mapsto G(x, \varepsilon) = G_\varepsilon(x)$ is a smooth mapping, and if $x_0 \in \mathbb{R}^m$ is such that $G_0(x_0) = 0$ and G_0 is a submersion at x_0, then for each sufficiently small ε the zero set $G_\varepsilon^{-1}(\{0\})$ is locally near x_0 a smooth $(m-n)$-dimensional submanifold.

Continuation packages such as AUTO [7] have been developed in order to implement this result numerically in the case $m = n + 1$, i.e. when the solution set of (2.1) consists of a single one-dimensional curve. In such case the so-called *pseudo-arclength method* [12] allows the continuation of the solution curve irrespective of the direction of this curve; for example, one can continue the curve around folds and the method also has no problems with "vertical" solution branches—something which will be important in our further story. The same packages can also be used when $m > n + 1$ on condition that we restrict x to appropriate subsets of $\mathbb{R}^m$, which means that we have to add further equations to (2.1). Such additional equations can be chosen globally or, more appropriately for many problems involving dynamical systems, they may be adapted from step to step in the calculation. For example, still assuming that $G_0(x_0) = 0$ and that $DG_0(x_0)$ is surjective, let $A = A_{x_0} \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^{m-n-1})$ be such that

$$A\left(\operatorname{Ker}\left(DG_0(x_0)\right)\right) = \mathbb{R}^{m-n-1}. \tag{2.2}$$

Define $G : \mathbb{R}^m \to \mathbb{R}^{m-1} = \mathbb{R}^n \times \mathbb{R}^{m-n-1}$ by

$$G(x) := (G_0(x), A(x - x_0)); \tag{2.3}$$

then $G(x_0) = 0$ and $DG(x_0) \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^{m-1})$ is surjective. Applying the foregoing continuation result to the equation

$$G(x) = 0 \tag{2.4}$$

gives us locally a single one-dimensional branch of solutions of (2.1) satisfying the additional condition $A(x - x_0) = 0$. Moreover, this branch can be computed numerically by using the continuation packages mentioned before. In such general formulation the way to choose A seems to be rather vague and arbitrary; however, in many applications (such as those in this paper) one can make a motivated choice for A, based on the particular aspects of the problem under consideration.

In order to illustrate how this continuation works for periodic orbits of autonomous ordinary differential equations let us consider a multi-parameter system

$$\dot{u} = g(u, \alpha), \tag{2.5}$$

with $u \in \mathbb{R}^n$, $\alpha \in \mathbb{R}^k$ and $g : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ smooth. In fact, we consider (2.5) as an unfolding (a perturbation) of the unperturbed equation

$$\dot{u} = g_0(u), \qquad \text{with } g_0(u) := g(u, 0). \tag{2.6}$$

Our aim is to calculate branches of periodic solutions of (2.5), for values of α close to 0, and starting from a given periodic solution of (2.6). The fact that (2.5) is autonomous leads to two additional problems:

(i) finding a periodic solution also involves finding its (minimal) period $T > 0$;

(ii) if $u(t)$ is a periodic solution then so is $u_\theta(t) := u(t+\theta)$ for each $\theta \in \mathbb{R}$; since such phase-shifted solutions have the same geometrical orbit we should avoid them in our continuation results or in our numerical calculations.

We will further explain how to deal with (ii); as for (i) we just rescale time and replace (2.5) by

$$\dot{u} = Tg(u, \alpha). \tag{2.7}$$

T-periodic solutions of (2.5) correspond to 1-periodic solutions of (2.7), and vice-versa. We denote by $\tilde{u}(t; p, T, \alpha)$ the solution of (2.7) such that $\tilde{u}(0; p, T, \alpha) = p$. Finding the 1-periodic solutions of (2.7) is then equivalent to solving the equation

$$G_0(p, T, \alpha) := \tilde{u}(1; p, T, \alpha) - p = 0. \tag{2.8}$$

This is an equation of the form (2.1), with $m = n + 1 + k$ and $x = (p, T, \alpha)$.

Next let $p_0 \in \mathbb{R}^n$ and $T_0 > 0$ be such that $x_0 := (p_0, T_0, 0)$ is a solution of (2.8), i.e. $u_0(t) := \tilde{u}(t; p_0, T_0, 0)$ is a 1-periodic solution of

$$\dot{u} = T_0 g_0(u). \tag{2.9}$$

We tacitly assume that 1 is the *minimal* period of $u_0(t)$; this implies that $\dot{u}_0(t) \neq 0$ for all $t \in \mathbb{R}$, and in particular $g_0(p_0) = T_0^{-1}\dot{u}_0(0) \neq 0$. As a first step in trying to continue this solution of (2.8) we calculate in detail the derivative $DG_0(p_0, T_0, 0)$, by considering separately the partial derivatives of G_0 in the variables p, T and α; we start with $D_pG_0(p_0, T_0, 0) = D_p\tilde{u}(1; p_0, T_0, 0) - I$.

The matrix function $V(t) := D_p\tilde{u}(t; p_0, T_0, 0) \in \mathcal{L}(\mathbb{R}^n)$ is a solution of the initial value problem

$$\dot{V}(t) = T_0 Dg_0(u_0(t))V(t), \qquad V(0) = I, \tag{2.10}$$

i.e. $V(t)$ is the fundamental matrix of the variational equation

$$\dot{v} = T_0 Dg_0(u_0(t))v. \tag{2.11}$$

It follows that $M := V(1)$ is the *monodromy* matrix of the periodic solution $u_0(t)$, and the eigenvalues of M are the Floquet multipliers; also

$$D_pG_0(p_0, T_0, 0) = M - I. \tag{2.12}$$

Differentiating the identity $\dot{u}_0(t) = T_0 g_0(u_0(t))$ shows that $\dot{u}_0(t)$ and hence also $g_0(u_0(t))$ are 1-periodic solutions of the variational equation (2.11); it follows that $g_0(u_0(t)) = V(t)g_0(p_0)$ and $g_0(p_0) = g_0(u_0(0)) = g_0(u_0(1)) = V(1)g_0(p_0) = Mg_0(p_0)$, or equivalently

$$g_0(p_0) \in \mathrm{Ker}\,(M - I). \tag{2.13}$$

This shows that 1 is always a multiplier, i.e. an eigenvalue of M. As we will see further on the geometric multiplicity m_g and (to a lesser extend also) the algebraic multiplicity m_a of this eigenvalue will play an important role; the foregoing shows that $m_a \geq m_g \geq 1$.

The function $w(t) := \dfrac{\partial \tilde{u}}{\partial T}(t; p_0, T_0, 0)$ satisfies the initial value problem

$$\dot{w}(t) = T_0 Dg_0(u_0(t))w(t) + g_0(u_0(t)), \qquad w(0) = 0;$$

the variation-of-constants formula and (2.13) then shows that

$$\frac{\partial G_0}{\partial T}(p_0,T_0,0)=\frac{\partial \tilde{u}}{\partial T}(1;p_0,T_0,0)=V(1)\int_0^1 V(s)^{-1}g_0(u_0(s))\,ds=M\int_0^1 g_0(p_0)\,ds=g_0(p_0).$$

In a similar way one finds

$$D_\alpha G_0(p_0,T_0,0)=D_\alpha\tilde{u}(1;p_0,T_0,0)=T_0M\int_0^1 V(s)^{-1}D_\alpha g(u_0(s),0)\,ds. \tag{2.14}$$

Combining these results we finally conclude that

$$DG_0(p_0,T_0,0)\cdot(p,T,\alpha)=(M-I)p+Tg_0(p_0)+D_\alpha G_0(p_0,T_0,0)\cdot\alpha, \tag{2.15}$$

with $D_\alpha G_0(p_0,T_0,0)$ given by (2.14). With this explicit expression for $DG_0(p_0,T_0,0)\in\mathcal{L}(\mathbb{R}^{n+1+k},\mathbb{R}^n)$ at hand we can now try to find conditions which ensure that G_0 is a submersion at $(p_0,T_0,0)$; however, before doing so we first solve the phase shift problem mentioned earlier in this section.

Theorem 1 *Using the notation introduced before, let $p_0\in\mathbb{R}^n$ and $T_0>0$ be such that $G_0(p_0,T_0,0)=0$ and G_0 is a submersion at $(p_0,T_0,0)$. Define $G:\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^k\to\mathbb{R}^n\times\mathbb{R}$ by*

$$G(p,T,\alpha):=(G_0(p,T,\alpha),\langle g_0(p_0),p-p_0\rangle), \tag{2.16}$$

where $\langle\cdot,\cdot\rangle$ is any scalar product on $\mathbb{R}^n$. Then $G(p_0,T_0,0)=0$, G is a submersion at the point $(p_0,T_0,0)$, and locally near $(p_0,T_0,0)$ the solution set of the equation

$$G(p,T,\alpha)=0 \tag{2.17}$$

forms a smooth k-dimensional submanifold of $\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^k$.

Proof The first statement is obvious, the second follows from the fact that $(g_0(p_0),0,0)\in\mathrm{Ker}\,(DG_0(p_0,T_0,0))$ (see (2.13) and (2.15)), and the third from our earlier statements on submersions and zero sets of mappings. ■

The solutions of equation (2.17) give us the initial points $p\in\mathbb{R}^n$ of T-periodic solutions of (2.5), subject to the *phase condition*

$$\langle g_0(p_0),p-p_0\rangle=0. \tag{2.18}$$

Geometrically this means that we are looking for fixed points of the (parametrized) Poincaré map for (2.5) on the transversal section $\Sigma:=\{p\in\mathbb{R}^n\mid\langle g_0(p_0),p-p_0)\rangle=0\}$ to the orbit $\{u_0(t)\mid t\in\mathbb{R}\}$. This way each periodic orbit corresponds to a unique solution of (2.17). We have chosen the particular phase condition (2.18) because it is relatively simple and has a direct geometrical meaning, but of course (2.18) can be replaced by other more involved conditions such as for example the integral condition used in [21]. An integral condition is also used as the default phase constraint in AUTO.

Next we turn to conditions on p_0 and T_0 which ensure that G_0 is a submersion at $(p_0,T_0,0)$. It follows from (2.13) that $+1$ is an eigenvalue of the monodromy matrix M, and $g_0(p_0)$ is an eigenvector. In general systems $+1$ will typically be a *simple* eigenvalue of M, i.e. $m_g=m_a=1$; this means that $\mathrm{Ker}\,(M-I)=\mathbb{R}g_0(p_0)$ and $\mathbb{R}^n=\mathrm{Im}\,(M-I)\oplus\mathbb{R}g_0(p_0)$. (We say that $u_0(t)$ is a *non-degenerate* periodic solution). It follows then immediately from (2.15) that $DG_0(p_0,T_0,0)$ is surjective; in fact already $D_{(p,T)}G_0(p_0,T_0,0)$ is surjective. This implies that the k-dimensional solution manifold given by theorem 1 can be parametrized by $\alpha\in\mathbb{R}^k$, giving us the following result.

Theorem 2 *Let $u_0(t)$ be a 1-periodic solution of (2.9), with $T_0 > 0$ and $g_0(p_0) \neq 0$ for $p_0 := u_0(0)$. Let $V(t)$ be the fundamental matrix solution of the variational equation (2.11), and suppose $+1$ is a simple eigenvalue of $M = V(1)$. Then there exist smooth functions $p^* : \mathbb{R}^k \to \mathbb{R}^n$ and $T^* : \mathbb{R}^k \to \mathbb{R}$, with $p^*(0) = p_0$ and $T^*(0) = T_0$, such that for all sufficiently small $\alpha \in \mathbb{R}^k$ the solution $\tilde{u}(t; p^*(\alpha), T^*(\alpha), \alpha)$ of (2.7) is 1-periodic. Moreover, up to phase shifts these are the only 1-periodic solutions of (2.7) with T near T_0 and with orbit near $\Gamma_0 := \{u_0(t) \mid t \in \mathbb{R}\}$.*

Remark 3 Geometrically the foregoing result can be explained as follows. For each sufficiently small α the vector field $g(\cdot, \alpha)$ induces a Poincaré map P_α on Σ; p_0 is a fixed point of P_0. If $u_0(t)$ is a non-degenerate periodic solution, then $+1$ is not an eigenvalue of the linearization $DP_0(p_0)$, and P_α has for each sufficiently small α a unique fixed point $p^*(\alpha)$ close to p_0. It follows in particular that for each sufficiently small α the periodic orbit $\Gamma_\alpha := \{\tilde{u}(t; p^*(\alpha), T^*(\alpha), \alpha) \mid t \in \mathbb{R}\}$ of (2.5) is *isolated*. These periodic orbits can be obtained by numerical continuation if one restricts to the case $k = 1$.

In case the multiplier $+1$ is not simple one can in general expect a bifurcation of periodic orbits at the orbit Γ_0 (such as for example a saddle-node of periodic orbits); here we will not discuss such bifurcations. There are also several particular classes of dynamical systems whose periodic orbits will in general not have $+1$ as a simple multiplier, such as equivariant systems (with a sufficiently large symmetry group), or time-reversible systems. In this paper we concentrate on the case of *conservative* systems, i.e. systems having one or more first integrals. In the next section we consider such conservative systems in general, in section 5 we then further specialize to hamiltonian systems.

3 Continuation in conservative systems

In this section we generalize the continuation results of Sepulchre and MacKay ([21]) to conservative systems having several (independent) first integrals. There are two aspects in which our approach differs from the one in [21], namely

(i) we restrict to finite-dimensional systems and do not work with equations on infinite-dimensional Banach spaces or Banach manifolds, and

(ii) we do not use the loop space approach of [21]; instead we identify periodic orbits with the zeros of the mappings G_0 or G introduced in section 2.

As we will see this simplified set-up allows us in particular to give a more transparent and less technical definition of normal periodic orbits. The reader who wishes to do so should have no difficulty translating our results to the framework of [21].

Using the notation of section 2 we fix a smooth vector field g_0 on $\mathbb{R}^n$ and a non-trivial periodic orbit $\Gamma_0 = \{u_0(t) \mid t \in \mathbb{R}\}$ of g_0, with minimal period $T_0 > 0$ and monodromy matrix M; we denote by $m_g := \dim \operatorname{Ker}(M - I)$ and $m_a := \dim \operatorname{Ker}((M - I)^n)$ the geometric, respectively algebraic multiplicity of the multiplier 1. A smooth function $F : \mathbb{R}^n \to \mathbb{R}$ is a *first integral* of g_0 if

$$DF(u) \cdot g_0(u) = 0, \qquad \forall u \in \mathbb{R}^n. \tag{3.1}$$

(Since our analysis is local near Γ_0 it is in fact sufficient to assume that (3.1) holds in a neighborhood of Γ_0). We denote by $\mathcal{F}$ the vector space of all first integrals of g_0.

Next we fix some $F \in \mathcal{F}$; using (3.1) we have then for all (t, p, T) that

$$\begin{aligned} \frac{d}{dt} F(\tilde{u}(t; p, T, 0)) &= DF(\tilde{u}(t; p, T, 0)) \cdot \dot{\tilde{u}}(t; p, T, 0) \\ &= TDF(\tilde{u}(t; p, T, 0)) \cdot g_0(\tilde{u}(t; p, T, 0)) \; = \; 0, \end{aligned}$$

and therefore

$$F(\tilde{u}(t; p, T, 0)) = F(p), \qquad \forall (t, p, T). \tag{3.2}$$

So orbits of g_0 stay on fixed level sets of F. To see what this implies for our periodic orbit Γ_0 we set

$$W := \{\nabla F(p_0) \mid F \in \mathcal{F}\} \qquad \text{and} \qquad k := \dim W. \tag{3.3}$$

(The gradient $\nabla F : \mathbb{R}^n \to \mathbb{R}^n$ of a smooth function $F : \mathbb{R}^n \to \mathbb{R}$ is defined in the usual way by the condition that $DF(u) \cdot \tilde{u} = \langle \nabla F(u), \tilde{u} \rangle$ for all $u, \tilde{u} \in \mathbb{R}^n$). We also choose $F_j \in \mathcal{F}$ $(1 \le j \le k)$ such that $\{\nabla F_j(p_0) \mid 1 \le j \le k\}$ forms a basis of W, and we set

$$f_0 = (f_{0,1}, f_{0,2}, \ldots, f_{0,k}) := (F_1(p_0), F_2(p_0), \ldots, F_k(p_0)) \in \mathbb{R}^k. \tag{3.4}$$

The following heuristic argument then shows why we should expect our periodic orbit Γ_0 to belong to a k-parameter family of periodic orbits.

Consider the section $\Sigma = \{p \in \mathbb{R}^n \mid \langle g_0(p_0), p - p_0 \rangle = 0\}$ which we introduced before, and the corresponding Poincaré map $P : \Sigma \to \Sigma$. We can foliate Σ by the level sets of the functions F_j; more precisely, for each $f = (f_1, f_2, \ldots, f_k) \in \mathbb{R}^k$ we set

$$\Sigma_f := \{p \in \Sigma \mid F_j(p) = f_j, \; 1 \le j \le k\}.$$

It follows from (3.1) that $g_0(p_0)$ is orthogonal to W, and therefore locally near p_0 and for f near f_0 the set Σ_f is a smooth submanifold of codimension $k+1$. It follows from (3.2) that the Poincaré map P leaves this foliation invariant, i.e. P decomposes as a k-parameter family of mappings $P_f : \Sigma_f \to \Sigma_f$. For $f = f_0$ the mapping P_{f_0} has the fixed point $p_0 \in \Sigma_{f_0}$, and one expects that under "generic conditions" for g_0 the linearization $DP_{f_0}(p_0)$ will not have $+1$ as an eigenvalue. When this is indeed the case then P_f has for each f near f_0 a unique fixed point $p^*(f) \in \Sigma_f$ near $p_0 = p^*(f_0)$. This gives us a k-parameter family of periodic orbits Γ_f, parametrized by the values $f = (f_1, f_2, \ldots, f_k)$ of the first integrals F_j $(1 \le j \le k)$. Denoting the period of Γ_f by $T^*(f)$ we have the identity $G_0(p^*(f), T^*(f), 0) = 0$; differentiating at $f = f_0$ gives then (see (2.15) and (2.13)):

$$(M - I)Dp^*(f_0) + g_0(p_0)DT^*(f_0) = 0 \qquad \text{and} \qquad (M - I)^2 Dp^*(f_0) = 0.$$

Using the definition of Σ and the fact that $p^*(f) \in \Sigma_f$ one can easily show that the image of $Dp^*(f_0) \in \mathcal{L}(\mathbb{R}^k; \mathbb{R}^n)$ is a k-dimensional subspace of $\mathbb{R}^n$ orthogonal to $g_0(p_0)$; the foregoing relations then prove that $\mathrm{Ker}\,((M-I)^2)$ is at least $(k+1)$-dimensional (since it contains $g_0(p_0)$ and $\mathrm{Im}\,(Dp^*(f_0))$). The conclusion is that the algebraic multiplicity m_a of the multiplier $+1$ is greater than or equal to $k+1$, and therefore Γ_0 will be degenerate (at least if $k \ge 1$). In proposition 8 we will show that the foregoing argument holds true precisely when $m_a = k+1$.

3.1 Normal periodic orbits

To make the foregoing argument rigorous we fix some $F \in \mathcal{F}$ and differentiate (3.2) with respect to the p-variable at the point $(t, p, T) = (t, p_0, T_0)$; this gives

$$DF(u_0(t)) \cdot V(t)p = DF(p_0) \cdot p, \qquad \forall t \in \mathbb{R},\ \forall p \in \mathbb{R}^n; \tag{3.5}$$

taking $t = 1$ then shows that

$$\mathrm{Im}\,(M - I) \subset \mathrm{Ker}\,(DF(p_0)), \qquad \forall F \in \mathcal{F}, \tag{3.6}$$

and hence $m_g \geq k$. The definition of a first integral also implies that

$$g_0(p_0) \in \mathrm{Ker}\,(DF(p_0)), \qquad \forall F \in \mathcal{F}. \tag{3.7}$$

Combining (3.6) and (3.7) with the definition (3.3) of W we conclude that

$$\mathrm{Im}\,(M - I) + \mathbb{R}g_0(p_0) \subset W^{\perp}. \tag{3.8}$$

From this it is easy to see that $m_a \geq k + 1$; indeed, if $m_g \geq k + 1$ then $m_a \geq m_g \geq k + 1$, and if $m_g = k$ then (3.8) implies that $\mathrm{Im}\,(M - I) = W^{\perp}$ and $g_0(p_0) \in \mathrm{Im}\,(M - I)$, such that $g_0(p_0) \in \mathrm{Im}\,(M - I) \cap \mathrm{Ker}\,(M - I)$ and therefore $m_a \geq \mathrm{codim}\,(\mathrm{Im}((M - I)^2)) \geq k + 1$. Now remember that it follows from (2.15) that $\mathrm{Im}\,(D_{(p,T)}G_0(p_0, T_0, 0)) = \mathrm{Im}\,(M - I) + \mathbb{R}g_0(p_0)$; also, the periodic orbit Γ_0 is non-degenerate if and only if $\mathrm{Im}\,(M - I) + \mathbb{R}g_0(p_0) = \mathbb{R}^n$, which is only possible if $W = \{0\}$, i.e. when there are no first integrals. This motivates then the following definition of a "normal" periodic orbit, a definition which was first introduced in [21] (for the case $k \leq 1$) and which generalizes the concept of non-degeneracy to conservative systems.

Definition 4 We say that the periodic solution $u_0(t)$ or the corresponding periodic orbit Γ_0 is *normal* if

$$\mathrm{Im}\,(M - I) + \mathbb{R}g_0(p_0) = W^{\perp}. \tag{3.9}$$

As we have already noticed this definition includes the case of a non-degenerate periodic orbit (when $k = 0$). It is not difficult to prove that the verification of (3.9) is independent of the choice of the point $p_0 \in \Gamma_0$, i.e. normality is a property of the periodic orbit Γ_0.

The next proposition characterizes normal periodic orbits in terms of the geometric and algebraic multiplicities m_g and m_a of the multiplier $+1$; it also shows that our definition of normality coincides with the one in [21] when $k \leq 1$.

Proposition 5 *The periodic orbit Γ_0 is normal if and only if either*

(i) $m_g = k$,

or

(ii) $m_g = k + 1$ *and* $g_0(p_0) \notin Im\,(M - I)$.

In particular, Γ_0 is normal if $m_a = k + 1$.

Proof The condition (3.9) will be satisfied if and only if either $\mathrm{Im}\,(M-I) = W^\perp$ or $\mathrm{Im}\,(M-I)$ has codimension 1 in $W^\perp$ and $g_0(p_0) \notin \mathrm{Im}\,(M-I)$. Because of (3.8) and $\dim W = k$ the condition $\mathrm{Im}\,(M-I) = W^\perp$ is equivalent to $m_g = \dim \mathrm{Ker}\,(M-I) = k$; similarly, $\mathrm{Im}\,(M-I)$ has codimension 1 in $W^\perp$ if and only if $m_g = \dim \mathrm{Ker}\,(M-I) = k+1$. This proves the first part of the proposition.

Next suppose that $m_a = k+1$; since $m_g \geq k$ it follows that $m_a \leq m_g + 1$ and

$$\mathbb{R}^n = \mathrm{Im}\,((M-I)^2) \oplus \mathrm{Ker}\,((M-I)^2), \qquad \text{with } \dim \mathrm{Ker}\,((M-I)^2) = m_a = k+1.$$

Moreover,

$$\mathrm{Im}\,((M-I)^2) \subset \mathrm{Im}\,(M-I) \subset W^\perp \tag{3.10}$$

and

$$g_0(p_0) \in \mathrm{Ker}\,(M-I) \cap W^\perp \subset \mathrm{Ker}\,((M-I)^2) \cap W^\perp.$$

A simple dimension argument then shows that

$$\mathrm{Ker}\,((M-I)^2) \cap W^\perp = \mathbb{R}g_0(p_0) \qquad \text{and} \qquad W^\perp = \mathrm{Im}\,((M-I)^2) \oplus \mathbb{R}g_0(p_0); \tag{3.11}$$

in combination with (3.10) this last equality gives $W^\perp = \mathrm{Im}\,(M-I) + \mathbb{R}g_0(p_0)$, i.e. Γ_0 is normal. ■

We have already observed that $m_g \geq k$ and $m_a \geq k+1$; for a typical ("generic") periodic orbit in a conservative system one expects these multiplicities to take their lowest possible values, so $m_g = k$ and $m_a = k+1$. According to proposition 5 such typical periodic orbit is normal.

3.2 Continuation of normal periodic orbits

Next we will show that normal periodic orbits indeed belong to k-parameter families of (normal) periodic orbits, as was expected from the heuristic argument given before. To prove this we will introduce a specific k-parameter unfolding g_α of the conservative vector field g_0, constructed in such a way that

(i) the mapping $G_0(p,T,\alpha)$ (defined as in (2.8)) is a submersion at $(p_0,T_0,0)$; and

(ii) the vector field g_α can only have periodic orbits near Γ_0 if $\alpha = 0$.

Using the property (i) we can then apply theorem 1 to obtain a k-parameter family of periodic orbits, which by property (ii) must be periodic orbits of the original conservative vector field g_0. The unfolding we have in mind is motivated by earlier treatments of conservative systems (see e.g. [23]) and by [21]; it takes the form

$$g_\alpha(u) = g(u,\alpha) := g_0(u) + \sum_{j=1}^{k} \alpha_j \nabla F_j(u), \qquad \forall u \in \mathbb{R}^n,\ \forall \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{R}^k. \tag{3.12}$$

Observe that this unfolding is no longer conservative for $\alpha \neq 0$; in fact, this is precisely what allows us to prove that g_α has the property (ii) mentioned above.

Lemma 6 *With $g_\alpha(u)$ given by (3.12), let $u(t)$ be a T-periodic solution of $\dot{u} = g_\alpha(u)$, with $p := u(0)$ sufficiently close to p_0. Then $\alpha = 0$, and $u(t)$ is a periodic solution of $\dot{u} = g_0(u)$.*

Proof Let $u(t)$ be a T-periodic solution of $\dot{u} = g_\alpha(u)$, for some $T > 0$ and some $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{R}^k$. Let $F(u) := \sum_{j=1}^k \alpha_j F_j(u)$; then $F \in \mathcal{F}$ and

$$\dot{u}(t) = g_0(u(t)) + \nabla F(u(t)), \qquad \forall t \in \mathbb{R}.$$

It follows that

$$\begin{aligned} 0 = F(u(T)) - F(u(0)) = \int_0^T \frac{d}{dt} F(u(t))\, dt &= \int_0^T DF(u(t)) \cdot (g_0(u(t)) + \nabla F(u(t)))\, dt \\ &= \int_0^T \|\nabla F(u(t))\|^2\, dt, \end{aligned}$$

which implies that $\nabla F(u(t)) = 0$ for all $t \in \mathbb{R}$. In particular

$$\sum_{j=1}^k \alpha_j \nabla F_j(p) = \nabla F(p) = \nabla F(u(0)) = 0.$$

Since the vectors $\nabla F_j(p_0)$ $(1 \le j \le k)$ are linearly independent the same will be true for the vectors $\nabla F_j(p)$ $(1 \le j \le k)$ if p is sufficiently close to p_0. We conclude that $\alpha_j = 0$ for $1 \le j \le k$, and the proof is complete. ■

Now we come to our main result which generalizes theorem 4 of [21].

Theorem 7 *Let Γ_0 be a normal periodic orbit of the conservative vector field g_0. Then this vector field possesses locally near Γ_0 a k-parameter family of normal periodic orbits, where k is given by (3.3). Moreover, this family persists under smooth conservative perturbations.*

Proof We will use theorem 1, which means that we have to show that the mapping $G_0(p, T, \alpha)$ given by (2.8) is a submersion at $(p_0, T_0, 0)$. Since Γ_0 is normal we have that $\mathrm{Im}\,(D_{(p,T)} G_0(p_0, T_0, 0)) = W^\perp$, and therefore it is sufficient to show that for $\alpha \in \mathbb{R}^k$ the condition $D_\alpha G_0(p_0, T_0, 0) \cdot \alpha \in W^\perp$ implies $\alpha = 0$; indeed, when this condition is satisfied then $\mathrm{Im}\,(D_\alpha G_0(p_0, T_0, 0))$ is a k-dimensional subspace of $\mathbb{R}^n$ complementary to $W^\perp$, and hence $DG_0(p_0, T_0, 0)$ is surjective.

So we fix some $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{R}^k$ and suppose that $D_\alpha G_0(p_0, T_0, 0) \cdot \alpha \in W^\perp$. Let $F(u) := \sum_{j=1}^k \alpha_j F_j(u)$; then $F \in \mathcal{F}$ and $D_\alpha g(u_0(s), 0) \cdot \alpha = \nabla F(u_0(s))$; also $\nabla F(p_0) \in W$ and therefore $DF(p_0) \cdot D_\alpha G_0(p_0, T_0, 0) \cdot \alpha = 0$. Using (2.14) this last relation takes the form

$$T_0 DF(p_0) M \int_0^1 V(s)^{-1} \nabla F(u_0(s))\, ds = 0;$$

because of (3.5) and (3.6) this further simplifies to

$$\int_0^1 DF(u_0(s)) \cdot \nabla F(u_0(s))\, ds = \int_0^1 \|\nabla F(u_0(s))\|^2\, ds = 0.$$

It follows that $\nabla F(u_0(s)) = 0$ for all $s \in \mathbb{R}$; taking $s = 0$ we find $\sum_{j=1}^k \alpha_j \nabla F_j(p_0) = \nabla F(p_0) = 0$, and hence $\alpha = 0$, since the vectors $\nabla F_j(p_0)$ $(1 \le j \le k)$ are linearly independent. We conclude that $DG_0(p_0, T_0, 0)$ is indeed surjective.

Theorem 1 then shows that locally near $(p_0, T_0, 0)$ the solution set of (2.17) forms a k-dimensional submanifold of $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^k$. For each (p, T, α) on this submanifold p is the initial point of a T-periodic solution of $\dot{u} = g_\alpha(u)$; it follows then from lemma 6 that $\alpha = 0$, and so p is the initial point of a T-periodic solution of the conservative equation $\dot{u} = g_0(u)$. Since $p \in \Sigma$ we get a k-parameter family of (geometrically different) periodic orbits of the vector field g_0. As long as (p, T) stays sufficiently close to (p_0, T_0) this periodic solution will also be normal; this can be seen as follows. With an appropriate choice of the parameter $\lambda \in \mathbb{R}^k$ we get smooth functions $p^* : \mathbb{R}^k \to \Sigma \subset \mathbb{R}^n$ and $T^* : \mathbb{R}^k \to \mathbb{R}$ with $p^*(0) = p_0$ and $T^*(0) = T_0$, and such that $G(p^*(\lambda), T^*(\lambda), 0) = 0$ for all (sufficiently small) $\lambda \in \mathbb{R}^k$. For each such λ there will be a monodromy matrix M_λ and a subspace W_λ such that

$$\mathrm{Im}\,(M_\lambda - I) + \mathbb{R} g_0(p^*(\lambda)) \subset W_\lambda^\perp. \tag{3.13}$$

By continuity the codimension of the subspace at the left hand side of (3.13) is at most equal to k (its value for $\lambda = 0$), while the dimension of W_λ is at least equal to k: indeed, the vectors $\nabla F_j(p^*(\lambda))$ $(1 \leq j \leq k)$ belong to W_λ and are linearly independent for small λ. It follows from this dimension argument that we must have equality in (3.13), i.e. $p^*(\lambda)$ generates a normal periodic orbit.

Since our arguments are based on the implicit function theorem it is clear that the result will persist for nearby conservative systems (having k first integrals $\tilde{F}_j$ close to F_j $(1 \leq j \leq k)$). We leave it to the interested reader to write out the details. ■

It is important to observe that next to the k-parameter family of periodic orbits given by theorem 7 there may be other periodic orbits near Γ_0, such as for example branches of subharmonic solutions (see section 8 for some examples); however such periodic orbits will have minimal periods which are away from the minimal period T_0 of Γ_0.

Next we turn to the question how we can parametrize the k-parameter family of periodic orbits given by theorem 7; the argument given at the beginning of this section suggests that we should try the values f_j of the first integrals F_j $(1 \leq j \leq k)$ along the periodic orbit. The next proposition shows that this is indeed possible if $m_a = k + 1$.

Proposition 8 *Let Γ_0 be a periodic orbit of the conservative vector field g_0, with minimal period $T_0 > 0$. Define W, k and $f_0 \in \mathbb{R}^k$ by (3.3) and (3.4), and assume that $m_a = k + 1$. Then the vector field g_0 has for each $f = (f_1, f_2 \ldots, f_k) \in \mathbb{R}^k$ sufficiently close to f_0 a unique periodic orbit Γ_f close to Γ_0, with period T_f close to T_0, and such that $F_j(p) = f_j$ $(1 \leq j \leq k)$ for each $p \in \Gamma_f$.*

Proof Consider the unfolding g_α of g_0 given by (3.12), and let $G_0 : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}^n$ be the associated map as given by (2.8). Then define $\widetilde{G} : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^k$ by

$$\widetilde{G}(p, T, \alpha, f) := \Big(G_0(p, T, \alpha), \langle g_0(p_0), p - p_0 \rangle, F_1(p) - f_1, \ldots, F_k(p) - f_k \Big).$$

Near $(p_0, T_0, 0, f_0)$ the equation $\widetilde{G}(p, T, \alpha, f) = 0$ will only have solutions for $\alpha = 0$ (lemma 6); each such solution $(p, T, 0, f)$ then gives us the initial point $p \in \Sigma$ of a T-periodic solution of $\dot{u} = g_0(u)$, and such that $F_j(p) = f_j$ $(1 \leq j \leq k)$. Therefore it is sufficient to show that

the equation $\widetilde{G}(p,T,\alpha,f)=0$ can be solved by the implicit function theorem for (p,T,α) as a function of f; since $\widetilde{G}(p_0,T_0,0,f_0)=0$ this means that we have to show that

$$D_{(p,T,\alpha)}\widetilde{G}(p_0,T_0,0,f_0)\cdot(p,T,\alpha)=0 \tag{3.14}$$

implies $(p,T,\alpha)=(0,0,0)$.

So suppose that (3.14) holds for some $(p,T,\alpha)\in\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^k$; it follows then from (2.15) and (3.8) that $D_\alpha G_0(p_0,T_0,0)\cdot\alpha\in W^\perp$, and the proof of theorem 7 shows that in that case we must have $\alpha=0$. Taking this into account (3.14) reduces to

$$(M-I)p+Tg_0(p_0)=0,\quad \langle g_0(p_0),p\rangle=0\quad\text{and}\quad DF_j(p_0)\cdot p=0\ \ (1\le j\le k). \tag{3.15}$$

Applying $(M-I)$ to the first of these equations gives $(M-I)^2p=0$, such that in combination with the last equation we can conclude that $p\in\mathrm{Ker}\,((M-I)^2)\cap W^\perp$. It follows from (3.11) that $p=\lambda g_0(p_0)$ for some $\lambda\in\mathbb{R}$, and bringing this in the second equation of (3.15) gives $\lambda=0$ and hence $p=0$. The first equation of (3.15) then finally gives $T=0$, and the proof is complete. ■

The examples worked out in sections 7 and 8 show that the conclusion of proposition 8 may fail when $m_a>k+1$. When $k=1$ it is also sometimes convenient to parametrize the one-dimensional branch of periodic orbits given by theorem 7 by the corresponding period T; this is possible if $m_g=1$, as the next proposition shows.

Proposition 9 *Let $p_0\in\mathbb{R}^n$ be a point on a periodic orbit Γ_0 of the conservative vector field g_0, with minimal period $T_0>0$, $\dim W=1$ and $m_g=1$. Then there exists a smooth mapping $p^*:\mathbb{R}\to\Sigma\subset\mathbb{R}^n$ with $p^*(T_0)=p_0$, such that for each T near T_0 the point $p^*(T)$ generates a T-periodic orbit Γ_T of g_0.*

Proof Since $m_g=k=1$ the periodic orbit Γ_0 is normal; in fact we have $\mathrm{Im}\,(M-I)=W^\perp$ and $\mathrm{Ker}\,(M-I)=\mathbb{R}g_0(p_0)$. Using theorem 7 (with $k=1$) it is then sufficient to show that the equations

$$G_0(p,T,\alpha)=0\qquad\text{and}\qquad\langle g_0(p_0),p-p_0\rangle=0$$

(with $\alpha\in\mathbb{R}$) can be solved for (p,α) as a function of T, i.e. we have to prove that for each $(p,\alpha)\in\mathbb{R}^n\times\mathbb{R}$ the conditions

$$D_{(p,\alpha)}G_0(p_0,T_0,0)\cdot(p,\alpha)=(M-I)p+\alpha\frac{\partial G_0}{\partial\alpha}(p_0,T_0,0)=0\quad\text{and}\quad\langle g_0(p_0),p\rangle=0 \tag{3.16}$$

imply $(p,\alpha)=(0,0)$. The same argument as in the proof of proposition 8 shows that the first equation of (3.16) implies $\alpha=0$; but then $p\in\mathrm{Ker}\,(M-I)=\mathbb{R}g_0(p_0)$, and the second equation of (3.16) implies $p=0$, as wanted. ■

Remark 10 Also when $k>1$ and $m_g=k$ we can use T as one of the parameters for parametrizing the k-parameter family of periodic orbits given by theorem 7; this follows from the fact that $m_g=k$ implies that

$$\frac{\partial G_0}{\partial T}(p_0,T_0,0)=g_0(p_0)\in\mathrm{Im}\,(M-I)=\mathrm{Im}\,(D_pG_0(p_0,T_0,0)),$$

such that the T-variable is not needed in order to make G_0 a submersion at $(p_0, T_0, 0)$. This is no longer the case when $m_g = k+1$ and $g_0(p_0) \notin \mathrm{Im}\,(M-I)$ (the other alternative when Γ_0 is normal). Indeed, denote the solutions of $G(p,T,\alpha)=0$ as $(p^*(\lambda), T^*(\lambda), 0)$ with $\lambda \in \mathbb{R}^k$ any admissible parameter; differentiating $G_0(p^*(\lambda), T^*(\lambda), 0) = 0$ at $\lambda = 0$ then gives $(M-I)Dp^*(0) + g_0(p_0)DT^*(0) = 0$, which implies $DT^*(0) = 0$ when $g_0(p_0) \notin \mathrm{Im}\,(M-I)$.

When $k = 1$ the foregoing results can be immediately implemented numerically. When $k > 1$ one has to impose further conditions in order to obtain one-dimensional branches of periodic orbits which one could attempt to calculate numerically. For example, under the conditions of proposition 8 one can look for periodic orbits along which $F_j = f_{0,j}$ for $k-1$ of the k independent first integrals F_j $(1 \le j \le k)$. Of course there are many ways in which one can impose such additional conditions, none of which however seems to be "natural". This situation changes when we restrict to Hamiltonian systems for which the additional structure provides a natural way to select a one-dimensional subbranch of periodic orbits. In section 5 we will show how this can be done; in the next section we first give a brief survey on Hamiltonian systems.

4 A brief survey on Hamiltonian systems and their symmetries

In Hamiltonian systems there is always an immediate first integral, namely the Hamiltonian itself. But in many cases there are also other first integrals, usually called "symmetries" in the Hamiltonian context, because of Noether's theorem — see theorem 11 below. In specializing the results of the foregoing section to Hamiltonian systems we will explicitly use the symplectic structure behind such systems; therefore we give here a very brief survey of some basic elements of the theory of Hamiltonian systems, emphasizing in particular the role of first integrals. We restrict to the elementary case of a symplectic vector space with the standard symplectic form; the extension to general symplectic manifolds should be straightforward.

We start with an even-dimensional phase space $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$, whose elements we denote as $u = (x, y)$ and on which we use the standard scalar product $\langle u, \tilde{u}\rangle = \sum_{j=1}^{2n} u_j \tilde{u}_j$. We denote by $J \in \mathcal{L}(\mathbb{R}^{2n})$ the standard symplectic matrix given by $J(x,y) := (y, -x)$; observe that J is anti-symmetric and such that $J^2 = -I$. We define a symplectic (i.e. anti-symmetric and non-degenerate bilinear) form $\omega : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ by

$$\omega(u, \tilde{u}) := \langle u, J\tilde{u}\rangle, \qquad \forall u, \tilde{u} \in \mathbb{R}^{2n}. \tag{4.1}$$

For each smooth function $H : \mathbb{R}^{2n} \to \mathbb{R}$ we define a corresponding Hamiltonian vector field $X_H : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by

$$DH(u) \cdot \tilde{u} = \omega(X_H(u), \tilde{u}), \qquad \forall u, \tilde{u} \in \mathbb{R}^{2n}, \tag{4.2}$$

or equivalently, by

$$X_H(u) = J\nabla H(u), \qquad \forall u \in \mathbb{R}^{2n}. \tag{4.3}$$

We call

$$\dot{u} = X_H(u) \tag{4.4}$$

the *Hamiltonian equation with Hamiltonian* H; we denote its flow by $\varphi_H(t,u) = \varphi_H^t(u)$ (with $t \in \mathbb{R}$ and $u \in \mathbb{R}^{2n}$). From $DH(u) \cdot X_H(u) = \omega(X_H(u), X_H(u)) = 0$ it is immediate that H is a first integral of (4.4). Differentiating (4.2) once more gives us in combination with the symmetry of $D^2H(u)$ the relation

$$\omega(DX_H(u) \cdot \bar{u}, \tilde{u}) = \omega(DX_H(u) \cdot \tilde{u}, \bar{u}), \qquad \forall u, \bar{u}, \tilde{u} \in \mathbb{R}^{2n}. \tag{4.5}$$

Next we consider symmetries of the Hamiltonian system (4.4); for this we need three ingredients. First there is the concept of a *symplectic diffeomorphism*, which is a diffeomorphism $\Psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that

$$\omega(D\Psi(u) \cdot \bar{u}, D\Psi(u) \cdot \tilde{u}) = \omega(\bar{u}, \tilde{u}), \qquad \forall u, \bar{u}, \tilde{u} \in \mathbb{R}^{2n}. \tag{4.6}$$

In particular, for each $t \in \mathbb{R}$ the diffeomorphism $\Psi(u) := \varphi_H^t(u)$ is symplectic: to see this just combine (4.5) with the fact that for each $u, \bar{u} \in \mathbb{R}^{2n}$ the mapping $t \mapsto D\varphi_H^t(u) \cdot \bar{u}$ is a solution of the variational equation

$$\dot{w} = DX_H(\varphi_H^t(u)) \cdot w$$

to show that

$$\frac{d}{dt}\, \omega\left(D\varphi_H^t(u) \cdot \bar{u}, D\varphi_H^t(u) \cdot \tilde{u}\right) = 0 \quad \Longrightarrow \quad \omega\left(D\varphi_H^t(u) \cdot \bar{u}, D\varphi_H^t(u) \cdot \tilde{u}\right) = \omega(\bar{u}, \tilde{u}).$$

Second, a *symmetry* of (4.4) is a symplectic diffeomorphism $\Psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that the Hamiltonian H is invariant under Ψ: $H(\Psi(u)) = H(u)$ for all $u \in \mathbb{R}^{2n}$. Differentiating this identity and using (4.2) and (4.6) we find then that $D\Psi(u)^{-1} X_H(\Psi(u)) = X_H(u)$ for all $u \in \mathbb{R}^{2n}$, which in turn implies that the flow φ_H^t of (4.4) commutes with Ψ:

$$\varphi_H^t(\Psi(u)) = \Psi\left(\varphi_H^t(u)\right), \qquad \forall t \in \mathbb{R},\ \forall u \in \mathbb{R}^{2n}.$$

Third, a smooth function $F : \mathbb{R}^{2n} \to \mathbb{R}$ is a *first integral* of (4.4) if $F(\varphi_H^t(u)) = F(u)$ for all $(t,u) \in \mathbb{R} \times \mathbb{R}^{2n}$, which means that φ_H^t is for each $t \in \mathbb{R}$ a symmetry of the Hamiltonian vector field X_F, and which is also equivalent to the condition

$$\{F, H\}(u) := DF(u) \cdot X_H(u) = \omega\left(X_F(u), X_H(u)\right) = 0, \qquad \forall u \in \mathbb{R}^{2n}.$$

The smooth function $\{F, H\} : \mathbb{R}^{2n} \to \mathbb{R}$, defined as indicated, is called the *Poisson bracket* of F and H; it is obviously anti-symmetric: $\{F, H\} = -\{H, F\}$, and hence we have the following result.

Theorem 11 *Let $F : \mathbb{R}^{2n} \to \mathbb{R}$ and $H : \mathbb{R}^{2n} \to \mathbb{R}$ be two smooth functions; then the following properties are equivalent:*

(i) $\{F, H\} \equiv 0$*;*
(ii) *F is a first integral for the Hamiltonian vector field X_H;*
(iii) *H is a first integral for the Hamiltonian vector field X_F;*
(iv) *φ_F^s is for each $s \in \mathbb{R}$ a symmetry of X_H;*
(v) *φ_H^t is for each $t \in \mathbb{R}$ a symmetry of X_F.*

Moreover, each of these properties implies that the flows of X_H and X_F commute:

$$\varphi_H^t\left(\varphi_F^s(u)\right) = \varphi_F^s\left(\varphi_H^t(u)\right), \qquad \forall t, s \in \mathbb{R},\ \forall u \in \mathbb{R}^{2n}. \tag{4.7}$$

Under fairly mild conditions (namely that either X_F or X_H has at least one bounded orbit) also the converse is true: (4.7) implies each of the properties (i)–(v).

Sometimes it is convenient to write Hamiltonian systems in complex coordinates; this can be done as follows. We identify the phase space $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ with $\mathbb{C}^n$, via the mapping

$$(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \ \longmapsto\ z = \frac{1}{\sqrt{2}}(y + ix) \in \mathbb{C}^n;$$

observe that this identification means that we consider $\mathbb{C}^n$ as a *real* vector space. For each $z, \tilde{z} \in \mathbb{C}^n$ we set $z \cdot \tilde{z} := \sum_{j=1}^n z_j \tilde{z}_j$. One can then directly verify that the scalar product $\langle \cdot, \cdot \rangle$ and the symplectic form $\omega(\cdot, \cdot)$ on $\mathbb{R}^{2n}$ take on $\mathbb{C}^n$ the respective forms

$$\left\langle z^{(1)}, z^{(2)} \right\rangle := z^{(1)} \cdot \bar{z}^{(2)} + \bar{z}^{(1)} \cdot z^{(2)}, \qquad \forall z^{(1)}, z^{(2)} \in \mathbb{C}^n, \tag{4.8}$$

and

$$\omega\left(z^{(1)}, z^{(2)}\right) := \left\langle z^{(1)}, i z^{(2)} \right\rangle, \qquad \forall z^{(1)}, z^{(2)} \in \mathbb{C}^n. \tag{4.9}$$

We identify a real-valued mapping $H : \mathbb{C}^n \to \mathbb{R}$ with the mapping $\tilde{H} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by $\tilde{H}(x, y) := H\left(1/\sqrt{2}\,(y + ix)\right)$; in particular, H is *smooth* if $\tilde{H}$ is smooth. If this is the case then we have for each $z = 1/\sqrt{2}\,(y + ix)$ and each $\zeta = 1/\sqrt{2}\,(\eta + i\xi)$ in $\mathbb{C}^n$ that

$$DH(z) \cdot \zeta = D_x \tilde{H}(x, y) \cdot \xi + D_y \tilde{H}(x, y) \cdot \eta = \frac{\partial H}{\partial z}(z) \cdot \zeta + \frac{\partial H}{\partial \bar{z}}(z) \cdot \bar{\zeta},$$

where $\dfrac{\partial H}{\partial z} : \mathbb{C}^n \to \mathbb{C}^n$ and $\dfrac{\partial H}{\partial \bar{z}} : \mathbb{C}^n \to \mathbb{C}^n$ are defined by

$$\frac{\partial H}{\partial z}(z) := \frac{1}{\sqrt{2}}\left(\nabla_y \tilde{H}(x, y) - i \nabla_x \tilde{H}(x, y)\right) \ \text{ and } \ \frac{\partial H}{\partial \bar{z}}(z) := \frac{1}{\sqrt{2}}\left(\nabla_y \tilde{H}(x, y) + i \nabla_x \tilde{H}(x, y)\right).$$

(So $\dfrac{\partial H}{\partial \bar{z}}(z)$ is the complex conjugate of $\dfrac{\partial H}{\partial z}(z)$). Using (4.8) and (4.9) we see that

$$DH(z) \cdot \tilde{z} = \left\langle \frac{\partial H}{\partial \bar{z}}(z), \tilde{z} \right\rangle = \omega\left(i \frac{\partial H}{\partial \bar{z}}(z), \tilde{z}\right), \qquad \forall z, \tilde{z} \in \mathbb{C}^n, \tag{4.10}$$

which shows that in this complex setting we should replace $\nabla H(u)$ and $X_H(u)$ by respectively $\dfrac{\partial H}{\partial \bar{z}}(z)$ and $i\dfrac{\partial H}{\partial \bar{z}}(z)$. In particular, the Hamilton equation corresponding to the Hamiltonian $H(z)$ takes the form

$$\dot{z} = i \frac{\partial H}{\partial \bar{z}}(z), \tag{4.11}$$

while the Poisson bracket of two smooth functions $F, G : \mathbb{C}^n \to \mathbb{R}$ is given by

$$\{F, G\}(z) := DF(z) \cdot \left(i \frac{\partial G}{\partial \bar{z}}(z)\right) = i\left(\frac{\partial F}{\partial z}(z) \cdot \frac{\partial G}{\partial \bar{z}}(z) - \frac{\partial F}{\partial \bar{z}}(z) \cdot \frac{\partial G}{\partial z}(z)\right). \tag{4.12}$$

5 Continuation in symmetric Hamiltonian systems

Now we return to our discussion of the continuation of periodic orbits, replacing the unperturbed vector field g_0 of section 3 by a Hamiltonian vector field $X_H(u)$, with $u \in \mathbb{R}^{2n}$ and $H \in C^\infty(\mathbb{R}^{2n};\mathbb{R})$. The space $\mathcal{F}$ then takes the form

$$\mathcal{F} := \left\{ F \in C^\infty(\mathbb{R}^{2n};\mathbb{R}) \mid \{F,H\} \equiv 0 \right\};$$

in particular we have $H \in \mathcal{F}$. As in section 3 we assume that $p_0 \in \mathbb{R}^{2n}$ generates a non-trivial periodic orbit of X_H, with minimal period $T_0 > 0$. Using the notations introduced before it is easily seen that

$$V(t) = D\varphi_H^{T_0 t}(p_0) \quad (\forall t \in \mathbb{R}) \qquad \text{and} \qquad M = D\varphi_H^{T_0}(p_0). \tag{5.1}$$

In order to determine the multiplicities m_g and m_a we also introduce the spaces

$$\mathcal{Z}_0 := \left\{ G \in \mathcal{F} \mid \{G,F\}(p_0) = 0,\ \forall F \in \mathcal{F} \right\} \qquad \text{and} \qquad Z_0 := \left\{ X_G(p_0) \mid G \in \mathcal{Z}_0 \right\}; \tag{5.2}$$

of course $H \in \mathcal{Z}_0$ and $X_H(p_0) \in Z_0$. As before we set $W := \left\{ \nabla F(p_0) \mid F \in \mathcal{F} \right\}$ and $k := \dim W$; clearly $Z_0 \subset JW = \left\{ J\nabla F(p_0) \mid F \in \mathcal{F} \right\} = \left\{ X_F(p_0) \mid F \in \mathcal{F} \right\}$. When choosing a collection $\{F_j \mid 1 \le j \le k\} \subset \mathcal{F}$ such that $\{\nabla F_j(p_0) \mid 1 \le j \le k\}$ forms a basis of W it seems advisable (although not strictly necessary) to include the Hamiltonian H, for example by taking $F_1 := H$.

We can then prove the following (see also K. Meyer [16]).

Proposition 12 *Under the foregoing conditions the monodromy matrix M is symplectic (i.e. $M^T J M = J$), $JW \subset Ker(M-I)$ and*

$$Im\,(M-I) + Z_0 \subset W^\perp. \tag{5.3}$$

As a consequence m_a is even, $m_g \ge k$ and $m_a \ge k + \dim Z_0$.

Proof It follows from (5.1) and the fact that $\varphi_H^{T_0}$ is symplectic that $\omega(Mu, M\tilde{u}) = \omega(u,\tilde{u})$, which proves that M is symplectic. As a consequence the eigenvalues of M come in quadruples $\{\lambda, \lambda^{-1}, \bar{\lambda}, \bar{\lambda}^{-1}\}$, with each of the eigenvalues in such quadruple having the same multiplicities; such quadruple reduces to a pair for $|\lambda| = 1$, $\lambda \neq \pm 1$, and to a singleton for $\lambda = \pm 1$. Moreover $\det M = \pm 1$, and since the non-singular matrix $V(t)$ homotopes $M = V(1)$ to the identity $I = V(0)$ we conclude that $\det M = 1$. Hence both the eigenvalues $+1$ and -1 of M must have an even algebraic multiplicity, and m_a is even.

Next let $F \in \mathcal{F}$; it follows then from theorem 11 and $\varphi_H^{T_0}(p_0) = p_0$ that

$$\varphi_H^{T_0}\left(\varphi_F^s(p_0)\right) = \varphi_F^s\left(\varphi_H^{T_0}(p_0)\right) = \varphi_F^s(p_0), \qquad \forall s \in \mathbb{R},$$

i.e. $t \mapsto \varphi_H^t(\varphi_F^s(p_0))$ forms for each $s \in \mathbb{R}$ a T_0-periodic solution of $\dot{u} = X_H(u)$. Differentiating with respect to s at $s = 0$ and using (5.1) then shows that $MX_F(p_0) = X_F(p_0)$. We conclude that

$$X_F(p_0) \in \mathrm{Ker}\left(M - I\right), \qquad \forall F \in \mathcal{F}, \tag{5.4}$$

or stated differently: $JW \subset \mathrm{Ker}\,(M-I)$. This directly implies that $m_g \ge k$.

Finally let $G \in \mathcal{Z}_0$; then (by definition of $\mathcal{Z}_0$)

$$\langle \nabla F(p_0), X_G(p_0) \rangle = DF(p_0) \cdot X_G(p_0) = \{F, G\}(p_0) = 0, \qquad \forall F \in \mathcal{F}.$$

This shows that $Z_0 \subset W^\perp$, which in combination with $\mathrm{Im}\,(M-I) \subset W^\perp$ (see (3.8)) gives us (5.3); moreover, from $\mathrm{Im}\,((M-I)^2) = (M-I)(\mathrm{Im}\,(M-I)) \subset (M-I)(W^\perp)$, $Z_0 \subset W^\perp$ and $Z_0 \subset JW \subset \mathrm{Ker}\,(M-I)$ it follows that $\mathrm{Im}\,((M-I)^2)$ has codimension at least equal to $k + \dim Z_0$, which implies $m_a \geq k + \dim Z_0$. ■

As an immediate consequence of Proposition 12 we see that Proposition 8 can not apply if Γ_0 is a normal periodic orbit of the Hamiltonian equation $\dot{u} = X_H(u)$ such that either $\dim Z_0 \geq 2$ or when k is even; in such case Γ_0 still belongs to a k-parameter family of normal periodic orbits, but this family cannot be parametrized by the values of the first integrals F_j $(1 \leq j \leq k)$. A further remark is that in certain cases Proposition 12 may not give the optimal lower bound for m_a. Suppose for example that $k > 1$, $\dim Z_0 = 1$, and that there exist $m < k$ first integrals $F_1, F_2, \ldots, F_m \in \mathcal{F}$ (including H) such that the following holds:

(i) $\{F_i, F_j\}(p_0) = 0$ for $1 \leq i, j \leq m$;

(ii) the vectors $\nabla F_i(p_0)$ $(1 \leq i \leq m)$ are linearly independent;

(iii) $2m > k+1$ if k is odd, or $2m > k+2$ if k is even.

Then the same arguments as in the proof of Proposition 12 will show that $m_a \geq 2m$, which by (iii) is strictly larger than the lower bound given by Proposition 12.

Now we come to the main result of this section; essentially this result says that along the k-parameter continuation of a normal periodic orbit of the Hamilton equation $\dot{u} = X_H(u)$ there is a natural way to select a one-parameter subfamily which is easily calculated numerically and which generates the full k-parameter family via the symmetries. More precisely, we have the following.

Theorem 13 *Assume that $\Gamma_0 = \{\varphi_H^t(p_0) \mid t \in \mathbb{R}\}$ is a normal T_0-periodic orbit of $\dot{u} = X_H(u)$; with W, k and F_j $(1 \leq j \leq k)$ as before, define $g : \mathbb{R}^{2n} \times \mathbb{R}^k \to \mathbb{R}^{2n}$ by*

$$g(u, \alpha) := X_H(u) + \sum_{j=1}^{k} \alpha_j \nabla F_j(u). \tag{5.5}$$

Denote the flow of $\dot{u} = Tg(u, \alpha)$ by $\tilde{u}(t; p, T, \alpha)$, and let

$$G_0(p, T, \alpha) := \tilde{u}(1; p, T, \alpha) - p, \qquad \forall (p, T, \alpha) \in \mathbb{R}^{2n} \times\,]0, \infty[\, \times \mathbb{R}^k.$$

Then near $(p_0, T_0, 0)$ the solution set of the equations

$$\begin{cases} G_0(p, T, \alpha) = 0, \\ \langle\, X_{F_j}(p_0), p - p_0 \,\rangle = 0, \qquad 1 \leq j \leq k, \end{cases} \tag{5.6}$$

consists of a smooth one-dimensional curve along which $\alpha \equiv 0$. This curve can be parametrized by the period T if $m_g = k$. Projecting this curve on the phase space $\mathbb{R}^{2n}$ and acting on the projection with the flows of the Hamiltonian vector fields X_{F_j} $(1 \leq j \leq k)$ generates a $(k+1)$-dimensional manifold invariant under the flow of X_H and foliated by a k-parameter family of normal periodic orbits of $\dot{u} = X_H(u)$.

Proof In the foregoing section we have shown that G_0 is a submersion at $(p_0, T_0, 0)$, and that $\alpha \equiv 0$ along any solution branch of $G_0(p, T, \alpha) = 0$. Moreover, we have that

$$\mathrm{Ker}\,(M - I) \times \{0\} \times \{0\} \subset \mathrm{Ker}\,(DG_0(p_0, T_0, 0))\,.$$

Now the phase conditions in (5.6) have the form

$$A \cdot (p - p_0, T - T_0, \alpha) = 0,$$

with $A \in \mathcal{L}(\mathbb{R}^{2n} \times \mathbb{R} \times \mathbb{R}^k; \mathbb{R}^k)$ given by

$$A \cdot (p, T, \alpha) := (\langle\, X_{F_1}(p_0), p\,\rangle, \langle\, X_{F_2}(p_0), p\,\rangle, \ldots, \langle\, X_{F_k}(p_0), p\,\rangle)\,.$$

According to the discussion in section 1 on one-dimensional solution curves of submersions and using the facts that $\dim(JW) = k$ and $JW \subset \mathrm{Ker}\,(M - I)$ it is sufficient to show that the restriction of A to $JW \times \{0\} \times \{0\}$ is injective; then the solution set of (5.6) is locally near $(p_0, T_0, 0)$ indeed a smooth one-dimensional curve, and all other statements follow easily from the foregoing discussions (in particular from Theorem 11 and Remark 10).

So let $F \in \mathcal{F}$, and suppose that $A \cdot (X_F(p_0), 0, 0) = 0$. Since $X_F(p_0)$ can be written as a linear combination of the vectors $X_{F_j}(p_0)$ $(1 \le j \le k)$ this implies that

$$\|X_F(p_0)\|^2 = \langle\, X_F(p_0), X_F(p_0)\,\rangle = 0.$$

Hence $X_F(p_0) = 0$, which proves the required injectivity of A. ■

As with all our continuation results also the foregoing one is robust under (appropriate) small perturbations. Since the equations (5.6) have a one-dimensional solution branch these equations can be used directly for numerical continuation of a given normal periodic orbit of the Hamiltonian vector field X_H. We refer to the next section for a more geometrical interpretation of Theorem 13 in the case of an integrable system. We continue this section with the discussion of a particular case and conclude with one further continuation result, this time using an external parameter.

In systems which have a so-called *scaling property* the application of Theorem 13 produces a one-parameter family of periodic orbits related by scaling. We say that the Hamiltonian vector field X_H has a scaling property if there exists a smooth mapping (the scaling matrix)

$$S :]0, \infty[\longrightarrow \mathcal{L}(\mathbb{R}^{2n}),\ c \longmapsto S_c$$

such that S_c is invertible for each $c > 0$, $S_1 = I$ and

$$\varphi_H^{ct}(S_c u) = S_c \varphi_H^t(u), \qquad \forall (t, u) \in \mathbb{R}^{2n} \times \mathbb{R},\ \forall c > 0, \tag{5.7}$$

or equivalently

$$cX_H(S_c u) = S_c X_H(u), \qquad \forall u \in \mathbb{R}^{2n},\ \forall c > 0. \tag{5.8}$$

For example the equations for the n-body problem have such property. Suppose then that X_H has some further symmetries, and that $p_1 \in \mathbb{R}^{2n}$ generates a normal periodic orbit $\Gamma_1 := \{\varphi_H^t(p_1) \mid t \in \mathbb{R}\}$ with minimal period $T_1 > 0$ and monodromy matrix M_1. Then (5.7) implies that for each $c > 0$ the point $p_c := S_c p_1$ generates a periodic orbit $\Gamma_c = S_c(\Gamma_1)$ with

minimal period $T_c = cT_1$, i.e. we have $G_0(p_c, cT_1, 0) = 0$ for all $c > 0$. Since the solution branch can be parametrized by the period we necessarily have that $m_g = k$ (otherwise differentiating $G_0(p_c, cT_1, 0) = 0$ in c at $c = 1$ leads to $T_1 = 0$). Moreover, differentiating (5.7) in u at $(t, u) = (T_1, p_1)$ shows that the monodromy matrix M_c of Γ_c is similar to M_1: $S_c^{-1} M_c S_c = M_1$. Therefore the multipliers stay fixed along the whole branch, and in particular each of the periodic orbits is normal, with $m_g = k$. Finally, different orbits along this branch can not be related by symmetry, since they have different periods. We conclude that up to phase shift, symmetry and reparametrization the solution branch given by an application of Theorem 13 must coincide with the branch $\{(p_c, cT_1, 0) \mid c > 0\}$ obtained from the scaling property. An explicit example of a Hamiltonian system with a scaling property is given by the n-body problem; in particular, all so-called figure-eight orbits of the 3-body problem with equal masses (see [3]) are related by scaling, and starting from one such orbit our continuation method will give the full branch of figure-eight orbits (see [6] for details).

We conclude this section with a continuation result which uses explicitly an external parameter and which can be applied to periodic orbits of symmetric Hamiltonian systems for which $m_g = k$ (so in particular to normal periodic orbits of Hamiltonian systems with a scaling property). We consider a one-parameter family X_{H_λ} of Hamiltonian vector fields, depending smoothly on the parameter $\lambda \in \mathbb{R}$, and we assume that $\{H_\lambda, F_{\lambda,j}\} \equiv 0$ for all $\lambda \in \mathbb{R}$ and for certain functions $F_{\lambda,j} \in C^\infty(\mathbb{R}^{2n}; \mathbb{R})$ $(2 \le j \le k)$ which also depend smoothly on λ. Suppose that $p_0 \in \mathbb{R}^{2n}$ generates a periodic orbit $\Gamma_0 := \{\varphi_{H_0}^t(p_0) \mid t \in \mathbb{R}\}$ of X_{H_0}, with minimal period $T_0 > 0$ and such that

(i) the vectors $\nabla H_0(p_0)$, $\nabla F_{0,2}(p_0)$, …, $\nabla F_{0,k}(p_0)$ are linearly independent; and

(ii) $m_g = k$.

We then set up the following scheme:

(1) define the unfolding

$$g(u, \lambda, \alpha) := X_{H_\lambda}(u) + \alpha_1 \nabla H_\lambda(u) + \sum_{j=2}^{k} \alpha_j \nabla F_{\lambda,j}(u);$$

(2) denote by $\tilde{u}(t; p, T, \lambda, \alpha)$ the flow of $\dot{u} = Tg(u, \lambda, \alpha)$;

(3) let $G_0(p, T, \lambda, \alpha) := \tilde{u}(1; p, T, \lambda, \alpha) - p$.

Then we have the following continuation result.

Theorem 14 *Under the foregoing conditions the set of equations*

$$\begin{cases} G_0(p, T, \lambda, \alpha) = 0, \\ \langle\, X_{H_0}(p_0), p - p_0 \,\rangle = 0, \\ \langle\, X_{F_{0,j}}(p_0), p - p_0 \,\rangle = 0, \quad (2 \le j \le k), \end{cases} \tag{5.9}$$

has for each (fixed) T near T_0 a unique one-dimensional solution branch near $(p, \lambda, \alpha) = (p_0, 0, 0)$; we have $\alpha \equiv 0$ along this branch, and the branch can be parametrized by λ.

Proof The condition $m_g = k$ implies that $\operatorname{Im}(M - I) = W^\perp$, and therefore the derivative $D_{(p,\alpha)} G_0(p_0, T_0, 0, 0)$ is surjective from $\mathbb{R}^{2n} \times \mathbb{R}^k$ onto $\mathbb{R}^{2n}$. The same arguments as in the proof of Theorem 13 show that the set of equations (5.9) has near $(p, T, \lambda, \alpha) = (p_0, T_0, 0, 0)$ a two-parameter family of solutions; also, $\alpha \equiv 0$ along this family which can be parametrized by (T, λ). Fixing the period T then proves the result. ∎

For an application of Theorem 14 to the 3-body problem we refer to [6] where it is shown how the method described above gives a numerical continuation of the figure-eight orbits (which appear when all masses are equal) when one of the masses is allowed to vary. An application to the spring pendulum is briefly described in [18].

6 Some particular cases

In this section we consider in some generality two particular situations which will appear in the examples given in sections 7 and 8. First we look at relative equilibria and show how our methods can be adapted to deal to some extent with such relative equilibria; then we consider integrable Hamiltonian systems with two degrees of freedom, using action-angle variables to explain the geometrical meaning of our continuation scheme.

6.1 Continuation of relative equilibria

To start consider a Hamiltonian system (4.4) with Hamiltonian $H \in C^\infty(\mathbb{R}^{2n};\mathbb{R})$, and fix some $F \in \mathcal{F}$, i.e. $F \in C^\infty(\mathbb{R}^{2n};\mathbb{R})$ and $\{H,F\} \equiv 0$. According to theorem 11 this condition means that the Hamiltonian vector field X_H is equivariant with respect to the one-dimensional symmetry group $\mathfrak{G} := \{\varphi_F^s \mid s \in \mathbb{R}\}$ of symplectic transformations generated by the Hamiltonian vector field X_F; that is, we have $D\varphi_F^s(u)\cdot X_H(u) = X_H(\varphi_F^s(u))$ for all $u \in \mathbb{R}^{2n}$. Then choose some $\Omega \in \mathbb{R}$ and set $u(t) = \varphi_F^{\Omega t}(v(t))$ in (4.4); one can easily see that $v(t)$ will satisfy the Hamilton equation

$$\dot v = X_H(v) - \Omega X_F(v) = X_{H-\Omega F}(v). \tag{6.1}$$

Since $u(0) = v(0)$ we conclude that

$$\varphi_H^t(u) = \varphi_F^{\Omega t}\left(\varphi_{H-\Omega F}^t(u)\right) \qquad \text{and} \qquad D\varphi_H^t(u) = D\varphi_F^{\Omega t}\left(\varphi_{H-\Omega F}^t(u)\right)\cdot D\varphi_{H-\Omega F}^t(u). \tag{6.2}$$

Next suppose that $p_0 \in \mathbb{R}^{2n}$ is an equilibrium of (6.1):

$$X_{H-\Omega F}(p_0) = X_H(p_0) - \Omega X_F(p_0) = 0; \tag{6.3}$$

then $\varphi_H^t(p_0) = \varphi_F^{\Omega t}(p_0)$ is a solution of (4.4) with the property that the corresponding flow-orbit is contained in (and if $\Omega \neq 0$ actually coincides with) a symmetry-orbit (a $\mathfrak{G}$-orbit). Such orbits (and the corresponding initial points p_0) are called *relative equilibria with respect to the group* $\mathfrak{G}$ since they correspond to equilibria in the reduced system obtained from (4.4) using the $\mathfrak{G}$-symmetry — see [16] or [15] for some of the earlier accounts on symmetry reduction in Hamiltonian systems, [13], [22], [2] and [20] for some more recent work, [14] or [19] for a general and comprehensive account of symmetry reductions, [5] for some explicitly worked out examples, or [25] for an introduction in the case where $\mathfrak{G}$ is isomorphic to the circle group S^1; we also refer to section 8 where for a particular system the reduction will be carried out explicitly. Now assume that the group orbit $\mathfrak{G}(p_0)$ is closed, meaning that p_0 generates a periodic orbit of X_F; then p_0 also generates a periodic solution of X_H. More precisely, if $s \mapsto \varphi_F^s(p_0)$ is periodic with minimal period τ_0 then $t \mapsto \varphi_H^t(p_0)$ is periodic with minimal period $T_0 = \tau_0/\Omega$. If moreover such periodic orbit $\Gamma_0 = \mathfrak{G}(p_0) = \{\varphi_H^t(p_0) \mid t \in \mathbb{R}\}$

is normal (as a periodic orbit of X_H) then our foregoing continuation schemes will give us a branch of periodic orbits of X_H containing the relative equilibrium p_0. Observe that when checking whether Γ_0 is normal the fact that $\nabla H(p_0)$ and $\nabla F(p_0)$ are proportional will influence the value of $k = \dim W$. Also, in order to calculate $M = D\varphi_H^{T_0}(p_0)$ we can use the fact that (6.2) and (6.3) imply that

$$D\varphi_H^t(p_0) = D\varphi_F^{\Omega t}(p_0) \cdot \exp\left(tDX_{H-\Omega F}(p_0)\right), \qquad \forall t \in \mathbb{R},$$

and

$$M = D\varphi_F^{T_0}(p_0) \cdot \exp\left(T_0 DX_{H-\Omega F}(p_0)\right). \tag{6.4}$$

The examples in sections 7 and 8 show that in a number of cases the continuation of such "periodic" relative equilibrium leads to a whole branch of relative equilibria. Moreover, in many applications the symmetry group $\mathfrak{G}$ is such that all its orbits are closed, which implies that all relative equilibria are periodic. In such cases one can consider the problem of the continuation of relative equilibria as part of the problem of the continuation of periodic orbits. We will show next that in fact the continuation approach of the foregoing sections can be adapted to the problem of the continuation of (certain) relative equilibria. But first we put the definition of a relative equilibrium in a somewhat broader and slightly different setting.

This time we start with a finite-dimensional subspace $\mathcal{G}$ of $C^\infty(\mathbb{R}^{2n};\mathbb{R})$ with the property that $\{F,G\} \in \mathcal{G}$ for all $F,G \in \mathcal{G}$ (i.e. $\mathcal{G}$ forms a finite-dimensional Lie algebra). Denote by $\mathfrak{G}$ the Lie group of diffeomorphisms on $\mathbb{R}^{2n}$ generated by the one-parameter groups $\{\varphi_F^s \mid s \in \mathbb{R}\}$ with $F \in \mathcal{G}$. Consider also a Hamiltonian vector field X_H with Hamiltonian $H \in C^\infty(\mathbb{R}^{2n};\mathbb{R})$ and such that $\mathcal{G} \subset \mathcal{F} := \{F \in C^\infty(\mathbb{R}^{2n};\mathbb{R}) \mid \{H,F\} \equiv 0\}$. Then the vector field X_H is equivariant with respect to the group $\mathfrak{G}$ (theorem 11), and a point $p_0 \in \mathbb{R}^{2n}$ is a *relative equilibrium* of X_H with respect to $\mathfrak{G}$ if $X_H(p_0) = X_F(p_0)$ for some $F \in \mathcal{G}$; more precisely, if we choose elements $F_i \in \mathcal{G}$ $(1 \le i \le m)$ such that $\{X_{F_i}(p_0) \mid 1 \le i \le m\}$ forms a basis of $Y := \{X_F(p_0) \mid F \in \mathcal{G}\}$, then p_0 is a relative equilibrium of X_H with respect to $\mathfrak{G}$ if there exist elements $\Omega_i \in \mathbb{R}$ such that

$$X_H(p_0) = \sum_{i=1}^{m} \Omega_i X_{F_i}(p_0). \tag{6.5}$$

Geometrically this means that the flow orbit $\{\varphi_H^t(p_0) \mid t \in \mathbb{R}\}$ of p_0 is contained in the symmetry orbit $\mathfrak{G}(p_0)$. Observe that together with p_0 also all other points of $\mathfrak{G}(p_0)$ are relative equilibria.

Next we formulate and prove a continuation result for relative equilibria under a restrictive condition for $\mathcal{G}$; we hope to give more general results in a later paper. We will assume that $\mathcal{G} \subset \mathcal{Z} := \{G \in \mathcal{F} \mid \{G,F\} \equiv 0, \forall F \in \mathcal{F}\}$; this implies in particular that the group $\mathfrak{G}$ is abelian. Fix some relative equilibrium $p_0 \in \mathbb{R}^{2n}$ of X_H with respect to $\mathfrak{G}$, i.e. assume that

$$X_H(p_0) = \sum_{i=1}^{m} \Omega_i^0 X_{F_i}(p_0) \tag{6.6}$$

for some $\Omega_i^0 \in \mathbb{R}$ $(1 \le i \le m)$ and with $F_i \in \mathcal{G}$ $(1 \le i \le m)$ such that $\{X_{F_i}(p_0) \mid 1 \le i \le m\}$ forms a basis of $Y = \{X_F(p_0) \mid F \in \mathcal{G}\}$. Also choose $F_j \in \mathcal{F}$ $(m+1 \le j \le k)$ such that

$\{\nabla F_j(p_0) \mid 1 \le j \le k\}$ forms a basis of $W = \{\nabla F(p_0) \mid F \in \mathcal{F}\}$, and set

$$L := DX_H(p_0) - \sum_{i=1}^{m} \Omega_i^0 DX_{F_i}(p_0) \in \mathcal{L}(\mathbb{R}^{2n}). \tag{6.7}$$

Lemma 15 *Under the foregoing conditions we have*

$$JW \subset Ker\,(L) \qquad and \qquad Im\,(L) + Y \subset W^{\perp}. \tag{6.8}$$

As a consequence zero is an eigenvalue of L with geometric multiplicity at least equal to k, and algebraic multiplicity at least equal to $(k+m)$.

Proof Let $H_0 := H - \sum_{i=1}^{m} \Omega_i^0 F_i$; then $L = DH_0(p_0)$, and it follows from $\mathcal{G} \subset \mathcal{Z}$ that $\{H_0, F\} \equiv 0$ for all $F \in \mathcal{F}$. Theorem 11 implies that $X_{H_0}(\varphi_F^s(p_0)) = D\varphi_F^s(p_0) \cdot X_{H_0}(p_0) = 0$ for all $F \in \mathcal{F}$ and all $s \in \mathbb{R}$; differentiation at $s = 0$ gives $L \cdot X_F(p_0) = 0$ for all $F \in \mathcal{F}$. This proves the first inclusion of (6.8) and the fact that zero is an eigenvalue of L with geometric multiplicity equal to $\dim \mathrm{Ker}\,(L) \ge \dim(JW) = k$. Differentiating the identity $DF(u) \cdot X_{H_0}(u) = \{F, H_0\}(u) = 0$ at $u = p_0$ and using $X_{H_0}(p_0) = 0$ gives $DF(p_0) \cdot L(u) = 0$ for all $u \in \mathbb{R}^{2n}$ and all $F \in \mathcal{F}$; this shows that $\mathrm{Im}\,(L) \subset W^{\perp}$. Combining this with the fact that $Y \subset W^{\perp}$ (an immediate consequence of $\mathcal{G} \subset \mathcal{Z}$) gives the second inclusion in (6.8), which then also implies $\mathrm{Im}\,(L^{\nu}) + Y \subset W^{\perp}$ for each $\nu \ge 1$. Moreover, for ν sufficiently large we have $\mathbb{R}^{2n} = \mathrm{Im}\,(L^{\nu}) \oplus \mathrm{Ker}\,(L^{\nu})$ and hence (since $Y \subset JW \subset \mathrm{Ker}\,(L)$)

$$\mathrm{Im}\,(L^{\nu}) \cap Y \subset \mathrm{Im}\,(L^{\nu}) \cap \mathrm{Ker}\,(L) \subset \mathrm{Im}\,(L^{\nu}) \cap \mathrm{Ker}\,(L^{\nu}) = \{0\}.$$

It follows that $\mathrm{Im}\,(L^{\nu})$ has codimension at least equal to $\dim Y = m$ in $W^{\perp}$, and codimension at least equal to $(k+m)$ in $\mathbb{R}^{2n}$; consequently $\dim \mathrm{Ker}\,(L^{\nu}) \ge k+m$, which proves that the algebraic multiplicity of the zero eigenvalue of L (which is equal to $\dim \mathrm{Ker}\,(L^{\nu})$) must be at least equal to $(k+m)$. ■

We will say that the relative equilibrium p_0 is *normal* if we have equality in the second inclusion of (6.8), i.e. if

$$\mathrm{Im}\,(L) + Y = W^{\perp}. \tag{6.9}$$

More in particular this condition will be satisfied when the zero eigenvalue of L has either geometric multiplicity equal to k, or algebraic multiplicity equal to $(k+m)$. Indeed, in the first case $\mathrm{Im}\,(L)$ has codimension k in $\mathbb{R}^{2n}$, and therefore $\mathrm{Im}\,(L) = W^{\perp}$, by (6.8) and $\dim W = k$. In the second case we have $\mathbb{R}^{2n} = \mathrm{Im}\,(L^{\nu}) \oplus \mathrm{Ker}\,(L^{\nu})$ and $\dim \mathrm{Ker}\,(L^{\nu}) = k+m$ for $\nu \ge 1$ sufficiently large (see the proof of lemma 15); hence $\mathrm{Im}\,(L^{\nu}) \subset \mathrm{Im}\,(L)$ has codimension m in $W^{\perp}$ and since $\mathrm{Im}\,(L^{\nu}) \cap Y = \{0\}$ it follows that $\mathrm{Im}\,(L^{\nu}) \oplus Y = W^{\perp}$ and $W^{\perp} = \mathrm{Im}\,(L) + Y$.

In order to continue such normal relative equilibrium p_0 we define a smooth mapping $G_0 : \mathbb{R}^{2n} \times \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^{2n}$ by

$$G_0(p, \Omega, \alpha) := X_H(p) - \sum_{i=1}^{m} \Omega_i X_{F_i}(p) + \sum_{j=1}^{k} \alpha_j \nabla F_j(p), \tag{6.10}$$

with $\Omega = (\Omega_1, \Omega_2, \ldots, \Omega_m) \in \mathbb{R}^m$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{R}^k$; if moreover we set $\Omega_0 := (\Omega_1^0, \Omega_2^0, \ldots, \Omega_m^0) \in \mathbb{R}^m$ then (6.6) can be rewritten as $G_0(p_0, \Omega_0, 0) = 0$. We have the following continuation result.

Theorem 16 *Assume that $\mathcal{G} \subset \mathcal{Z}$, and let $p_0 \in \mathbb{R}^{2n}$ be a normal relative equilibrium of X_H with respect to $\mathfrak{G}$. Define m, k, F_j $(1 \leq j \leq k)$, G_0 and Ω_0 as indicated before. Then the solution set near $(p_0, \Omega_0, 0)$ of the system of equations*

$$\begin{cases} G_0(p, \Omega, \alpha) = 0, \\ \langle X_{F_j}(p_0), p - p_0 \rangle = 0, \quad (1 \leq j \leq k), \end{cases} \tag{6.11}$$

consists of a smooth m-dimensional submanifold of $\mathbb{R}^{2n} \times \mathbb{R}^m \times \mathbb{R}^k$ along which $\alpha \equiv 0$. If $(p, \Omega, 0) \in \mathbb{R}^{2n} \times \mathbb{R}^m \times \mathbb{R}^k$ is such solution then $X_H(p) = \sum_{i=1}^m \Omega_i X_{F_i}(p)$ and p is a normal relative equilibrium of X_H with respect to $\mathfrak{G}$.

Proof We have

$$DG_0(p_0, \Omega_0, 0) \cdot (p, \Omega, \alpha) = L(p) - \sum_{i=1}^m \Omega_i X_{F_i}(p_0) + \sum_{j=1}^k \alpha_j \nabla F_j(p_0),$$

and it follows then immediately from (6.9) that $DG_0(p_0, \Omega_0, 0) \in \mathcal{L}(\mathbb{R}^{2n} \times \mathbb{R}^m \times \mathbb{R}^k; \mathbb{R}^{2n})$ is surjective, i.e. G_0 is a submersion at $(p_0, \Omega_0, 0)$. Also the linear mapping $A \in \mathcal{L}(\mathbb{R}^{2n}; \mathbb{R}^k)$ given by

$$A \cdot p := \Big(\langle X_{F_1}(p_0), p \rangle, \langle X_{F_2}(p_0), p \rangle, \ldots, \langle X_{F_k}(p_0), p \rangle \Big)$$

is surjective; more precisely, the restriction of A to the k-dimensional subspace $JW = \{X_F(p_0) \mid F \in \mathcal{F}\}$ is injective and hence surjective, by the same argument as in the last part of the proof of theorem 13. Since $JW \subset \mathrm{Ker}\,(L)$ we conclude that locally near $(p_0, \Omega_0, 0)$ the solution set of (6.11) is an m-dimensional submanifold of $\mathbb{R}^{2n} \times \mathbb{R}^m \times \mathbb{R}^k$.

Next suppose that $G_0(p, \Omega, \alpha) = 0$ for some $(p, \Omega, \alpha) \in \mathbb{R}^{2n} \times \mathbb{R}^m \times \mathbb{R}^k$ with p close to p_0. Taking the scalar product of $G_0(p, \Omega, \alpha) = 0$ with $\sum_{j=1}^k \alpha_j \nabla F_j(p)$ gives

$$\left\| \sum_{j=1}^k \alpha_j \nabla F_j(p) \right\|^2 = 0 \quad \Longrightarrow \quad \sum_{j=1}^k \alpha_j \nabla F_j(p) = 0 \quad \Longrightarrow \quad \alpha_j = 0 \ \ (1 \leq j \leq k),$$

since the vectors $\nabla F_j(p)$ $(1 \leq j \leq k)$ are linearly independent for p close to p_0. We conclude that p is a relative equilibrium of X_H with respect to $\mathfrak{G}$; this relative equilibrium is normal by a perturbation argument similar to the one given in the proof of theorem 7. ■

In case $m = 1$ the foregoing theorem gives us a one-dimensional branch of normal relative equilibria; we can then use numerical continuation packages to solve the system (6.11) and calculate such branches. We refer to sections 7 and 8 for some explicit examples.

6.2 Integrable Hamiltonian systems

The second special case we want to consider in more detail is that of integrable Hamiltonian systems with two degrees of freedom; we will show how for such systems the continuation results of the foregoing section can be interpreted in terms of action-angle variables. The set-up is as follows. We start with a Hamiltonian system X_H on $\mathbb{R}^4$, corresponding to

the Hamilton function $H \in C^\infty(\mathbb{R}^4;\mathbb{R})$; such Hamiltonian system is *integrable* if there is at least one further first integral which is independent of H, it is *superintegrable* if next to H there are at least two more independent first integrals. A well-known example of a superintegrable system is that of the Kepler motion in the plane: next to the energy and the angular momentum there is a third constant of motion, namely (an appropriate component of) the Laplace-Runge-Lenz vector (see [11]). In the superintegrable case we will have $k = 3$ and $m_a = 4$ along each periodic orbit Γ_0, and hence each such periodic orbit is normal (see Proposition 12 and Proposition 5). Moreover it follows from Theorem 7 that all orbits near Γ_0 will be periodic.

In the remainder of this subsection we exclude the superintegrable case and assume that next to the Hamiltonian H there is just one further independent constant of motion $F \in C^\infty(\mathbb{R}^4;\mathbb{R})$, i.e. $\{H, F\} \equiv 0$. For some $(h_0, f_0) \in \mathbb{R}^2$ let $\mathcal{T}_0$ be a bounded connected component of the set $\{u \in \mathbb{R}^4 \mid (H(u), F(u)) = (h_0, f_0)\}$ such that the vectors $\nabla H(u)$ and $\nabla F(u)$ are linearly independent for all $u \in \mathcal{T}_0$. It is shown in [5] that under these conditions $\mathcal{T}_0$ is a two-dimensional torus, and that there exists an open neighborhood of $\mathcal{T}_0$ in which one can introduce canonical coordinates $(I_1, I_2, \theta_1, \theta_2) \in \mathbb{R} \times \mathbb{R} \times S^1 \times S^1$ (with $S^1 := \mathbb{R}/\mathbb{Z}$) such that the following holds:

(i) in the new coordinates $\mathcal{T}_0$ is given by $\{I_1^0\} \times \{I_2^0\} \times S^1 \times S^1$, for some $(I_1^0, I_2^0) \in \mathbb{R}^2$;

(ii) for $(I_1, I_2) \in \mathbb{R}^2$ close to (I_1^0, I_2^0) the torus $\mathcal{T}_{(I_1,I_2)} := \{I_1\} \times \{I_2\} \times S^1 \times S^1$ is invariant under the flow of X_H;

(iii) near $\mathcal{T}_0$ and in the new coordinates the system X_H takes the form

$$\dot{I}_1 = 0, \quad \dot{I}_2 = 0, \quad \dot{\theta}_1 = \Omega_1(I_1, I_2), \quad \dot{\theta}_2 = \Omega_2(I_1, I_2), \tag{6.12}$$

with

$$\Omega_i(I_1, I_2) = \frac{\partial K}{\partial I_i}(I_1, I_2), \qquad (i = 1, 2)$$

for some $K \in C^\infty(\mathbb{R}^2;\mathbb{R})$.

The variables (I_i, θ_i) $(i = 1, 2)$ are called (local) *action-angle variables*. The property (iii) means that in these action-angle variables the system is still Hamiltonian, with a Hamilton function K which depends only on the action variables I_i and is independent of the angle variables θ_i; in the new variables the coordinate functions I_1 and I_2 are two independent constants of motion. It is easy to write down the flow of (6.12), namely

$$\varphi_K^t(I_1, I_2, \theta_1, \theta_2) = \Big(I_1, I_2, \theta_1 + \Omega_1(I_1, I_2)\, t, \theta_2 + \Omega_2(I_1, I_2)\, t\Big), \tag{6.13}$$

where the angle variables should of course be taken modulo $\mathbb{Z}$. We set $\Omega_i^0 := \Omega_i(I_1^0, I_2^0)$ $(i = 1, 2)$ and assume that $(\Omega_1^0, \Omega_2^0) \neq (0, 0)$. For fixed $(I_1, I_2) \in \mathbb{R}^2$ near (I_1^0, I_2^0) the flow on the torus $\mathcal{T}_{(I_1,I_2)}$ will be periodic if and only if there exist some $T > 0$ and some $k_1, k_2 \in \mathbb{Z}$ such that $\Omega_1(I_1, I_2)T = k_1$, $\Omega_2(I_1, I_2)T = k_2$ and $\gcd(|k_1|, |k_2|) = 1$. This condition in turn is equivalent to the existence of some $k_1, k_2 \in \mathbb{Z}$ such that

$$\gcd(|k_1|, |k_2|) = 1 \qquad \text{and} \qquad k_2\Omega_1(I_1, I_2) - k_1\Omega_2(I_1, I_2) = 0; \tag{6.14}$$

the minimal period T is then given by $|k_1|/|\Omega_1(I_1, I_2)|$ or $|k_2|/|\Omega_2(I_1, I_2)|$, depending on whether $\Omega_1(I_1, I_2) \neq 0$ or $\Omega_2(I_1, I_2) \neq 0$. We can consider $(k_1, k_2) \in \mathbb{Z}^2$ as the *winding numbers* of the periodic orbits on the torus $\mathcal{T}_{(I_1,I_2)}$.

Now suppose that the flow on the torus $\mathcal{T}_0$ is periodic, with minimal period $T_0 > 0$, and let $k_1, k_2 \in \mathbb{Z}$ be such that $\Omega_1^0 T_0 = k_1$ and $\Omega_2^0 T_0 = k_2$. It follows from (6.13) that the corresponding monodromy matrix M_0 is given by

$$M_0 \cdot (I_1, I_2, \theta_1, \theta_2) = (I_1, I_2, \theta_1, \theta_2) + (0, 0, \delta_{11} I_1 + \delta_{12} I_2, \delta_{21} I_1 + \delta_{22} I_2),$$

with

$$\delta_{ij} := T_0 \frac{\partial \Omega_i}{\partial I_j} \left(I_1^0, I_2^0\right), \quad (i, j = 1, 2).$$

Since the coordinate functions I_1 and I_2 are two independent constants of motion we have $k = 2$ and $W = \{(I_1, I_2, 0, 0) \mid I_1, I_2 \in \mathbb{R}\}$. Also, let D be the 2×2-matrix with matrix-elements δ_{ij} $(i, j = 1, 2)$. Then the periodic orbits on $\mathcal{T}_0$ will be normal if either $\det D \neq 0$, or if $D \neq 0$, $\det D = 0$ and $(\Omega_1^0, \Omega_2^0) \notin \mathrm{Im}\,(D)$. Some elementary algebra (using $\Omega_1^0 T_0 = k_1$ and $\Omega_2^0 T_0 = k_2$) shows that these conditions are equivalent to the condition

$$\left(k_2 \frac{\partial \Omega_1}{\partial I_1}(I_1^0, I_2^0) - k_1 \frac{\partial \Omega_2}{\partial I_1}(I_1^0, I_2^0), k_2 \frac{\partial \Omega_1}{\partial I_2}(I_1^0, I_2^0) - k_1 \frac{\partial \Omega_2}{\partial I_2}(I_1^0, I_2^0) \right) \neq (0, 0). \tag{6.15}$$

Assuming that the flow on $\mathcal{T}_0$ is periodic and normal we next apply the continuation theorem 13, starting from an arbitrary point $p_0 = (I_1^0, I_2^0, \theta_1^0, \theta_2^0) \in \mathcal{T}_0$. Using (6.12) the equations (5.6) take the explicit form

$$\alpha_1 T = 0, \;\; \alpha_2 T = 0, \;\; \Omega_1(I_1, I_2) T = k_1, \;\; \Omega_2(I_1, I_2) T = k_2, \;\; \theta_1 = \theta_1^0, \;\; \theta_2 = \theta_2^0,$$

to be solved for $(I_1, I_2, \theta_1, \theta_2, T, \alpha_1, \alpha_2)$ near $(I_1^0, I_2^0, \theta_1^0, \theta_2^0, T_0, 0, 0)$. As expected, we have $\alpha_1 = \alpha_2 = 0$, while the phase conditions fix the angle variables θ_1 and θ_2. The remaining two equations for T, I_1 and I_2, namely

$$\Omega_1(I_1, I_2) = \frac{k_1}{T}, \qquad \Omega_2(I_1, I_2) = \frac{k_2}{T}, \tag{6.16}$$

can be solved for I_1 and I_2 as a function of T if $\det D \neq 0$, i.e. if $m_g = 2 = k$ (compare with theorem 13). In general, the normality condition (6.15) shows that the equivalent equations

$$T = \frac{k_i}{\Omega_i(I_1, I_2)}, \qquad k_2 \Omega_1(I_1, I_2) - k_1 \Omega_2(I_1, I_2) = 0, \tag{6.17}$$

(where $i \in \{1, 2\}$ is chosen such that $\Omega_i^0 \neq 0$) can either be solved for (T, I_1) as a function of I_2, or for (T, I_2) as a function of I_1. We obtain a one-dimensional curve in the (I_1, I_2)-plane; the flow on the tori corresponding to points on this curve is periodic and normal; moreover, the *winding numbers* (k_1, k_2) of the periodic orbits on these tori (see (6.16)) remain constant along the curve. We conclude that in the case of an integrable Hamiltonian system with two degrees of freedom our method allows the continuation of periodic tori with given winding numbers. In this context it is important to observe that in practice it is usually impossible to obtain the action-angle variables explicitly; however, our method also works when we use the original coordinates or any other (canonical) coordinate system that is convenient.

7 Example 1

In this and in the next section we consider two explicit examples on which we apply our foregoing results. These examples are relatively simple and are chosen such that they clearly illustrate some particular features of the theory. Both examples consist of 4-dimensional Hamiltonian systems with one additional constant of motion, i.e. the systems are integrable. The first example can be solved explicitly; it contains three different types of branches of periodic orbits, and we show that our continuation results can be used to reconstruct each of these branches. The second example (given in the next section) derives from a physical application and will be treated partly analytically, partly numerically. In both cases we use the complex notation for Hamiltonian systems such as discussed in section 4.

For our first example we take $n = 2$, set $z = (a, b) \in \mathbb{C}^2$ and consider the Hamiltonian

$$H(a, b) := a\bar{a} - b\bar{b} + \frac{1}{2}(a\bar{a})^2 . \tag{7.1}$$

The solution of the corresponding Hamilton equations

$$\begin{cases} \dot{a} = i(1 + |a|^2)\, a \\ \dot{b} = -ib \end{cases} \tag{7.2}$$

with initial values (a_0, b_0) is given by

$$\tilde{a}(t; a_0, b_0) = a_0 e^{i(1+|a_0|^2)t}, \qquad \tilde{b}(t; a_0, b_0) = b_0 e^{-it}, \qquad \forall a_0, b_0 \in \mathbb{C}. \tag{7.3}$$

The system (7.2) has the functions $G_1(a, b) := a\bar{a} = |a|^2$ and $G_2(a, b) := b\bar{b} = |b|^2$ as constants of motion: one can verify this directly from (7.3), or by checking that $\{H, G_1\} \equiv 0$ and $\{H, G_2\} \equiv 0$. The Hamiltonian H is functionally dependent on G_1 and G_2, since $H = G_1 - G_2 + 1/2\, G_1^2$. Also the function $F(a, b) := a\bar{a} + b\bar{b} = G_1(a, b) + G_2(a, b)$ is a first integral; the corresponding Hamiltonian vector field $i\dfrac{\partial F}{\partial \bar{z}}(a, b) = i(a, b)$ generates an S^1-action on $\mathbb{C}^2$, explicitly given by

$$(\theta, a, b) \in (\mathbb{R} \bmod 2\pi) \times \mathbb{C}^2 = S^1 \times \mathbb{C}^2 \;\longmapsto\; \left(e^{i\theta}a, e^{i\theta}b\right) \in \mathbb{C}^2. \tag{7.4}$$

The equations (7.2) and the flow (7.3) are equivariant with respect to this S^1-symmetry. As a consequence we can for this particular case consider the equation (6.1) as the original Hamiltonian equation (7.2) in a frame rotating with angular velocity Ω; relative equilibria are actual equilibria in an appropriate rotating frame, and are automatically periodic solutions of (7.2). The system has a unique equilibrium at the origin $(a, b) = (0, 0)$, and, as one can easily see from (7.3), three different types of periodic solutions. We now analyse these periodic solutions one by one, and show what happens when we continue them using the approach of theorem 13 or theorem 16.

Case 1. Solutions with initial value $p_0 = (a_0, 0) \neq (0, 0)$ are periodic with minimal period $T_0 = 2\pi(1 + |a_0|^2)^{-1}$. Along these periodic orbits the gradients of H and F are linearly dependent, i.e. these periodic orbits are relative equilibria with respect to the S^1-action (take $\Omega = (1 + |a_0|^2)$ in (6.3)). Using phase shifts or the S^1-symmetry we can restrict our

attention to the case where a_0 is real and strictly positive, i.e. we can w.l.o.g. assume that $a_0 = \rho_0$ for some $\rho_0 \in \mathbb{R}$, $\rho_0 > 0$. The monodromy matrix M of the corresponding periodic solution can be obtained by differentiating the mapping $(a, b) \mapsto (\tilde{a}(T_0; a, b), \tilde{b}(T_0; a, b))$ at the initial point $(\rho_0, 0)$; we find

$$M \cdot (a, b) = \left(a + i\rho_0^2 \, T_0 \, (a + \bar{a}\,) \, , be^{i\,T_0} \right), \qquad \text{with } \; T_0 = 2\pi \left(1 + \rho_0^2 \right)^{-1}.$$

Hence $\mathrm{Ker}\,(M - I) = i\mathbb{R} \times \{0\}$ and $m_g = 1$. Moreover, W contains the nonzero vector $\nabla H(p_0) = (\rho_0 + \rho_0^3, 0)$ and $JW \subset \mathrm{Ker}\,(M - I)$ by Proposition 12; so $k = \dim W = 1 = m_g$ and the periodic orbit is normal. In order to apply the continuation scheme of theorem 13 we choose $F_1 = G_1$. We have to find initial points $p = (a_0, b_0)$ near $(\rho_0, 0)$ which generate 1-periodic solutions of

$$\begin{cases} \dot{a} = iT(1 + |a|^2)a + \alpha T a, \\ \dot{b} = -iTb, \end{cases} \tag{7.5}$$

with (T, α) near $(T_0, 0)$, and subject to the phase condition

$$\Big\langle (i\rho_0, 0), (a_0 - \rho_0, b_0) \Big\rangle = -i\rho_0(a_0 - \bar{a}_0) = 0 \qquad \Longrightarrow \qquad a_0 = \bar{a}_0.$$

So $a_0 = \rho_a \in \mathbb{R}$ with $\rho_a > 0$ near ρ_0. Using polar coordinates to integrate the first equation of (7.5) with initial value $a = \rho_a$ shows that the solution has the form

$$\tilde{a}(t) = \rho_a e^{\alpha T t + iT\tilde{\phi}(t)}, \qquad \tilde{\phi}(t) = \int_0^t \left(1 + \rho_a^2 e^{2\alpha T \tau} \right) d\tau;$$

imposing the periodicity condition $\tilde{a}(1) = \rho_a$ and using the fact that (T, ρ_a) must be close to (T_0, ρ_0) we get then

$$\alpha = 0 \qquad \text{and} \qquad T = \frac{2\pi}{1 + \rho_a^2},$$

with $\rho_a > 0$ arbitrary (but close to ρ_0). Imposing the periodicity condition on the solution of the second equation of (7.5) (with initial value b_0 near 0 and T near T_0) gives $b_0 = 0$. We conclude that the continuation method gives us the one-dimensional branch $A := \{(\rho_a, 0) \mid \rho_a \in \mathbb{R},\, \rho_a > 0\}$ of initial points which generate normal periodic orbits of (7.2) which are relative equilibria and which have minimal period $T = 2\pi \, (1 + \rho_a^2)^{-1}$. Observe that the branch A can be parametrised by the period T if one wishes to do so. The same branch A of normal relative equilibria can also be obtained by an application of theorem 16; we leave the details to the reader.

Case 2. Also initial points of the form $(0, b_0)$ with $b_0 \neq 0$ lead to periodic orbits which are relative equilibria; using phase shifts or the S^1-symmetry we can w.l.o.g. restrict to the case where $b_0 = \rho_0$ for some $\rho_0 \in \mathbb{R}$, $\rho_0 > 0$. First we show that $p_0 = (0, \rho_0)$ is a normal relative equilibrium. We have $X_H(p_0) = -X_F(p_0)$ (i.e. $\Omega_0 = -1$), $X_F(p_0) = (0, i\rho_0) \in Y$, and the matrix L defined by (6.7) is here given by $L = DX_H(p_0) + DX_F(p_0)$, or explicitly by $L \cdot (a, b) = (2ia, 0)$, such that $\mathrm{Im}\,(L) = \mathbb{C} \times \{0\}$. It follows then from (6.8) that

$$\mathbb{C} \times \{0\} + \{0\} \times i\mathbb{R} \subset \mathrm{Im}\,(L) + Y \subset W^{\perp};$$

since $\nabla F(p_0) = (0, \rho_0) \in W$ we conclude that we must have equality in the foregoing inclusion, and that p_0 is a normal relative equilibrium. Moreover, $W = \{0\} \times \mathbb{R}$ and

$k = \dim W = 1$. We can continue the relative equilibrium p_0 by the procedure given in theorem 16; for the implementation of this procedure we set $m = k = 1$, $\Omega_0 = -1$ and $F_1 = F$. According to theorem 16 we should look for solutions $(a, b, \Omega, \alpha) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R} \times \mathbb{R}$ near $(0, \rho_0, -1, 0)$ of the set of equations

$$\big(i(1+|a|^2)a, -ib\big) - \Omega\,(ia, ib) + \alpha\,(a, b) = (0, 0), \qquad \Big\langle (0, i\rho_0), (a, b-\rho_0) \Big\rangle = 0. \tag{7.6}$$

It follows easily that b must be real (i.e. $b = \rho_b \in \mathbb{R}$ near ρ_0), $a = 0$, $\alpha = 0$ and $\Omega = -1$. Hence the continuation method of theorem 16 will give us the branch $B = \{(0, \rho_b) \mid \rho_b \in \mathbb{R},\ \rho_b > 0\}$ of normal relative equilibria. Now observe that each of these relative equilibria generates a 2π-periodic orbit of (7.2) which is *not* normal: indeed, the minimal period is $T_0 = 2\pi$ and the monodromy matrix is just the identity ($M \cdot (a, b) = (a, b)$); hence $m_g = \dim \mathrm{Ker}\,(M - I) = 4 > k + 1 = 2$, and the periodic orbit generated by $(0, \rho_0)$ is not normal.

Case 3. When both a_0 and b_0 are nonzero then according to (7.3) the solution of (7.2) with initial value (a_0, b_0) is periodic if and only if we can find a $T > 0$ and strictly positive integers $\tilde{p}$ and $\tilde{q}$ such that

$$\big(1 + |a_0|^2\big)\, T = 2\pi\tilde{p} \qquad \text{and} \qquad T = 2\pi\tilde{q}, \tag{7.7}$$

that is if and only if $(1 + |a_0|^2)$ is rational, or equivalently, if and only if $|a_0|^2$ is rational. If this is the case we write

$$|a_0|^2 = \frac{p}{q}, \qquad \text{with } p, q \in \mathbb{N} \text{ and } \gcd(p, q) = 1;$$

then $\Gamma_0 = \{(\tilde{a}(t; a_0, b_0), \tilde{b}(t; a_0, b_0)) \mid t \in \mathbb{R}\}$ is periodic with minimal period $T = 2\pi q$, and (7.7) is satisfied for $(\tilde{p}, \tilde{q}) = (p + q, q)$. Applying the S^1-symmetry to Γ_0 gives us the 2-torus

$$T_{a_0,b_0} := \Big\{ \Big(a_0 e^{i\theta}, b_0 e^{i\vartheta} \Big) \mid \theta, \vartheta \in \mathbb{R} \Big\}$$

which is invariant under the flow of (7.2) and under the symmetry, and which is foliated by periodic orbits of (7.2). These orbits wind $\tilde{p} = p + q$ times around T_{a_0,b_0} in the a-direction and $\tilde{q} = q$ times in the negative b-direction before closing up; we say that the orbits have *winding numbers* equal to $(p + q, -q)$. The flow on the torus T_{a_0,b_0} is quasi-periodic when $|a_0|^2$ is irrational.

We now apply the continuation method of theorem 13 to a particular such periodic solution; using phase shifts and symmetry we can w.l.o.g. assume that both initial coordinates a_0 and b_0 are real and strictly positive. To fix the notation we consider the periodic orbit Γ_0 through the point $p_0 = (\beta^{1/2}, \rho_0)$, with $\rho_0 \in \mathbb{R}$, $\rho_0 > 0$, $\beta \in \mathbb{Q}$ and $\beta > 0$; writing $\beta = p/q$ with $\gcd(p, q) = 1$ the minimal period T_0 is equal to $2\pi q$. The gradients of G_1 and G_2 at p_0 are given by respectively $(\beta^{1/2}, 0)$ and $(0, \rho_0)$; they are linearly independent, and hence $k \geq 2$ and $\dim W^\perp \leq 2$. The monodromy matrix is given by

$$M \cdot (a, b) = (a + iT_0\beta\,(a + \bar{a}\,), b) = (a + 2\pi p i\,(a + \bar{a}\,), b),$$

from which it follows that $\mathrm{Ker}\,(M - I) = i\,\mathbb{R} \times \mathbb{C}$, $m_g = 3$ and $\mathrm{Im}\,(M - I) = i\,\mathbb{R} \times \{0\}$. Since $g_0(p_0) = (i(1 + \beta)\beta^{1/2}, -i\rho_0)$ does not belong to $\mathrm{Im}\,(M - I)$ it follows that the subspace $\mathrm{Im}\,(M - I) + \mathbb{R}g_0(p_0)$ is two-dimensional; this subspace is contained in $W^\perp$ (by (3.8)), and

since $\dim W^\perp \leq 2$ we conclude that $\mathrm{Im}\,(M-I) + \mathbb{R}g_0(p_0) = W^\perp$ and that Γ_0 is a normal periodic orbit, with $k=2$.

For the implementation of theorem 13 we take $F_1 = G_1$ and $F_2 = G_2$, and consider the system

$$\begin{cases} \dot{a} = iT(1+|a|^2)a + \alpha_1 T a, \\ \dot{b} = -iTb + \alpha_2 T b, \end{cases} \tag{7.8}$$

with (T, α_1, α_2) near $(T_0, 0, 0)$. We have to look for initial values (a_0, b_0) near $(\beta^{1/2}, \rho_0)$ such that the corresponding solution of (7.8) is 1-periodic; moreover, these initial values must satisfy the phase conditions

$$\big\langle (i\beta^{1/2}, 0), (a_0 - \beta^{1/2}, b_0 - \rho_0) \big\rangle = \big\langle (0, i\rho_0), (a_0 - \beta^{1/2}, b_0 - \rho_0) \big\rangle = 0,$$

which implies that both a_0 and b_0 must be real. Therefore we set $a_0 = \rho_a$ and $b_0 = \rho_b$, with ρ_a and ρ_b real and close to respectively $\beta^{1/2}$ and ρ_0 (in particular both ρ_a and ρ_b are strictly positive). Solving the second equation of (7.8) with initial value $\rho_b > 0$ and imposing the periodicity condition gives us

$$\rho_b e^{(-i+\alpha_2)T} = \rho_b \qquad \Longrightarrow \qquad \alpha_2 = 0 \ \text{ and } \ T = T_0 = 2\pi q.$$

Next we set $T = T_0$ in the first equation of (7.8), integrate this equation using polar coordinates, and impose the periodicity condition; we find

$$\rho_a e^{(\alpha_1 + i\phi)T_0} = \rho_a, \qquad \text{with } \ \phi = \int_0^1 \left(1 + \rho_a^2 e^{2\alpha_1 T_0 \tau}\right) d\tau.$$

This implies $\alpha_1 = 0$ and $(1+\rho_a^2)T_0 = (1+\rho_a^2)2\pi q \in 2\pi\mathbb{Z}$; since ρ_a^2 must be close to $\beta = p/q$ we get $(1+\rho_a^2)q = p+q$ and hence $\rho_a = \beta^{1/2}$. We conclude that the continuation method gives us a one-dimensional branch $C_\beta = \{(\beta^{1/2}, \rho_b) \mid \rho_b \in \mathbb{R},\ \rho_b > 0\}$ of initial points which generate (under the flow of (7.2) and the S^1-symmetry) invariant tori on which the flow is periodic, with constant period $2\pi q$ and constant winding numbers $(p+q, -q)$. Such branch exists for each strictly positive $\beta \in \mathbb{Q}$.

It is interesting to draw the branches A, B and C_β ($\beta \in \mathbb{Q}$, $\beta > 0$) in a (H,F)-diagram (see Fig. 1). First observe that the mapping $(\rho_a, \rho_b) \longmapsto (H,F) := (\rho_a^2 - \rho_b^2 + \frac{1}{2}\rho_a^4, \rho_a^2 + \rho_b^2)$ is bijective from $\mathbb{R}_+ \times \mathbb{R}_+$ onto $\mathcal{C} := \{(H,F) \in \mathbb{R}^2 \mid F \geq 0,\ -F \leq H \leq F + \frac{1}{2}F^2\}$. The branches A and B of relative equilibria correspond to respectively $\{(H,F) \mid F > 0,\ H = F + \frac{1}{2}F^2\}$ and $\{(H,F) \mid F > 0,\ H = -F\}$; together with the equilibrium at the origin they form the boundary of $\mathcal{C}$. Each interior point of $\mathcal{C}$ corresponds to an invariant torus. The curves C_β (with $\beta > 0$) take the form $\{(H,F) \mid F > \beta,\ H = -F + 2\beta + \frac{1}{2}\beta^2\}$; the corresponding tori are periodic if β is rational, and quasi-periodic if β is irrational.

Observe also that the branch C_β ($\beta > 0$, $\beta = p/q \in \mathbb{Q}$) has the boundary point $(\beta^{1/2}, 0)$ on the branch A; this boundary point generates a relative equilibrium with minimal period $T_\beta := 2\pi(1+\beta)^{-1} = 2\pi q/(p+q)$. Since the minimal period $2\pi q$ along C_β is an integer multiple of T_β we can consider C_β as a branch of subharmonic solutions bifurcating from the branch A at $(\beta^{1/2}, 0)$ (see also Fig. 1). Going into some further detail we see that the monodromy matrix M at $(\beta^{1/2}, 0)$ (see case 1) has next to the eigenvalue 1 (with $m_g = 1$ and $m_a = 2$) also the pair of simple eigenvalues

$$e^{\pm iT_\beta} = e^{\pm 2\pi i q/(p+q)} = e^{\pm 2\pi i \hat{p}/\hat{q}}, \qquad \text{with } \hat{p} = q \text{ and } \hat{q} = p+q;$$

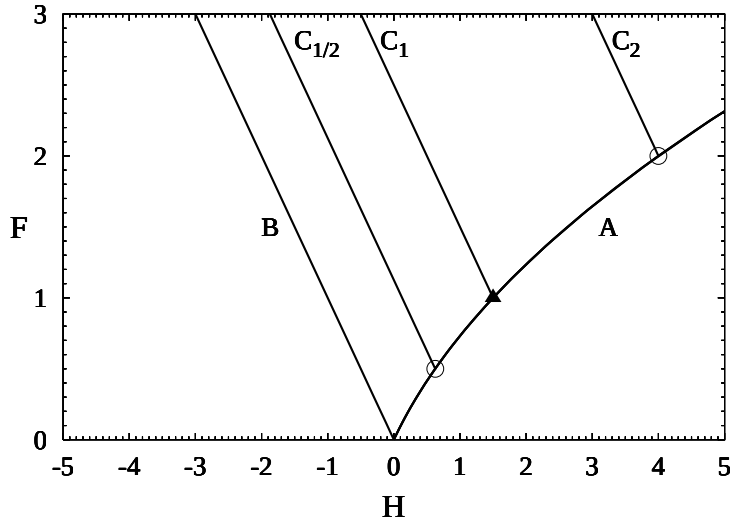

Figure 1: Branches of periodic orbits for the system (7.2); the branches C_β are shown for $\beta = 1/2$, $\beta = 1$ and $\beta = 2$. At the point ▲ there is a 2-subharmonic bifurcation (period-doubling), at both points ○ a 3-subharmonic bifurcation.

these multipliers (of the periodic orbit generated by $(\beta^{1/2}, 0)$) are $\hat{q}$-th roots of unity, and at the same time $\hat{q}$ is also equal to the quotient of the minimal period along C_β (which is equal to $2\pi q = 2\pi\hat{p}$) and the minimal period $T_\beta = 2\pi\hat{p}/\hat{q}$ at $(\beta^{1/2}, 0)$. So along the branch A we have bifurcation of a branch of $\hat{q}$-subharmonics precisely at those points where there is a pair of multipliers which are $\hat{q}$-th roots of unity; except for the fact that there is only a single branch of bifurcating subharmonics this agrees with the general theory on subharmonic bifurcation such as developed e.g. in [26] for the general case, and in [4] for the Hamiltonian case. The discrepancy in the number of bifurcating branches is due to the symmetry; we refer to some future work for a general treatment of subharmonic bifurcation in symmetric Hamiltonian systems.

In the foregoing example we do not really need our continuation techniques to obtain the different branches of periodic orbits: we could have obtained those directly and more easily from the formula (7.3) for the solutions of (7.2); we nevertheless included a detailed analysis because it clearly and explicitly illustrates certain aspects of our continuation method:

(i) The unfolding parameters α_i (with $i = 1, 2$ in this example) are indeed zero along the calculated solution branches.

(ii) When a relative equilibrium p_0 generates a periodic orbit (as is the case in this example) then p_0 may be normal as a relative equilibrium while the corresponding periodic orbit is not normal (see case 2). In such situation one can apply theorem 16 to continue the periodic orbit.

(iii) One can parametrize by the period when $m_g = k$ (case 1), while there is isochrony (constant period) when m_g stays equal to $k+1$ along a branch (case 3). These are two extremes of the more general situation where one will find $m_g = k$ along a branch, except at isolated points where $m_g = k+1$; these exceptional points typically correspond to maxima and minima of the period.

(iv) Along the branch A we have $m_a = 2 = k + 1$, and it is possible to parametrize this branch by the value of either H or F, which illustrates proposition 8. This same pro-

position is not applicable (and indeed the conclusion of the proposition is not true) along the branches C_β where $m_a = 4 = k + 2$.

(v) The system we have considered here is an integrable Hamiltonian system with two degrees of freedom. The branches C_β (with $\beta \in \mathbb{Q}$ and $\beta > 0$) form an illustration of what we have found in section 6 for such systems, namely one-dimensional curves of invariant and periodic tori, with fixed winding numbers for the orbits.

(vi) The dimension k of W (and hence the number of unfolding parameters) depends on the starting solution but stays constant along the continuation, at least as long as the solution remains normal.

(vii) Although branches do not intersect (by the uniqueness part of our continuation results) it may be that a given branch has a limit point on a different branch (e.g. each of the C_β branches limits on the A branch); the first branch can then be seen as a branch of subharmonics bifurcating from the second branch. Such bifurcating branches of subharmonics may be detected by monitoring the multipliers along a given branch, more precisely by looking for multipliers which are roots of unity. This idea will be used to explore the periodic orbits in the example of the next section.

(viii) Both branches A and B bifurcate from the equilibrium at the origin. The linearization of (7.2) at the origin has the eigenvalues $\pm i$; these eigenvalues are double and semisimple. Hence we can not directly apply the Lyapunov Center Theorem, since this requires simple purely imaginary eigenvalues — see [23]. However, a more detailed bifurcation analysis using the S^1-equivariance reveals that at least two branches of relative equilibria should bifurcate from the origin; along one of these branches the limiting value of Ω (see 6.3) should be $+1$, along the other branch this limiting value should be -1. It is clear that these branches are precisely the branches A and B, respectively.

8 Example 2

Keeping the same notation as in the foregoing section we consider the Hamiltonian

$$H(a,b) := a\bar{b} + \bar{a}b - \frac{1}{2}\left(b\bar{b}\right)^2, \tag{8.1}$$

with corresponding Hamilton equations

$$\begin{cases} \dot{a} = ib, \\ \dot{b} = ia - i\,|b|^2 b. \end{cases} \tag{8.2}$$

This system forms a simplified version of a mean field approximation of a model of coupled quantum wells; it represents the evolution of the charge accumulation in a two-site system for which the electrostatic interaction is non-negligible in one of the sites (see [8] and [9] for more details).

8.1 Branches of relative equilibria

As in our first example also here the function $F(a,b) := |a|^2 + |b|^2$ is a first integral, generating an S^1-action which is given by (7.4) and which commutes with (8.2). This time it is no

longer possible to write down explicitly the flow of (8.2), but we can still find some special solutions, namely the equilibria and relative equilibria. The origin $(a, b) = (0, 0)$ is the only equilibrium of (8.2), and the linearization of (8.2) at this equilibrium has the double and semisimple eigenvalues $\pm i$. In a similar way as outlined in remark (viii) at the end of the foregoing section one can show that at least two branches of relative equilibria bifurcate from the origin, with limiting values of Ω equal to $+1$ and -1, respectively. To find these branches explicitly we look for the equilibria of the system (6.1) which here takes the form

$$\begin{cases} \dot{a} = i(b - \Omega a), \\ \dot{b} = i(a - \Omega b) - i\,|b|^2 b, \end{cases} \tag{8.3}$$

with $v = (a, b) \in \mathbb{C}^2$. A nontrivial equilibrium $(a, b) \neq (0, 0)$ of (8.3) must satisfy

$$b = \Omega a \qquad \text{and} \qquad |b|^2 = g(\Omega) := \frac{1}{\Omega} - \Omega, \tag{8.4}$$

and hence such equilibria can only exist if $g(\Omega) > 0$, i.e. if Ω belongs either to the interval $I_- :=]-\infty, -1[$ or to the interval $I_+ :=]0, 1[$. Using the S^1-symmetry we can without loss of generality assume that b is real and strictly positive; then (8.4) gives us for each $\Omega \in I_- \cup I_+$ a unique relative equilibrium

$$v_\Omega := \sqrt{g(\Omega)}\left(\frac{1}{\Omega}, 1\right);$$

this relative equilibrium generates a periodic orbit of (8.2), given by

$$\Gamma_\Omega := \left\{\varphi_H^t(v_\Omega) \mid t \in \mathbb{R}\right\} = \left\{e^{i\Omega t} v_\Omega \mid t \in \mathbb{R}\right\},$$

and with minimal period equal to $T_\Omega = 2\pi/|\Omega|$. To determine the monodromy matrix M_Ω corresponding to this periodic orbit we use (6.4) and observe that X_F is linear: $X_F(v) = iv$ and $\varphi_F^s(v) = \exp(is)v$ for all $s \in \mathbb{R}$. Hence $D\varphi_F^{\Omega T_\Omega}(v_\Omega) = \exp(i\Omega T_\Omega)I = \exp(\pm 2\pi i)I = I$ and (6.4) reduces to

$$M_\Omega = D\varphi_H^{T_\Omega}(v_\Omega) = \exp(T_\Omega L_\Omega), \qquad \text{with} \qquad L_\Omega := DX_{H-\Omega F}(v_\Omega). \tag{8.5}$$

We obtain the explicit form of L_Ω from (8.3):

$$L_\Omega(a, b) = \left(i(b - \Omega a), i(a - \Omega b) - 2ig(\Omega)b - ig(\Omega)\bar{b}\right), \qquad \forall (a, b) \in \mathbb{C}^2.$$

Observe that L_Ω coincides with the matrix L associated with the relative equilibrium v_Ω, as defined by (6.7). We know from lemma 15 that L_Ω has the eigenvalue zero with algebraic multiplicity at least equal to two, since $k \geq m = 1$ and hence $k + m \geq 2$. To determine the other eigenvalues we use the fact that (4.5) implies $L_\Omega^T = -JL_\Omega J^{-1}$; hence, if $\mu \in \mathbb{C}$ is an eigenvalue of L_Ω then so is $-\mu$. Therefore the characteristic polynomial of L_Ω has the form $\lambda^4 - \mu^2\lambda^2$, with $\pm\mu$ the nontrivial eigenvalues of L_Ω. A simple calculation (using the explicit form of L_Ω and splitting a and b into their real and imaginary part) gives

$$\mu^2 = -\left(\Omega^2 + \frac{3}{\Omega^2}\right) \qquad \Longrightarrow \qquad \mu = \pm i\sqrt{\Omega^2 + \frac{3}{\Omega^2}}.$$

It follows that the zero eigenvalue of L_Ω has algebraic multiplicity equal to two; consequently we must have $k = 1$, and the relative equilibrium v_Ω is normal for all $\Omega \in I_- \cup I_+$ (see the

discussion after (6.9)). Starting the continuation procedure of theorem 16 at v_{Ω_0} will give us the branch $A_+ := \{v_\Omega \mid \Omega \in I_+\}$ of relative equilibria if $\Omega_0 \in I_+$, and the branch $A_- := \{v_\Omega \mid \Omega \in I_-\}$ if $\Omega_0 \in I_-$.

Going back to the monodromy matrix we see from (8.5) that M_Ω has next to the eigenvalue 1 also the eigenvalues

$$\exp\left(\pm 2\pi i \lambda(\Omega)\right), \qquad \text{with } \lambda(\Omega) := \sqrt{1 + \frac{3}{\Omega^4}}\,. \tag{8.6}$$

For $\Omega \in I_-$ we have $1 < \lambda(\Omega) < 2$ and $\exp\left(\pm 2\pi i \lambda(\Omega)\right) \neq 1$; it follows that the nontrivial multipliers are simple, while the multiplier 1 has algebraic multiplicity $m_a = 2$. From proposition 5 and the fact that $k = 1$ at a relative equilibrium of (8.2) we conclude that for $\Omega \in I_-$ the periodic orbit Γ_Ω is normal. Starting the continuation procedure of theorem 13 at any such relative equilibrium v_Ω will give us again the full branch A_- of relative equilibria. For $\Omega \in I_+$ we have $\lambda(\Omega) > 2$, and both nontrivial multipliers will be equal to 1 if $\lambda(\Omega) \in \mathbb{N}$; for such Ω one can easily verify that $m_g = 3 > k + 1$, and therefore the periodic orbit Γ_Ω will not be normal. Setting $\Omega_m := ((m^2-1)/3)^{-1/4}$ for $m \geq 3$ we see that the branch A_+ gets subdivided into a countable number of subbranches $A_+^{(m)} := \{v_\Omega \mid \Omega_{m+1} < \Omega < \Omega_m\}$ $(m \geq 3)$ along which the periodic orbit Γ_Ω is normal. Starting the continuation procedure of theorem 13 at some v_Ω with $\Omega_{m+1} < \Omega < \Omega_m$ will give only the subbranch $A_+^{(m)}$, while the procedure of theorem 16 gives the full branch A_+. This again illustrates remark (ii) at the end of section 7.

It is interesting to depict the results so far in an (H, F)-diagram (see Fig. 2), obtained by mapping points $z = (a, b) \in \mathbb{C}^2$ onto $(H(z), F(z)) \in \mathbb{R}^2$. In such diagram the branches A_- and A_+ of relative equilibria get the form

$$A_- = \{(H(v_\Omega), F(v_\Omega)) \mid \Omega \in I_-\} \qquad \text{and} \qquad A_+ = \{(H(v_\Omega), F(v_\Omega)) \mid \Omega \in I_+\}\,, \tag{8.7}$$

with the following explicit expressions for $H(v_\Omega)$ and $F(v_\Omega)$:

$$H(v_\Omega) = \frac{1}{2\Omega^2}\left(3 + \Omega^2\right)\left(1 - \Omega^2\right) \qquad \text{and} \qquad F(v_\Omega) = \frac{1}{\Omega^3} - \Omega. \tag{8.8}$$

We have $H(v_\Omega) < 0$ for $\Omega \in I_-$ and $H(v_\Omega) > 0$ for $\Omega \in I_+$; also $(H(v_\Omega), F(v_\Omega)) \to (0, 0)$ as $\Omega \to 1$ or $\Omega \to -1$. The mapping $\Omega \mapsto F(v_\Omega)$ is a diffeomorphism from I_- onto $]0, +\infty[$, and also from I_+ onto $]0, +\infty[$; this allows us to parametrize the branches A_- and A_+ by the value f of $F(v_\Omega)$, as follows. For each $f > 0$ there exist unique elements $\Omega_-(f) \in I_-$ and $\Omega_+(f) \in I_+$ such that $F(v_{\Omega_-(f)}) = F(v_{\Omega_+(f)}) = f$; setting $h_-(f) := H(v_{\Omega_-(f)}) < 0$ and $h_+(f) := H(v_{\Omega_+(f)}) > 0$ (and $h_-(0) = h_+(0) = 0$) the branches A_- and A_+ obtain in the (H, F)-plane the form

$$A_- = \{(h_-(f), f) \mid f > 0\} \qquad \text{and} \qquad A_+ = \{(h_+(f), f) \mid f > 0\}\,.$$

Moreover, we have that

$$(H(z), F(z)) \in \mathcal{R} := \{(h, f) \mid f \geq 0,\ h_-(f) \leq h \leq h_+(f)\}\,, \qquad \forall z \in \mathbb{C}^2. \tag{8.9}$$

To prove (8.9) take some $z \in \mathbb{C}^2$ $(z \neq 0)$, and let $f := F(z)$. The continuous function $H : \mathbb{C}^2 \to \mathbb{R}$ maps the compact and connected subset $F^{-1}(f) = \{\tilde{z} \in \mathbb{C}^2 \mid F(\tilde{z}) = f\}$

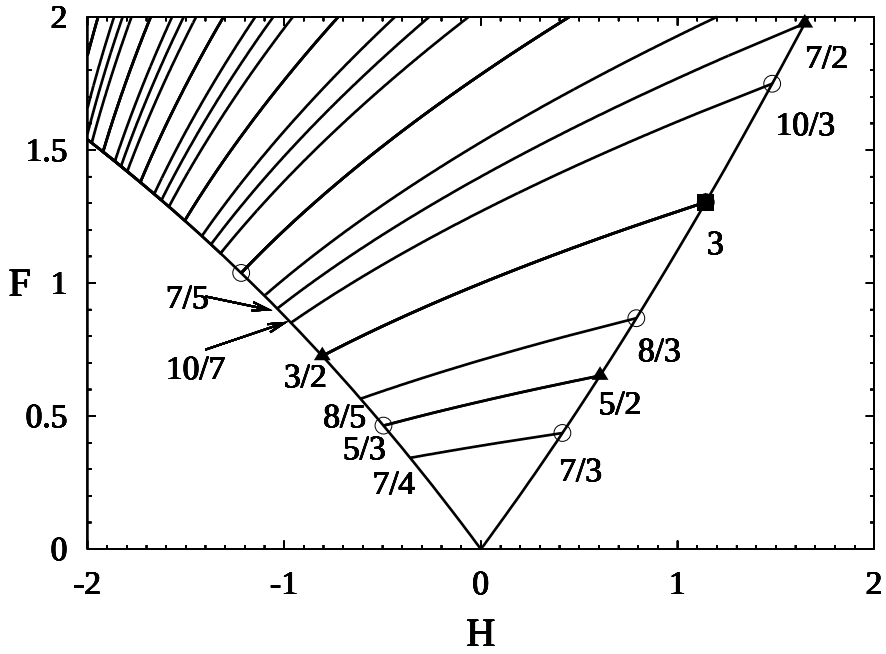

Figure 2: Branches of periodic orbits for the system (8.2). The labels along the branches of relative equilibria indicate the value of $\lambda(\Omega)$ (see (8.6)).

of $\mathbb{C}^2$ onto a compact interval $[\alpha, \beta]$, and therefore $\alpha \leq H(z) \leq \beta$. Let $z_- \in \mathbb{C}^2$ be such that $F(z_-) = f$ and $H(z_-) = \alpha$; then H attains at the point z_- its minimum on $F^{-1}(f)$, and hence we have $\nabla H(z_-) = \Omega \nabla F(z_-)$ for some $\Omega \in \mathbb{R}$. It follows that z_- is a relative equilibrium for the Hamiltonian vector field X_H, and therefore $z_- = e^{i\theta} v_\Omega$ for some $\theta \in \mathbb{R}$ and some $\Omega \in I_- \cup I_+$. From $f = F(z_-) = F(v_\Omega)$ we get $\Omega = \Omega_-(f)$ or $\Omega = \Omega_+(f)$, and then $\alpha = H(z_-) = H(v_\Omega) = \min\{H(\Omega_-(f)), H(\Omega_+(f))\} = \min\{h_-(f), h_+(f)\} = h_-(f)$. In the same way one proves that $\beta = h_+(f)$, and (8.9) follows. In subsection 8.2 we will show that for each point (h, f) in the interior of the region $\mathcal{R}$ the set

$$T_{h,f} := \{z \in \mathbb{C}^2 \mid (H(z), F(z)) = (h, f)\} \tag{8.10}$$

is a 2-torus invariant under the flows of X_H and X_F; this same property also follows from general considerations — see for example [5].

To obtain periodic solutions which do not correspond to relative equilibria we have to find those $(h, f) \in \text{int}(\mathcal{R})$ for which the flow on the torus $T_{h,f}$ is periodic; our discussion on integrable systems in section 6 suggests that in (H, F)-space we should find one-dimensional curves of points corresponding to periodic tori. One way to get a grip on such curves of periodic tori is to look for subharmonic bifurcations from the branches of relative equilibria — compare with remark (vii) at the end of section 7. A numerical study using AUTO indeed detects lots of such subharmonic bifurcations, both from the branch A_+ and from the branch A_-. Fixing one such bifurcation point (say on A_+), switching branches and continuing the subharmonic periodic orbit using the continuation scheme of theorem 13 (with $k = 2$, $F_1 = H$ and $F_2 = F$) gives a one-dimensional curve in the (H, F)-diagram which terminates on a relative equilibrium along the branch A_- (see Fig. 2), i.e. numerically we find "bridges" between the branches A_- and A_+ of relative equilibria.

In the remainder of this section we will first work out a symmetry-reduction for the system (8.2) and then use the reduced system to (a) prove the existence of invariant 2-tori, (b) introduce the rotation number for such torus, (c) give a theoretical proof for the existence of subharmonic bifurcations from the branches of relative equilibria, and (d) obtain a formula relating the points on A_- and A_+ which are connected by a bridge.

8.2 The symmetry reduction, invariant tori and rotation numbers

Following Cushman and Bates [5] we define a mapping $w : \mathbb{C}^2 \to \mathbb{R}^3$ by

$$\left\{ \begin{array}{rcll} w_1 & = & w_1(z) & := i(a\bar{b} - \bar{a}b), \\ w_2 & = & w_2(z) & := a\bar{b} + \bar{a}b, \qquad\qquad \forall z = (a,b) \in \mathbb{C}^2; \\ w_3 & = & w_3(z) & := a\bar{a} - b\bar{b}, \end{array} \right. \tag{8.11}$$

together with $F(z)$ the scalar functions $w_1(z)$, $w_2(z)$ and $w_3(z)$ form a basis for the quadratic invariants under the S^1-action on $\mathbb{C}^2$. Observe that $F(z)$ and $w_1(z)^2+w_2(z)^2+w_3(z)^2 = F(z)^2$ are constants of motion for the system (8.2), which implies that in w-space all spheres around the origin are flow-invariant. Using polar coordinates for the components a and b of $z \in \mathbb{C}^2$ we can rewrite (8.11) as

$$w(z) = \left(2\rho_a\rho_b \sin(\beta-\alpha), 2\rho_a\rho_b \cos(\beta-\alpha), \rho_a^2 - \rho_b^2\right), \qquad \forall z = \left(\rho_a e^{i\alpha}, \rho_b e^{i\beta}\right) \in \mathbb{C}^2. \tag{8.12}$$

It follows from this formula that for each $\bar{w} \in \mathbb{R}^3 \setminus \{0\}$ the inverse image $w^{-1}(\bar{w}) = \{z \in \mathbb{C}^2 \mid w(z) = \bar{w}\}$ is an S^1-orbit in $\mathbb{C}^2$. Indeed, from

$$\bar{w}_3 = \rho_a^2 - \rho_b^2 \qquad \text{and} \qquad \sqrt{\bar{w}_1^2 + \bar{w}_2^2 + \bar{w}_3^2} = \rho_a^2 + \rho_b^2$$

we can uniquely determine $\rho_a \geq 0$ and $\rho_b \geq 0$, and the relations $w_1(z) = \bar{w}_1$ and $w_2(z) = \bar{w}_2$ then give us $(\beta - \alpha)$ modulo 2π (at least if $\rho_a > 0$ and $\rho_b > 0$ — one has to consider separately the cases $\rho_a = 0$ or $\rho_b = 0$); this proves our claim. For later use we also write out the explicit expression for $\tilde{w}(\Omega) := w\left(v_\Omega\right)$, namely

$$\tilde{w}(\Omega) = \left(0, \frac{2(1-\Omega^2)}{\Omega^2}, \frac{(1-\Omega^2)^2}{\Omega^3}\right), \qquad \forall \Omega \in I_- \cup I_+. \tag{8.13}$$

Next we consider in more detail the flow on the sphere $S_f := \{w \in \mathbb{R}^3 \mid w_1^2 + w_2^2 + w_3^2 = f^2\}$ with radius $f > 0$; an easy calculation shows that this flow takes the form

$$\left\{ \begin{array}{rcl} \dot{w}_1 & = & 2w_3 - \frac{1}{2}\, w_2(f - w_3), \\ \dot{w}_2 & = & \frac{1}{2}\, w_1(f - w_3), \qquad\qquad \text{or equivalently:} \qquad \dot{w} = -2w \times \nabla_w H_f(w), \\ \dot{w}_3 & = & -2w_1, \end{array} \right. \tag{8.14}$$

with $H_f : \mathbb{R}^3 \to \mathbb{R}$ given by

$$H_f(w) := w_2 - \tfrac{1}{8}(f - w_3)^2, \qquad \forall w = (w_1, w_2, w_3) \in \mathbb{R}^3. \tag{8.15}$$

Since $H_f(w(z)) = H(z)$ for each z on the level set $\{z \in \mathbb{C}^2 \mid F(z) = f\}$ it follows that the restriction of H_f to the sphere S_f is a first integral for the flow on that sphere; this can also

be seen directly from (8.14). The discussion in subsection 8.1 implies that for $f > 0$ and $h \in \mathbb{R}$ the set $\Lambda_{h,f} := \{w \in S_f \mid H_f(w) = h\}$ is nonempty if and only if $h_-(f) \leq h \leq h_+(f)$, i.e. if and only if $(h, f) \in \mathcal{R}$. In Fig. 3 we depict (for a fixed value of $f > 0$ and a few values of h between $h_-(f)$ and $h_+(f)$) the projection of $\Lambda_{h,f}$ onto the (w_2, w_3)-plane. The sets $\Lambda_{h_-(f),f}$ and $\Lambda_{h_+(f),f}$ consist each of a single point, namely

$$\Lambda_{h_-(f),f} = \{\tilde{w}(\Omega_-(f))\} = w(\Gamma_{\Omega_-(f)}) \qquad \text{and} \qquad \Lambda_{h_+(f),f} = \{\tilde{w}(\Omega_+(f))\} = w(\Gamma_{\Omega_+(f)});$$

in these points the sphere S_f is tangent to the level sets $\{w \in \mathbb{R}^3 \mid H_f(w) = h_-(f)\}$ and $\{w \in \mathbb{R}^3 \mid H_f(w) = h_+(f)\}$, respectively. It follows that the points $\tilde{w}(\Omega_-(f))$ and $\tilde{w}(\Omega_+(f))$ are the critical points of the restriction of H_f to S_f and the equilibria for the flow on S_f; these equilibria correspond to the relative equilibria of the original system (8.2). For $h_-(f) < h < h_+(f)$ the set $\Lambda_{h,f}$ is a closed curve on S_f, invariant under the reduced flow (8.14); as a consequence the set $w^{-1}(\Lambda_{h,f}) = \{z \in \mathbb{C}^2 \mid F(z) = f \text{ and } H(z) = h\} = T_{h,f}$ is a 2-torus which is invariant under the flow (8.2) and under the S^1-action. In order to determine whether the flow on this torus is periodic or quasi-periodic we first introduce the rotation number for this flow.

For each $(h, f) \in \text{int}\,(\mathcal{R})$ the closed curve $\Lambda_{h,f}$ forms a periodic orbit for the reduced system (8.14), with minimal period $\tau(h, f) > 0$; this means that for each point $z \in T_{h,f}$ we have $w\left(\varphi_H^{\tau(h,f)}(z)\right) = w(z)$, which in turn implies that

$$\varphi_H^{\tau(h,f)}(z) = \varphi_F^{2\pi\Theta(h,f)}(z), \qquad \forall z \in T_{h,f}, \tag{8.16}$$

for some real number $\Theta(h, f)$ which is uniquely determined modulo 1. This number $\Theta(h, f)$ is independent of the choice of the point $z \in T_{h,f}$, as follows immediately from the fact that φ_H and φ_F commute and that $T_{h,f} = \{\varphi_H^t(\varphi_F^s(z)) \mid t, s \in \mathbb{R}\}$ for each $z \in T_{h,f}$. We call

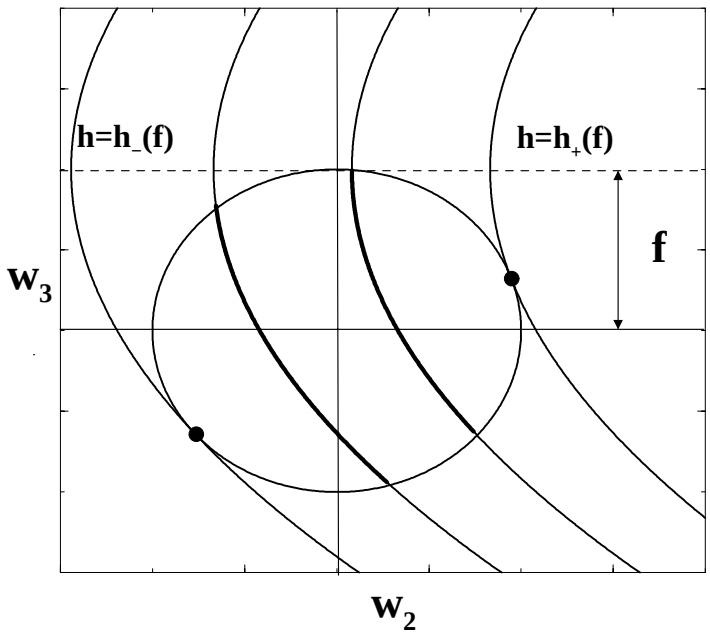

Figure 3: The sets $\Lambda_{h,f}$ projected onto the (w_2, w_3)-plane.

$\Theta(h,f)$ the **rotation number** of the flow on $T_{h,f}$. The same argument also shows that either all points on $T_{h,f}$ generate a periodic orbit of X_H (all with the same period), or no orbit on $T_{h,f}$ is closed (the quasi-periodic case). If the flow on $T_{h,f}$ is periodic then the minimal period is necessarily an integer multiple of $\tau(h,f)$, say $p\,\tau(h,f)$ for some $p \geq 1$. If $p = 1$ then $\Theta(h,f) = 0 \pmod{\mathbb{Z}}$ (this case seems not to appear in our example), if $p > 1$ then $p\,\Theta(h,f) \in \mathbb{Z}$, i.e.

$$\Theta(h,f) = \frac{q}{p} \pmod{\mathbb{Z}}, \qquad \text{with } 0 < q < p \text{ and } \gcd(q,p) = 1. \tag{8.17}$$

Conversely, if either $\Theta(h,f) = 0 \pmod{\mathbb{Z}}$ or if (8.17) holds, then the flow on $T_{h,f}$ is periodic, with minimal period equal to $\tau(h,f)$ in the first case and to $p\,\tau(h,f)$ in the second case. A careful analysis shows that both $\tau(h,f)$ and $\Theta(h,f)$ depend smoothly on $(h,f) \in \text{int}\,(\mathcal{R})$; moreover, if $(h_0,f_0) \in \text{int}\,(\mathcal{R})$ is such that the orbits on T_{f_0,h_0} are periodic and normal, then the continuation of one of these orbits using the approach of theorem 13 will result in a smooth one-dimensional curve contained in the interior of $\mathcal{R}$ and such that each point on this curve corresponds to a periodic torus and hence to a rational rotation number. It follows that $\Theta(h,f)$ will be constant along such curve (compare with the analysis in subsection 6.2, where we found constant winding numbers along curves of periodic orbits). In particular, the rotation number should be constant along the bridges which we found numerically and which are displayed in Fig. 2.

We conclude this subsection with the observation that the reduced system

$$\begin{cases} \dot{w}_1 &= 2w_3 - \dfrac{1}{2}\,w_2\left(\sqrt{w_1^1 + w_2^2 + w_3^2} - w_3\right), \\ \dot{w}_2 &= \dfrac{1}{2}\,w_1\left(\sqrt{w_1^1 + w_2^2 + w_3^2} - w_3\right), \\ \dot{w}_3 &= -2w_1, \end{cases} \tag{8.18}$$

(obtained from (8.14) by replacing f by $\sqrt{w_1^1 + w_2^2 + w_3^2}$) is also time-reversible, since it is invariant under the transformation $(w_1,w_2,w_3,t) \mapsto (-w_1,w_2,w_3,-t)$. This is in agreement with the fact that for each $(h,f) \in \text{int}\,(\mathcal{R})$ the periodic orbit $\Lambda_{h,f}$ of (8.18) has precisely two intersection points with the symmetry plane $w_1 = 0$; these two points are under the flow of (8.18) precisely half a period (i.e. $\frac{1}{2}\tau(h,f)$) away from each other.

8.3 Subharmonic bifurcations from the relative equilibria

In this subsection we show that a branch of subharmonic solutions bifurcates from the branches A_- and A_+ of relative equilibria at points where the nontrivial multipliers are roots of unity. The nontrivial multipliers of the relative equilibrium v_Ω (with $\Omega \in I_- \cup I_+$) are equal to $\exp(\pm 2\pi i \lambda(\Omega))$, with $\lambda(\Omega) > 1$ given by (8.6). Let $\Omega_0 \in I_- \cup I_+$ be such that

$$\lambda(\Omega_0) = \sqrt{1 + \frac{3}{\Omega_0^4}} = \frac{p}{q}, \qquad \text{with } p,q \in \mathbb{N},\ 0 < q < p \text{ and } \gcd(p,q) = 1. \tag{8.19}$$

Then we can expect at v_{Ω_0} the bifurcation of so-called q-subharmonics, i.e. periodic orbits near Γ_{Ω_0} with minimal period near $q\,T_{\Omega_0} = 2\pi q/|\Omega_0|$. Due to the S^1-symmetry each subharmonic solution generates a torus $T_{h,f}$ filled with subharmonic solutions, all with the same

minimal period and related to each other by symmetry; the parameters (h, f) identifying the torus must be close to $(F(v_{\Omega_0}), H(v_{\Omega_0}))$, the rotation number $\Theta(h, f)$ must be rational, and the minimal period for the flow on the torus must be close to $2\pi q/|\Omega_0|$. In order to determine the bifurcating subharmonics we will first find the appropriate rotation number, and then solve an equation such as (8.17) for (h, f) near $(F(v_{\Omega_0}), H(v_{\Omega_0}))$. Unfortunately, to work this out we cannot use the obvious variables (h, f); indeed, the point $(F(v_{\Omega_0}), H(v_{\Omega_0}))$ is at the boundary of the region $\text{int}(\mathcal{R})$ where the functions $\tau(h, f)$ and $\Theta(h, f)$ are defined, and this makes it difficult to apply the implicit theorem to solve (8.17). Instead we will use the variables (w_2, w_3), as follows.

As we have seen in subsection 8.2 each of the tori $T_{h,f}$ takes in w-space the form of a curve $\Lambda_{h,f}$ which has two intersection points with the plane $w_1 = 0$; for relative equilibria the curve $\Lambda_{h,f}$ shrinks down to a point in the same plane. The idea is now to use the coordinates (w_2, w_3) of these intersection points to parametrize the tori $T_{h,f}$. Taking (8.15) into account we consider the continuous mapping

$$(w_2, w_3) \mapsto (H^*(w_2, w_3), F^*(w_2, w_3)) := \left(w_2 - \frac{1}{8}\left(\sqrt{w_2^2 + w_3^2} - w_3\right)^2, \sqrt{w_2^2 + w_3^2} \right) \quad (8.20)$$

which is smooth outside of the origin and maps the (w_2, w_3)-plane onto the region $\mathcal{R}$ in the (h, f)-plane. The mapping (8.20) has a fold along the curve

$$A^* := \left\{ (w_2, w_3) \mid 4w_3 = w_2 \left(\sqrt{w_2^2 + w_3^2} - w_3\right) \right\}$$

and maps the complement of this curve in a two-to-one way onto the interior of $\mathcal{R}$; the curve A^* itself represents the equilibria of (8.18) and is mapped in a one-to-one way onto the boundary of $\mathcal{R}$. We can split A^* as $A^* = A^*_- \cup \{(0, 0)\} \cup A^*_+$, where A^*_- and A^*_+ represent respectively A_- and A_+ in the (w_2, w_3)-plane (compare with (8.13)):

$$A^*_\pm = \{(\tilde{w}_2(\Omega), \tilde{w}_3(\Omega)) \mid \Omega \in I_\pm\} = \left\{ \left(\frac{2(1-\Omega^2)}{\Omega^2}, \frac{(1-\Omega^2)^2}{\Omega^3} \right) \mid \Omega \in I_\pm \right\}.$$

For each $(w_2, w_3) \in \mathbb{R}^2$ not on A^* we set

$$\tau^*(w_2, w_3) := \tau(F^*(w_2, w_3), H^*(w_2, w_3)) \quad \text{and} \quad \Theta^*(w_2, w_3) := \Theta(F^*(w_2, w_3), H^*(w_2, w_3));$$

$\tau^*(w_2, w_3)$ is the minimal period of the periodic solution of (8.18) which starts at the point $(0, w_2, w_3)$, and a local analysis of (8.18) near its equilibria shows that $\tau^*(w_2, w_3)$ can be extended to a smooth function on $\mathbb{R}^2 \backslash \{(0, 0)\}$. The same then also holds true for the function $\Theta^*(w_2, w_3)$. Next we will calculate $\tau^*(w_2, w_3)$ and $\Theta^*(w_2, w_3)$ for $(w_2, w_3) \in A_- \cup A_+$.

Fix some $\Omega_0 \in I_- \cup I_+$; in order to determine $\tau^*(\tilde{w}_2(\Omega_0), \tilde{w}_3(\Omega_0))$ we set $f = F(v_{\Omega_0})$ in (8.14) and linearize this conservative and reversible system at its equilibrium $\tilde{w}(\Omega_0)$; the resulting matrix

$$\begin{pmatrix} 0 & -\dfrac{1-\Omega_0^2}{\Omega_0^2} & \dfrac{1+\Omega_0^2}{\Omega_0^2} \\ \dfrac{1-\Omega_0^2}{\Omega_0^2} & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}$$

has eigenvalues $\mu = 0$ and $\mu = \pm i\,|\Omega_0|\,\lambda(\Omega_0)$ (with $\lambda(\Omega)$ given by (8.6)). We conclude that

$$\tau^*(\tilde{w}_2(\Omega_0), \tilde{w}_3(\Omega_0)) = \frac{2\pi}{|\Omega_0|\,\lambda(\Omega_0)}. \tag{8.21}$$

Moreover, we have that $\varphi_H^t(v_{\Omega_0}) = \varphi_F^{\Omega_0 t}(v_{\Omega_0})$, which gives us via (8.16) the rotation number

$$\Theta^*(\tilde{w}_2(\Omega_0), \tilde{w}_3(\Omega_0)) = \frac{\Omega_0}{2\pi}\,\tau^*(\tilde{w}_2(\Omega_0), \tilde{w}_3(\Omega_0)) = \frac{\operatorname{sgn}\Omega_0}{\lambda(\Omega_0)} \pmod{\mathbb{Z}}. \tag{8.22}$$

Now let us fix some $\Omega_0 \in I_- \cup I_+$ such that (8.19) holds; then

$$\tau^*(\tilde{w}_2(\Omega_0), \tilde{w}_3(\Omega_0)) = \frac{2\pi q}{p\,|\Omega_0|} \quad \text{and} \quad \Theta^*(\tilde{w}_2(\Omega_0), \tilde{w}_3(\Omega_0)) = (\operatorname{sgn}\Omega_0)\frac{q}{p} \pmod{\mathbb{Z}}. \tag{8.23}$$

Bifurcating q-subharmonic solutions correspond to values $(w_2, w_3) \in \mathbb{R}^2 \setminus A^*$ which are close to $(\tilde{w}_2(\Omega_0), \tilde{w}_3(\Omega_0)$, for which $\Theta^*(w_2, w_3)$ is rational, say $\Theta^*(w_2, w_3) = \tilde{q}/\tilde{p} \pmod{\mathbb{Z}}$ with $0 < \tilde{q} < \tilde{p}$ and $\gcd(\tilde{p}, \tilde{q}) = 1$, and for which $\tilde{p}\,\tau^*(w_2, w_3)$ is close $2\pi q/|\Omega_0|$. In view of (8.23) and the continuity of τ^* and Θ^* this implies $\tilde{p} = p$ and $\tilde{q} = (\operatorname{sgn}\Omega_0)\,q \pmod{p}$; therefore we will find all bifurcating q-subharmonics by solving the equation

$$\Theta^*(w_2, w_3) = (\operatorname{sgn}\Omega_0)\frac{q}{p} \tag{8.24}$$

for (w_2, w_3) near $(\tilde{w}_2(\Omega_0), \tilde{w}_3(\Omega_0)$. Differentiation of (8.22) shows that

$$\frac{d}{d\Omega}\,\Theta^*(\tilde{w}_2(\Omega), \tilde{w}_3(\Omega))\Big|_{\Omega=\Omega_0} = \frac{6\operatorname{sgn}\Omega_0}{\Omega_0^5\,\lambda(\Omega_0)^3} \neq 0;$$

it follows that $D_{(w_2,w_3)}\Theta^*(\tilde{w}_2(\Omega_0), \tilde{w}_3(\Omega_0)) \in \mathcal{L}(\mathbb{R}^2; \mathbb{R})$ is surjective, and that the solution set of (8.24) forms near $(\tilde{w}_2(\Omega_0), \tilde{w}_3(\Omega_0)$ a smooth curve which is transversal to A^* at the same point. It is important to notice that the solutions $(w_2, w_3) \notin A^*$ of (8.24) come in pairs; the two solutions of each such pair are related by the fact that they parametrize the two intersection points of the same closed orbit $\Lambda_{h,f}$ of (8.18) with the plane $w_1 = 0$ (see the remark at the end of the foregoing subsection). When we return to the (H, F)-plane two such related solutions are mapped on the same point; as a consequence, the branch of subharmonics which in (w_2, w_3)-space crosses A_-^* or A_+^* at $(\tilde{w}_2(\Omega_0), \tilde{w}_3(\Omega_0))$ will be folded onto a half branch bifurcating from A_- or A_+ at $(H(v_{\Omega_0}), F(v_{\Omega_0}))$. It is precisely these kind of branches which we have obtained numerically and some of which are indicated on Fig. 2.

One may observe that the result which we have found here differs from the typical subharmonic branching behaviour in Hamiltonian or reversible systems (see e.g. [4] and [24]), where in general *two* branches of subharmonics bifurcate from branches of periodic orbits at points where there are some multipliers which are roots of unity. The reason for the non-typical behaviour of our example is the S^1-symmetry; we hope to give in the near future a more general treatment of subharmonic bifurcation in symmetric Hamiltonian or reversible systems.

8.4 Relative equilibria connected by bridges

We finish our discussion of the example (8.2) by finding out which relative equilibria can be connected by a bridge of subharmonics such as indicated in Fig. 2; we do not prove the existence of such bridges, but assume their existence based on the numerical evidence. Let $\Omega_1, \Omega_2 \in I_- \cup I_+$ be such that

$$\lambda(\Omega_i) = \frac{p_i}{q_i}, \qquad \text{with } 0 < q_i < p_i \text{ and } \gcd(p_i, q_i) = 1, \quad (i = 1, 2).$$

The branch of subharmonic solutions bifurcating from A_- or A_+ at v_{Ω_i} has rotation number (see (8.23))

$$\Theta_i = \frac{\operatorname{sgn} \Omega_i}{\lambda(\Omega_i)} = (\operatorname{sgn} \Omega_i)\frac{q_i}{p_i}, \qquad (i = 1, 2).$$

The two branches starting at respectively v_{Ω_1} and v_{Ω_2} can only be connected if $\Theta_1 = \Theta_2 \pmod{\mathbb{Z}}$. We see from (8.6) that $1 < \lambda(\Omega) < 2$ if $\Omega \in I_-$, and $\lambda(\Omega) > 2$ if $\Omega \in I_+$; therefore we have $-1 < \Theta_i < -\frac{1}{2}$ if $\Omega_i \in I_-$, and $0 < \Theta_i < \frac{1}{2}$ if $\Omega_i \in I_+$. It follows that no bridge can exist between two different relative equilibria on A_-, or between two different relative equilibria on A_+. If there is a bridge connecting $v_{\Omega_1} \in A_-$ to $v_{\Omega_2} \in A_+$ then necessarily

$$-\frac{q_1}{p_1} = \frac{q_2}{p_2} \pmod{\mathbb{Z}} \quad \Longrightarrow \quad -\frac{q_1}{p_1} = \frac{q_2}{p_2} - 1 \quad \Longrightarrow \quad p_1 = p_2 \text{ and } q_1 = p_2 - q_2, \qquad (8.25)$$

i.e. we must have that $\lambda(\Omega_1) = \lambda(\Omega_2)/(\lambda(\Omega_2) - 1)$. This shows that for each relative equilibrium on A_+ where the nontrivial multipliers are roots of unity there exists a unique relative equilibrium on A_- such that both relative equilibria can be connected by a bridge of subharmonics; our numerical results indicate that such bridge indeed exists. These numerical results also confirm the rule (8.25) which relates connected relative equilibria (see Fig. 2).

At this point a natural topological question arises regarding the invariant tori along a subharmonic bridge: how does the flow on these tori change as we move from one end of the bridge to the other? Remember that the flow induces opposite orientations on the relative equilibria at both ends of the bridge. Introducing appropriate angle coordinates one obtains explicit formulae which verify the "-1"-jump in (8.25) and which explain the reversal of the orientation; again the numerics confirm the analytical results. Details on this particular aspect of the model will be given elsewhere (see [27]).

We finish this rather long discussion of the system (8.2) with the observation that in this example, as in many others, numerical continuation techniques give us some very valuable global information which nicely complements the analytical results which usually are only local.

9 Discussion

Hamiltonian systems with symmetry arise abundantly when modeling the most diverse kind of applications; understanding the structure of the periodic orbits and invariant tori of such systems forms a first but indispensable step towards understanding the full system. We believe that the continuation method elaborated in the foregoing sections can in many cases

help to unraffle the sometimes complicated branching behaviour of these periodic orbits; together with local analytical results it can provide a kind of "skeleton" of the full system. Moreover, the method is relatively easy to implement; in setting up the system there is no need for special choices of coordinates, and all the required routines are provided by well tested packages such as AUTO.

Beside the (more didactical) examples worked out in this paper we have applied the method to a number of other systems; the results of these applications have or will be reported elsewhere. First there is the mean field model for four coupled quantum wells (see [8] and [9]). This is an 8-dimensional system which is both Hamiltonian and reversible and which has an S^1-symmetry; the example (8.2) discussed in Section 8 forms in fact a scaled down version with only two coupled quantum wells. The method has also been applied extensively on two particular instances of the 3-body problem (see [6] and [10]), namely the circular restricted three-body problem and the continuation of the Chenciner-Montgomery solution [3] of the 3-body problem with equal masses when one of the masses is allowed to vary. Further models on which the method has been tested successfully include the spring pendulum, the spherical pendulum and the normal form system at a Hamiltonian Hopf bifurcation. The approach can also be adapted to handle the Kepler problem which is an example of a superintegrable system due to the fact that next to the linear and the angular momentum there is an additional first integral, namely the Lagrange-Runge-Lenz vector.

While one can think of many more applications, also some further theoretical work remains to be done. In particular, it would be nice to overcome the restriction to abelian symmetry groups in the continuation results for relative equilibria as given in subsection 6.1; recent discussions with Juan-Pablo Ortega and James Montaldi give us some hope that this is indeed possible.

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