Bifurcation of Periodic Points in Reversible Diffeomorphisms

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Abstract In this paper we survey a general Liapunov-Schmidt type of reduction for the study of the bifurcation of periodic points from a symmetric fixed point in families of reversible diffeomorphisms. The approach is strongly interwoven with normal form theory for reversible mappings, and also addresses the stability problem for the bifurcating periodic points. The paper concludes with an application to subharmonic bifurcation in reversible vectorfields.

Keywords Reversible diffeomorphisms, periodic points, Liapunov-Schmidt reduction, normal forms, subharmonic bifurcation

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1 Introduction

For the study of subharmonic bifurcations from a given periodic orbit in a parametrized family of autonomous systems it is a standard approach to introduce the Poincaré map associated to the periodic orbit and to study the bifurcation of periodic points in this map. This can be done either by a normal form approach or by an appropriate Lyapunov-Schmidt type of reduction (see e.g. [5]). When the system is reversible (in a sense to be defined further on) and the periodic orbit is symmetric then also the Poincaré map will be reversible, and this additional structure should be taken into account when performing the reduction and analysing the bifurcations. The aim of this note is to give a brief survey on how this can be done; for more details we refer to the Ph. D. thesis [1] of the first author and to the papers [2] and [3]. In the same thesis [1] and in the paper [4] one can find similar results on symplectic diffeomorphisms.

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2 Preliminaries

We start with the definition of a reversible diffeomorphism. Let V be a (finite-dimensional) state space and $\Gamma \subset \mathcal{L}(V)$ a compact group of linear operators acting on V. Also, let $\chi : \Gamma \longrightarrow \mathbb{Z}_2 := \{1, -1\}$ be a non-trivial group-homomorphism (such group-homomorphism is usually called a *character* of Γ). A diffeomorphism $\Phi \in C^\infty(V)$ is then called Γ-*reversible* if

$$\Phi(\gamma \cdot x) = \gamma \cdot \Phi^{\chi(\gamma)}(x), \qquad \forall x \in V,\ \forall \gamma \in \Gamma.$$

The basic case appears when $\Gamma = \{I_V, R\}$, with $R \in \mathcal{L}(V)$ a linear involution (i.e. $R^2 = I_V$), and when χ is given by $\chi(I_V) = 1$ and $\chi(R) = -1$; a diffeomorhism $\Phi \in C^\infty(V)$ is then R-*reversible* if

$$R \circ \Phi \circ R = \Phi^{-1}. \tag{1}$$

For the sake of simplicity we will in this paper restrict to this simple case.

Such R-reversible diffeomorphisms arise for example as stroboscopic maps for periodic non-autonomous time-reversible systems: Φ is then the time-T-map associated to a T-periodic system of the form

$$\dot{x} = f(t, x), \tag{2}$$

with $f : \mathbb{R} \times V \to V$ such that $f(t+T, x) = f(t, x)$ and $f(-t, Rx) = -Rf(t, x)$. The flow $\varphi(t, x)$ of this system will be such that $\varphi(t, \varphi(T, x)) = \varphi(t + T, x)$ and $\varphi(t, Rx) = R\varphi(-t, x)$, and hence $\Phi := \varphi(T, \cdot)$ will satisfy (1). In a similar way one can consider an autonomous reversible system on a finite-dimensional space X, of the form

$$\dot{x} = f(x), \tag{3}$$

with $f : X \to X$ smooth and such that $f(R_0 x) = -R_0 f(x)$ for some linear involution $R_0 \in \mathcal{L}(X)$. A periodic orbit κ of (3) is *symmetric* if $R_0(\kappa) = \kappa$; one can show that these symmetric periodic orbits are precisely those orbits which have two intersection points p_1 and p_2 with $\mathrm{Fix}(R_0) := \{x \in X \mid R_0 x = x\}$. To construct a Poincaré-map associated with κ we take two R_0-invariant transversal sections Σ_1 and Σ_2 to κ, at respectively p_1 and p_2. The Poincaré-map $\Phi : \Sigma_1 \to \Sigma_1$ can then be written as

$$\Phi = \Phi_{2\to1} \circ \Phi_{1\to2},$$

where $\Phi_{1\to2}$ is the "halfway" Poincaré-map from Σ_1 to Σ_2, and $\Phi_{2\to1}$ the second halfway Poincaré-map starting at Σ_2 and arriving at Σ_1. The reversibility of (3) together with $R_0(\Sigma_1) = \Sigma_1$ and $R_0(\Sigma_2) = \Sigma_2$ imply that $\Phi_{2\to1} = R_0 \circ \Phi_{1\to2}^{-1} \circ R_0$ and

$$\Phi = R_0 \circ \Phi_{1\to2}^{-1} \circ R_0 \circ \Phi_{1\to2}.$$

Now identify Σ_1 with a vectorspace V in such a way that $p_1 \in \Sigma_1$ corresponds to $0 \in V$, and denote the restriction of R_0 to V by R; the mapping $\Phi : V \to V$

is then an R-reversible diffeomorphism, with $\Phi(0) = 0$. Periodic points of Φ bifurcating from the fixed point $x = 0$ correspond to subharmonic periodic orbits of (3) bifurcating from the symmetric periodic orbit κ. We will return to this problem of subharmonic bifurcation in Section 6.

Consider now a parametrized family of R-reversible diffeomorphisms with a symmetric fixed point; more precisely, let $\Phi : V \times \mathbb{R}^m \to \mathbb{R}$ (with $m \geq 0$) be C^∞-smooth and such that

(H1) (i) $\Phi(0, \lambda) = 0$ for all $\lambda \in \mathbb{R}^m$; let $A_\lambda := D_x\Phi(0, \lambda)$;

(ii) $A_0 \in \mathcal{L}(V)$ is invertible; hence $\Phi_\lambda = \Phi(\cdot, \lambda)$ is a local diffeomorphism for small λ;

(iii) $R \circ \Phi_\lambda \circ R = \Phi_\lambda^{-1}$, with $R \in \mathcal{L}(V)$ a linear involution.

Fix some integer $q \geq 1$, and consider the following problem:

($\mathbf{P}_q$) Find, for all small $\lambda \in \mathbb{R}^m$, all q-periodic points of Φ_λ close to the fixpoint $x = 0$.

Solving ($\mathbf{P}_q$) means to find all solutions $(x, \lambda) \in V \times \mathbb{R}^m$ near $(0, 0)$ of the fixpoint equation

$$\Phi_\lambda^{(q)}(x) = x, \qquad \text{with } \Phi_\lambda^{(q)} := \Phi \circ \cdots \circ \Phi \quad (q \text{ times}). \tag{4}$$

The problem ($\mathbf{P}_q$) has what we call an *implicit D_q-symmetry*. To explain what we mean by that let us denote by $\mathcal{S}_\lambda$ the solution set:

$$\mathcal{S}_\lambda := \left\{ x \in V \mid \Phi_\lambda^{(q)}(x) = x \right\}.$$

Clearly $x \in \mathcal{S}_\lambda$ implies $\Phi_\lambda(x) \in \mathcal{S}_\lambda$, and also $Rx \in \mathcal{S}_\lambda$, as can be seen from $\Phi_\lambda^{(q)}(Rx) = R\Phi_\lambda^{(-q)}(x)$. Now observe that $\Phi_\lambda^{(q)}$ acts as the identity on $\mathcal{S}_\lambda$, and hence Φ_λ generates a $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$-action on $\mathcal{S}_\lambda$. The operator R on the other hand generates a $\mathbb{Z}_2$-action, as follows from $R^2 = I$. Finally, since $R \circ \Phi_\lambda = \Phi_\lambda^{-1} \circ R$ we conclude that Φ_λ and R generate a D_q-action on $\mathcal{S}_\lambda$. (As a reminder: D_q is the symmetry group of a regular q-gone; it contains $2q$ elements and can be generated by the rotation over $2\pi/q$ and a reflection, both in the plane). We call this D_q-symmetry implicit because it appears only on the (yet to determine!) solution set $\mathcal{S}_\lambda$.

The approach to the problem ($\mathbf{P}_q$) which we describe in the next sections is based on a Liapunov-Schmidt type of reduction which lowers the dimension of the problem and leads to algebraic bifurcation equations; this will be done in such way that the D_q-symmetry becomes *explicit* and can be used to bring the bifurcation equations in a kind of canonical form (at least in the simplest cases). Moreover we explain the relation with normal form theory for reversible diffeomorphisms, and show how these normal forms can be used to obtain some stability properties for the bifurcating periodic points. In the final section we apply these results to the problem of subharmonic bifurcation in reversible vectorfields.

3 Orbit space formulation and reduction

The basic idea behind our approach is to replace the equation (4) for the q-periodic points of Φ_λ by an equivalent equation in an appropriate orbit space, and to perform the Liapunov-Schmidt reduction on this equivalent equation. The starting point is the observation that a q-periodic point $x \in V$ of Φ_λ generates a q-periodic orbit

$$\left(\Phi_\lambda^i(x)\right)_{i\in\mathbb{Z}} \in V^{\mathbb{Z}}, \qquad \text{with } \Phi_\lambda^{i+q}(x) = \Phi_\lambda^i(x),\ \forall i \in \mathbb{Z}.$$

This motivates us to define an *orbit space* Y_q by

$$Y_q := \left\{ y = (y_i)_{i\in\mathbb{Z}} \in V^{\mathbb{Z}} \mid y_{i+q} = y_i,\ \forall i \in \mathbb{Z} \right\} \cong V^q. \tag{5}$$

The mapping Φ_λ can be lifted to this orbit space by defining $\widehat{\Phi}_\lambda : Y_q \to Y_q$ as

$$\widehat{\Phi}_\lambda(y) := \left(\Phi_\lambda(y_i)\right)_{i\in\mathbb{Z}}, \qquad \forall y \in Y_q. \tag{6}$$

We also need the *shift operator* $\sigma \in \mathcal{L}(Y_q)$ given by

$$(\sigma \cdot y)_i := y_{i+1}, \qquad \forall i \in \mathbb{Z}, \forall y \in Y_q. \tag{7}$$

With these definitions it is easy to prove the following result.

Lemma 3.1 *Let $\lambda \in \mathbb{R}^m$, and let $x \in V$ be a solution of (4). Define $y \in Y_q$ by $y_i := \Phi_\lambda^{(i)}(x)$ for all $i \in \mathbb{Z}$. Then*

$$\widehat{\Phi}_\lambda(y) = \sigma \cdot y. \tag{8}$$

Conversely, if $y \in Y_q$ solves (8) (for some $\lambda \in \mathbb{R}^m$) then $x := y_0 \in V$ is a q-periodic point of Φ_λ, i.e. x satisfies (4).

Lemma 3.1 gives a one-to-one relation between the q-periodic points of Φ_λ and the solutions of (8); using this relation the problem $(\mathbf{P}_q)$ amounts to finding all solutions $(y, \lambda) \in Y_q \times \mathbb{R}^m$ of (8) in a neighborhood of $(0, 0)$.

An important property of (8) is that this equation is $\mathbb{Z}_q$-*equivariant*: it follows from $\sigma^q = I$ that σ generates a $\mathbb{Z}_q := \mathbb{Z}/q\mathbb{Z}$-action on Y_q, and the definitions immediately imply that

$$\widehat{\Phi}_\lambda \circ \sigma = \sigma \circ \widehat{\Phi}_\lambda, \qquad \forall \lambda \in \mathbb{R}^m. \tag{9}$$

Also, the reversibility of Φ_λ is inherited by $\widehat{\Phi}_\lambda$. Indeed, we have

$$\widehat{\Phi}_\lambda \circ \rho = \rho \circ \widehat{\Phi}_\lambda^{-1}, \tag{10}$$

with the linear operator $\rho \in \mathcal{L}(Y_q)$ defined by

$$(\rho \cdot y)_i := R y_{-i}, \qquad \forall i \in \mathbb{Z},\ \forall y \in Y_q. \tag{11}$$

Since $\rho^2 = I$ it follows from (10) that $\widehat{\Phi}_\lambda$ is ρ-reversible. Moreover, the relation $\rho \circ \sigma = \sigma^{-1} \circ \rho$ implies that σ and ρ generate a D_q-action on the orbit space Y_q.

As a first step towards solving (8) we consider the linearized equation

$$\widehat{A}_0 \cdot y = \sigma \cdot y, \tag{12}$$

with $\widehat{A}_0 := D\widehat{\Phi}_{\lambda=0}(0) \in \mathcal{L}(Y_q)$ the lift of $A_0 := D\Phi_{\lambda=0}(0) \in \mathcal{L}(V)$. The reversibility of $\Phi_{\lambda=0}$ implies that of A_0 and $\widehat{A}_0$: $RA_0R = A_0^{-1}$ and $\rho\widehat{A}_0\rho = \widehat{A}_0^{-1}$. A result similar to that of Lemma 3.1 shows that $\mathrm{Ker}\,(\widehat{A}_0 - \sigma)$ (the solution space of (12)) and $\mathrm{Ker}\,(A_0^q - I)$ are isomorphic. For a straightforward application of the Liapunov-Schmidt reduction to (8) we have to determine complementary subspaces of respectively $\mathrm{Ker}\,(\widehat{A}_0 - \sigma)$ and $\mathrm{Im}\,(\widehat{A}_0 - \sigma)$ in Y_q; since at this point in the analysis we do not want to impose any spectral conditions on A_0 (except the fact that A_0 must be invertible) this can not be done in general. In particular, we do not want to exclude the possibility that A_0 and $\widehat{A}_0$ are not semisimple, and therefore we can not assume that we have the splittings

$$V = \mathrm{Ker}\,(A_0^q - I) \oplus \mathrm{Im}\,(A_0^q - I) \quad \text{and} \quad Y_q = \mathrm{Ker}\,(\widehat{A}_0 - \sigma) \oplus \mathrm{Im}\,(\widehat{A}_0 - \sigma).$$

which are frequently taken as a starting point for a Liapunov-Schmidt reduction. The remedy for this difficulty consists in using the semisimple part S_0 of A_0 to obtain appropriate splittings of the spaces V and Y_q. We now describe how this can be done.

We know from elementary algebra that A_0 has a unique *semisimple-nilpotent decomposition* (S-N-decomposition for short), which means that there exist unique linear operators $S_0 \in \mathcal{L}(V)$ and $N_0 \in \mathcal{L}(V)$ such that $A_0 = S_0 + N_0$, S_0 is semisimple (i.e. complex diagonisable), N_0 is nilpotent and $S_0N_0 = N_0S_0$. Then RS_0R is the semisimple part of RA_0R, and S_0^{-1} that of A_0^{-1}; combined with the reversibilty of A_0 and the uniqueness of the S-N-decomposition this implies that also S_0 is R-reversible. Some further straightforward algebra and the fact that σ is semisimple (since $\sigma^q = I$) gives the following.

Lemma 3.2 *Let $A_0 = S_0 + N_0$ be the S-N-decomposition of A_0. Then*

- *S_0 is R-reversible: $RS_0R = S_0^{-1}$;*
- *S_0^q is the semisimple part of A_0^q;*
- *$Ker(A_0^q - I) \subset Ker(S_0^q - I)$;*
- *$\widehat{A}_0 = \widehat{S}_0 + \widehat{N}_0$ is the S-N-decomposition of $\widehat{A}_0$;*
- *$\rho \circ \widehat{S}_0 \circ \rho = \widehat{S}_0^{-1}$;*
- *$(\widehat{S}_0 - \sigma)$ is the semisimple part of $(\widehat{A}_0 - \sigma)$;*
- *$Ker(\widehat{A}_0 - \sigma) \subset Ker(\widehat{S}_0 - \sigma) \cong Ker(S_0^q - I)$;*
- *$Y_q = Ker(\widehat{S}_0 - \sigma) \oplus Im(\widehat{S}_0 - \sigma)$;*
- *$(\widehat{A}_0 - \sigma)$ is invertible on $Im(\widehat{S}_0 - \sigma)$.*

Next we introduce the *reduced phase space* for the problem $(\mathbf{P}_q)$; this is the subspace U of V given by

$$U := \mathrm{Ker}\,(S_0^q - I), \tag{13}$$

i.e. U is the *generalized eigenspace* of A_0 corresponding to those eigenvalues which are q-th roots of unity. These eigenvalues are the *resonant eigenvalues* for the problem $(\mathbf{P}_q)$. Observe that in a standard Liapunov-Schmidt reduction one would work with the *eigenspace* corresponding to the resonant eigenvalues, while here we will use the generalized eigenspace. The following lemma summarizes the main properties of U.

Lemma 3.3 *Let U be the reduced phase space defined by (13), and let $\zeta : U \to Y_q$ be the linear operator given by*

$$\zeta(u) := \left(S_0^i u\right)_{i\in\mathbb{Z}}, \qquad \forall u \in U. \tag{14}$$

Then the following holds:

- *S_0 generates a $\mathbb{Z}_q$-action on U;*
- *U is invariant under R;*
- *ζ is an isomorphism of U onto Ker $(\widehat{S}_0 - \sigma)$;*
- *$\zeta(S_0 u) = \sigma \cdot \zeta(u)$ for all $u \in U$;*
- *$\zeta(Ru) = \rho \cdot \zeta(u)$ for all $u \in U$.*

Now we have all ingredients needed to perform our reduction of the equation (8); this reduction will be based on the splitting

$$Y_q = \zeta(U) \oplus \mathrm{Im}\,(\widehat{S}_0 - \sigma) \tag{15}$$

of the orbit space Y_q. Since $(\widehat{S}_0 - \sigma) \in \mathcal{L}(Y_q)$ is semisimple we have $Y_q = \mathrm{Ker}\,(\widehat{S}_0-\sigma)\oplus\mathrm{Im}\,(\widehat{S}_0-\sigma)$, which combined with $\mathrm{Ker}\,(\widehat{S}_0-\sigma) = \zeta(U)$ gives (15). It follows that each $y \in Y_q$ can be written in a unique way as $y = \zeta(u) + w$, with $u \in U$ and $w \in W := \mathrm{Im}\,(\widehat{S}_0 - \sigma)$. Then $\sigma \cdot y = \zeta(S_0 u) + \sigma \cdot w$, and the equation (8) splits as

$$\begin{aligned} &(a) \quad \Psi_\lambda(u, w) = S_0 u, \\ &(b) \quad \Theta_\lambda(u, w) = \sigma \cdot w; \end{aligned} \tag{16}$$

the mappings $\Psi_\lambda : U \times W \to U$ and $\Theta_\lambda : U \times W \to W$ are uniquely determinated from the relation

$$\Phi_\lambda(\zeta(u) + w) = \zeta(\Psi_\lambda(u, w)) + \Theta_\lambda(u, w), \qquad \forall (u, w, \lambda) \in U \times W \times \mathbb{R}^m.$$

Equation (16b) can (locally near $(u, w, \lambda) = (0, 0, 0)$) be solved for $w = w_\lambda^*(u)$ by the implicit function theorem; bringing this solution into (16a) gives the *determining equation*

$$\Phi_{r,\lambda}(u) = S_0 u, \tag{17}$$

with $\Phi_{r,\lambda} : U \to U$ defined by

$$\Phi_{r,\lambda}(u) := \Psi_\lambda(u, w^*_\lambda(u)), \qquad \forall (u,\lambda) \in U \times \mathbb{R}^m. \tag{18}$$

We call $\Phi_{r,\lambda}$ the *reduced mapping* for the problem $(\mathbf{P}_q)$. We refrain from calling (17) the "bifurcation equation" since in case some of the resonant eigenvalues are non-semisimple it is possible to make a further reduction to the subspace $\mathrm{Ker}\,(A_0^q - I)$ (which is then a proper subspace of U).

The next Lemma gives some basic properties of $\Phi_{r,\lambda}$; they follow easily from the definitions.

Lemma 3.4 *The reduced mapping $\Phi_{r,\lambda}$ has the following properties:*

- $\Phi_{r,\lambda}(0) = 0$ *for all* λ*;*
- $D\Phi_{r,\lambda=0}(0) = A_0\big|_U$*;*
- $\Phi_{r,\lambda}$ *is* $\mathbb{Z}_q$*-equivariant:* $\Phi_{r,\lambda} \circ S_0 = S_0 \circ \Phi_{r,\lambda}$*;*
- $\Phi_{r,\lambda}$ *is* R*-reversible:* $R \circ \Phi_{r,\lambda} \circ R = \Phi_{r,\lambda}^{-1}$.

The main result is of course the relation between the solutions of $(\mathbf{P}_q)$ and the solutions of the determing equation.

Theorem 3.5 *Under the foregoing conditions there exists a smooth mapping $x^* : U \times \mathbb{R}^m \to V$, with $x^*(0,\lambda) = 0$, $D_u x^*(0,0) \cdot u = u$ and $x^*(Ru,\lambda) = Rx^*(u,\lambda)$ for all (u,λ), and such that the following holds: for all sufficiently small $(x,\lambda) \in V \times \mathbb{R}^m$ we have $\Phi_\lambda^{(q)}(x) = x$ if and only if $x = x^*(u,\lambda)$ for some small $u \in U$ for which the determining equation $\Phi_{r,\lambda}(u) = S_0 u$ is satisfied; moreover, we have for such (u,λ) that $x^*(S_0 u,\lambda) = \Phi_\lambda(x^*(u,\lambda))$.*

Observe that the $\mathbb{Z}_q$-equivariance of $\Phi_{r,\lambda}$ implies that for each solution $u \in U$ of (17) also the other points $S_0 u$, $S_0^2 u$, $\ldots$, S_0^{q-1} on the $\mathbb{Z}_q$-orbit of u solve the same equation; the last statement of Theorem 3.5 says that the mapping $x^*(\cdot,\lambda)$ lifts this solution orbit of (17) to a full q-periodic orbit of Φ_λ.

It is interesting to consider for a moment the special case where we assume that the mappings Φ_λ are such that

$$\Phi_\lambda \circ S_0 = S_0 \circ \Phi_\lambda. \tag{19}$$

One can then verify that

$$x^*(u,\lambda) = u \qquad \text{and} \qquad \Phi_{r,\lambda} = \Phi_\lambda\big|_U,$$

and the reduction result of Theorem 3.5 takes the following form: *for sufficiently small $(x,\lambda) \in V \times \mathbb{R}^m$ we have that x is a q-periodic point of Φ_λ if and only if $x = u \in U$ and $\Phi_\lambda(u) = S_0 u$.*

Returning to the general case we can look for the q-periodic points of the reduced mapping $\Phi_{r,\lambda}$, and apply Theorem 3.5 to this new problem (i.e. we perform our reduction on the equation $\Phi_{r,\lambda}^{(q)}(u) = u$). It follows from Lemma 3.4 that $\Phi_{r,\lambda}$ satisfies the condition (27), and one can easily see that the reduced phase space for $\Phi_{r,\lambda}$ is the space U itself (there is no further reduction). The reduction result for the special case then leads to the following conclusion: *for each sufficiently small* $(u, \lambda) \in U \times \mathbb{R}^m$ *we have that* u *is a q-periodic point of* $\Phi_{r,\lambda}$ *if and only if* u *satisfies the determining equation* $\Phi_{r,\lambda}(u) = S_0 u$. *The q-periodic orbits of* $\Phi_{r,\lambda}$ *are therefore necessarily also orbits under the natural* $\mathbb{Z}_q$*-action on* U. This conclusion allows us to reformulate the reduction theorem as follows.

Theorem 3.6 (Main Reduction Theorem) *Let* Φ_λ $(\lambda \in \mathbb{R}^m)$ *be a parametrized family of R-reversible diffeomorphisms, satisfying the hypotheses (H1). Let* $q \geq 1$, *and define the reduced phase space* U *by (13). Then there exists a parametrized family of reduced R-reversible diffeomorphisms* $\Phi_{r,\lambda} : U \to U$ *such that for each sufficiently small* $\lambda \in \mathbb{R}^m$ *there is a* 1*-to-*1 *relation between the small q-periodic orbits of* Φ_λ *and the small q-periodic orbits of* $\Phi_{r,\lambda}$. *Moreover,* $\Phi_{r,\lambda}$ *is equivariant with respect to the natural* $\mathbb{Z}_q$*-action on* U*, and all small q-periodic orbits of* $\Phi_{r,\lambda}$ *are necessarily* $\mathbb{Z}_q$*-orbits; they are given by the solutions of the determining equation (17).*

We have observed before that the problem $(\mathbf{P}_q)$ has an implicit D_q-symmetry, in the sense that there is a natural D_q-action on the solution set. With the reduction the $\mathbb{Z}_q$-part of this implicit symmetry has become *explicit*: indeed, the determining equation (17) is equivariant with respect to the Z_q-action generated by S_0 on U. To get a full D_q-equivariance one has to go yet one step further, by showing that for small $(u, \lambda) \in U \times \mathbb{R}^m$ the equation (17) is equivalent to the equation

$$\mathcal{G}(u, \lambda) := S_0^{-1} \Phi_{r,\lambda}(u) - S_0 \Phi_{r,\lambda}^{-1}(u) = 0. \tag{20}$$

Using the $\mathbb{Z}_q$-equivariance and the R-reversibility of $\Phi_{r,\lambda}$ and S_0 it is easily seen that the mapping $\mathcal{G} : U \times \mathbb{R}^m \to U$ is D_q-equivariant:

$$\mathcal{G}(S_0 u, \lambda) = S_0 \mathcal{G}(u, \lambda) \qquad \text{and} \qquad \mathcal{G}(Ru, \lambda) = -R\mathcal{G}(u, \lambda).$$

Replacing the determining equation (17) by the equivalent D_q-equivariant equation (20) makes the full D_q-symmetry explicit. As will be illustrated by the examples of Section 6 this D_q-equivariance is important when solving (20) explicitly.

4 Normal form of reversible diffeomorphisms

In order to apply the reduction result of Theorem 3.6 on concrete examples one needs some method to calculate or approximate the reduced diffeomorphism $\Phi_{r,\lambda}$. One possible approach is just to follow the reduction procedure as outlined in the foregoing section; since the reduction is based on splitting the original equations and applying the implicit function theorem it is in principle possible to obtain the Taylor expansion of $\Phi_{r,\lambda}(u)$ at $(u, \lambda) = (0, 0)$ up to any desired order. A different approach which we briefly describe in this section consists in first bringing the original mapping Φ_λ in an appropriate normal form; it appears that it is then very easy to obtain (a good approximation of) the reduced mapping $\Phi_{r,\lambda}$ from this normal form.

We first consider normal forms for general diffeomorphisms. Let us assume that $\Phi : V \times \mathbb{R}^m \to V$ is a smooth mapping satisfying the following properties:

(H2) (i) $\Phi(0,\lambda) = 0$ for all $\lambda \in \mathbb{R}^m$;

(ii) $A_0 := D_x\Phi(0,0) \in \mathcal{L}(V)$ is invertible; hence $\Phi_\lambda = \Phi(\cdot,\lambda)$ is a local diffeomorphism (near $x = 0$) for small λ.

Next to the S-N-decomposition $A_0 = S_0 + N_0$ of A_0 there is also the so-called *semisimple-unipotent decomposition*

$$A_0 = S_0 e^{\mathcal{N}_0}, \tag{21}$$

with $S_0 \in \mathcal{L}(V)$ semisimple, $\mathcal{N}_0 \in \mathcal{L}(V)$ nilpotent, and $S_0\mathcal{N}_0 = \mathcal{N}_0 S_0$. This decomposition is unique, with S_0 the same as in the S-N-decomposition and with $e^{\mathcal{N}_0} = I + S_0^{-1}N_0$. Fix some $k \geq 1$; starting from (21) and using Taylor expansions at $x = 0$ one can then determine order by order a parameter-dependent polynomial vectorfield $Z_\lambda : V \to V$ with $Z_\lambda(0) = 0$ and $DZ_{\lambda=0}(0) = 0$, and such that

$$\Phi_\lambda = S_0 e^{\mathcal{N}_0 + Z_\lambda} + \mathcal{O}(\|\cdot\|^{k+1}); \tag{22}$$

here the exponential of $\mathcal{N}_0 + Z_\lambda$ stands for the time-one-map corresponding to the vectorfield $\mathcal{N}_0 + Z_\lambda$. The normal form reduction then consists in using near-identity transformations to bring the vectorfield Z_λ in a form which satisfies certain additional conditions. In order to do so we need the following technical result.

Lemma 4.1 *Given a semisimple $S_0 \in \mathcal{L}(V)$ there exists a scalar product $\langle\cdot,\cdot\rangle$ on V such that for each $A \in \mathcal{L}(V)$ we have $AS_0 = S_0A$ if and only if $A^TS_0 = S_0A^T$ (with the transpose taken w.r.t. $\langle\cdot,\cdot\rangle$).*

Using this scalar product one can prove the following.

Theorem 4.2 (Normal Form Theorem) *Assume that Φ_λ satisfies (H2). Then there exists for each $k \geq 1$ a parameter-dependent near-identity transformation $\Psi_{k,\lambda} : V \to V$ such that*

$$\Psi_{k,\lambda}^{-1} \circ \Phi_\lambda \circ \Psi_{k,\lambda} = S_0 e^{\mathcal{N}_0 + Z_\lambda} + \mathcal{O}(\|\cdot\|^{k+1}), \tag{23}$$

with $Z_\lambda(0) = 0$, $DZ_{\lambda=0}(0) = 0$ and such that

$$S_0 \circ Z_\lambda = Z_\lambda \circ S_0 \qquad \text{and} \qquad e^{t\mathcal{N}_0^T} \circ Z_\lambda = Z_\lambda \circ e^{t\mathcal{N}_0^T}, \ \ \forall t \in \mathbb{R}. \tag{24}$$

Under the foregoing conditions one calls

$$\Phi_\lambda^{NF} := S_0 e^{\mathcal{N}_0 + Z_\lambda}$$

the *normal form* of Φ_λ up to order k. The first condition in (24) implies that Φ_λ^{NF} commutes with S_0, while the second condition is equivalent with

$$DZ_\lambda(x) \cdot \mathcal{N}_0^T x = \mathcal{N}_0^T Z_\lambda(x), \qquad \forall (x,\lambda) \in V \times \mathbb{R}^m. \tag{25}$$

In case Φ_λ is R-reversible we find that $\mathcal{N}_0$ and Z_λ in (22) will be R-*reversible vectorfields*:

$$R\mathcal{N}_0 = -\mathcal{N}_0 R \qquad \text{and} \qquad R \circ Z_\lambda = -Z_\lambda \circ R. \tag{26}$$

Moreover, there exists a scalar product on V which next to the property given by Lemma 4.1 is also such that R is orthogonal: $RR^T = I$. The near-identity transformation $\Psi_{k,\lambda}$ given by the Normal Form Theorem can then be chosen such that it commutes with R, and as a consequence the reversibility of Φ_λ is maintained after the normal form transformation. It follows that modulo a near-identity transformation we can assume that

$$\Phi_\lambda = \Phi_\lambda^{NF} + \mathcal{O}(\|\cdot\|^{k+1}) \tag{27}$$

with (in particular)

$$S_0 \circ \Phi_\lambda^{NF} = \Phi_\lambda^{NF} \circ S_0 \qquad \text{and} \qquad R \circ \Phi_\lambda^{NF} \circ R = (\Phi_\lambda^{NF})^{-1}. \tag{28}$$

One can also arrange to have (25) satisfied, but this will be of no particular help in determinig the reduced diffeomorphism $\Phi_{r,\lambda}$; it may however be very helpful in solving the determining equation (see Section 6 for an example).

Theorem 4.3 *Assume (H1), (27) and (28). Then the mappings x^* and $\Phi_{r,\lambda}$ given by Theorem 3.5 are of the form*

$$x^*(u,\lambda) = u + \mathcal{O}(\|u\|^{k+1}) \qquad \textit{and} \qquad \Phi_{r,\lambda}(u) = \Phi_\lambda^{NF}(u) + \mathcal{O}(\|u\|^{k+1}). \tag{29}$$

This means that we obtain an approximation of $\Phi_{r,\lambda}$ by first bringing Φ_λ in normal form up to the desired order and then restricting the normal form to the reduced phase space U. Setting $\Psi_\lambda^{NF} := S_0^{-1}\Phi_\lambda^{NF} = e^{\mathcal{N}_0 + Z_\lambda}$ one has

$$\Phi_\lambda = S_0\Psi_\lambda^{NF} + \mathcal{O}(\|\cdot\|^{k+1}) \qquad \text{and} \qquad \Phi_{r,\lambda}(u) = S_0\Psi_\lambda^{NF}(u) + \mathcal{O}(\|u\|^{k+1}). \tag{30}$$

Up to terms of order k the determining equation (17) takes the form

$$\Psi_\lambda^{NF}(u) = u, \tag{31}$$

which for sufficiently small (u,λ) is equivalent to

$$\mathcal{N}_0(u) + Z_\lambda(u) = 0. \tag{32}$$

This means that the solutions of the determining equation (i.e. the q-periodic points of $\Phi_{r,\lambda}$) can be approximated by the equilibria $u \in U$ of the normal form vectorfield $\mathcal{N}_0 + Z_\lambda(\cdot)$ (both this vectorfield and Ψ_λ^{NF} leave the subspace U invariant). Observe also that due to (24) and (26) the equation (32) is D_q-equivariant.

5 Stability of bifurcating periodic points

In this section we describe how the foregoing reduction and normal form results can also be used to determine the stability properties of bifurcating periodic orbits. Let $x \in V$ be a q-periodic point of Φ_λ, i.e. $\Phi_\lambda^{(q)}(x) = x$; the (linear) stability of the corresponding periodic orbit $y = (y_i)_{i\in\mathbb{Z}} = (\Phi_\lambda^{(i)}(x))_{i\in\mathbb{Z}} \in Y_q$ is then determined by the eigenvalues of $D\Phi_\lambda^{(q)}(x) \in \mathcal{L}(V)$. Observe that by the chain rule

$$D\Phi_\lambda^{(q)}(y_i) = D\Phi_\lambda(y_{i+q-1}) \cdot D\Phi_\lambda(y_{i+q-2}) \cdots D\Phi_\lambda(y_{i+1}) \cdot D\Phi_\lambda(y_i), \qquad (33)$$

from which it follows that the spectrum of $D\Phi_\lambda^{(q)}(x_i)$ is independent of $i \in \mathbb{Z}$; therefore we can just take $i = 0$ and study $D\Phi_\lambda^{(q)}(x)$. For bifurcating periodic orbits we have $x = x^*(u,\lambda)$ and $y_i = x^*(S_0^i u, \lambda)$, with $(u,\lambda) \in U \times \mathbb{R}^m$ small and such that $\Phi_{r,\lambda}(u) = S_0 u$; substituting this into (33) (with $i = 0$) shows that the stability of such bifurcating periodic orbit is determined by the eigenvalues of $D\Phi_\lambda^{(q)}(x^*(u,\lambda) = \mathcal{D}(u,\lambda) \in \mathcal{L}(V)$, with $\mathcal{D} : U \times \mathbb{R}^m \to \mathcal{L}(V)$ defined by

$$\mathcal{D}(u,\lambda) := D\Phi_\lambda(x^*(S_0^{q-1}u,\lambda)) \cdots D\Phi_\lambda(x^*(S_0 u,\lambda)) \cdot D\Phi_\lambda(x^*(u,\lambda)). \qquad (34)$$

Moreover, as we will see in Section 6, periodic orbits bifurcating from a symmetric fixed point (here taken to be $x = 0$) will typically themselves also be symmetric; this means that $Ru = S_0^j u$ and $Rx^*(u,\lambda) = \Phi_\lambda^{(j)}(x^*(u,\lambda))$ for some $j \in \mathbb{Z}$. For convenience of formulation we will say that $u \in U$ is *symmetric* if the $\mathbb{Z}_q$-orbit through u is invariant under R, i.e. if $Ru = S_0^j u$ for some $j \in \mathbb{Z}$. If (u,λ) solves the determining equation (17) (or the equivalent equation (20)) and if u is symmetric then the corresponding q-periodic orbit of Φ_λ is also symmetric. A straightforward calculation shows the following result for such symmetric orbits.

Lemma 5.1 *Let $(u,\lambda) \in U \times \mathbb{R}^m$ be sufficiently small and such that u is symmetric. Then there exists a linear involution $T \in \mathcal{L}(V)$ (i.e. $T^2 = I$) such that*

$$T\mathcal{D}(u,\lambda)T = \mathcal{D}(u,\lambda)^{-1}.$$

This implies that if $\mu \in \mathbb{C}$ is an eigenvalue of $\mathcal{D}(u,\lambda)$ with u symmetric then also μ^{-1} is an eigenvalue. As a consequence we can only have a weak form of stability for symmetric periodic orbits, namely when all eigenvalues of $\mathcal{D}(u,\lambda)$ are on the unit circle; the only alternative is to have eigenvalues both inside and outside the unit circle, in which case the periodic orbit is unstable. In what follows we restrict the discussion to such symmetric periodic orbits.

For $(u,\lambda) = (0,0)$ we find $\mathcal{D}(0,0) = A_0^q$, which implies that $\mu = 1$ is an eigenvalue of $\mathcal{D}(0,0)$ with algebraic multiplicity equal to $\dim U$, and with geometric multiplicity equal to the sum of the geometric multiplicities of the resonant eigenvalues of A_0. Now we make an additional hypothesis:

(H3) All non-resonant eigenvalues μ of A_0 (i.e. $\mu^q \neq 1$) are simple and lie on the unit circle (i.e. $|\mu| = 1$).

For small (u, λ) the eigenvalues of $\mathcal{D}(u, \lambda)$ are close to those of $\mathcal{D}(0, 0)$; in particular, if (H3) holds then the eigenvalues of $\mathcal{D}(u, \lambda)$ which are not close to $\mu = 1$ will be simple. If moreover u is symmetric this can be combined with Lemma 5.1 to conclude that the eigenvalues of $\mathcal{D}(u, \lambda)$ which are not close to $\mu = 1$ will be simple and on the unit circle. Consequently, under the hypothesis (H3) the stability of bifurcating symmetric periodic orbits will be determined by the *critical eigenvalues* of $\mathcal{D}(u, \lambda)$, that is by those eigenvalues which are close to $\mu = 1$. The total multiplicity of these critical eigenvalues is equal to $\dim U$.

To calculate the critical eigenvalues of $\mathcal{D}(u, \lambda)$ one can put $\mathcal{D}(u, \lambda)$ in block form using the splitting $V = \mathrm{Ker}\,(S_0^q - I) \oplus \mathrm{Im}\,(S_0^q - I) = U \oplus \mathrm{Im}\,(S_0^q - I)$, and prove that there exists a similarity transformation which makes this block form diagonal; the critical eigenvalues are then the eigenvalues of the block corresponding to the subspace U. This diagonalization procedure is relatively easy to work out when Φ_λ is in normal form up to a sufficiently high order, namely when $\Phi_\lambda = S_0 \Psi_\lambda^{NF} + \mathcal{O}(\| \cdot \|^{k+1})$ with $\Psi_\lambda^{NF} = S_0^{-1} \Phi_\lambda^{NF} = e^{\mathcal{N}_0 + Z_\lambda}$ such that

$$S_0 \circ \Psi_\lambda^{NF} = \Psi_\lambda^{NF} \circ S_0 \qquad \text{and} \qquad R \circ \Psi_\lambda^{NF} \circ R = (\Psi_\lambda^{NF})^{-1}. \tag{35}$$

In this case the procedure outlined above gives the following result.

Theorem 5.2 *Assume that Φ_λ satisfies (H1) and is in normal form up to order $k \geq 2$, i.e. we have (30) and (35). Then there exists a smooth mapping $\widehat{\mathcal{D}} : U \times \mathbb{R}^m \to \mathcal{L}(U)$, with*

$$\widehat{\mathcal{D}}(u, \lambda) = D\Psi_\lambda^{NF}(u)\big|_U + \mathcal{O}(\|u\|^k),$$

such that for all sufficiently small solutions (u, λ) of the determing equation (17) the critical eigenvalues of $\mathcal{D}(u, \lambda)$ are given by the q-th powers of the eigenvalues of $\widehat{\mathcal{D}}(u, \lambda)$.

At the end of Section 4 we noticed that up to terms of order k the solutions of (17) are given by the fixed points of $\Psi_\lambda^{NF}\big|_U$, or equivalently, by the zero's of $(\mathcal{N}_0 + Z_\lambda)\big|_U$. According to the foregoing result and under the hypothesis (H3) the stability of the corresponding periodic orbit of Φ_λ is determined up to terms of order $(k - 1)$ by the eigenvalues of

$$D\Psi_\lambda^{NF}(u)\big|_U = e^{(\mathcal{N}_0 + DZ_\lambda(u))|_U};$$

this means that up to higher order terms the stability properties of the bifurcating periodic solutions of Φ_λ are the same as those of the corresponding equilibria of the normal vectorfield $\mathcal{N}_0 + Z_\lambda$ restricted to the reduced phase space U. If u is symmetric (i.e. $RS_0^j u = u$ for some $j \in \mathbb{Z}$) then

$$(RS_0^j) \circ D\Psi_\lambda^{NF}(u) \circ (RS_0^j) = (D\Psi_\lambda^{NF}(u))^{-1}$$

and

$$(RS_0^j) \circ DZ_\lambda(u) \circ (RS_0^j) = -DZ_\lambda(u),$$

which again leads to either a weak form of stability or instability. In applications the challenge will usually be to take the approximation order k sufficiently large such that the higher order terms do not disturb the picture obtained from the study of the normal form.

6 Application

In this last section we show how the foregoing methods and techniques can be used to study the problem of subharmonic bifurcation from a symmetric periodic orbit κ in the reversible system (3). As described in Section 2 this problem can be put into the form (4) by introducing a Poincaré-map P associated to κ. It is possible to construct P such that it inherits the reversibility of the vectorfield f; the fixed point of P corresponding to κ will then also be symmetric. Periodic points bifurcating from this fixed point correspond to bifurcating subharmonic solutions. We first describe the set-up and hypotheses in detail.

Let X be an even-dimensional state space ($\dim X = 2n$), and $R_0 \in \mathcal{L}(X)$ a linear involution with $\dim(\mathrm{Fix}\, R_0) = n$. Consider a smooth vectorfield $f : X \to X$ which is R_0-reversible ($f(R_0 x) = -R_0 F(x)$ for all $x \in X$), and denote by $\phi_t(x)$ the flow of the system

$$\dot{x} = f(x). \tag{36}$$

The reversibility of f implies that $\phi_t(R_0 x) = R_0 \phi_{-t}(x)$ for all $(t, x) \in \mathbb{R} \times X$. Let $p_1 \in \mathrm{Fix}\, R_0$ and $T_0 > 0$ be such that $\phi_{T_0}(p_1) = p_1$ and $\phi_t(p_1) \neq p_1$ for all t such that $0 < t < T_0$; then p_1 generates a symmetric T_0-periodic orbit $\kappa := \{\phi_t(p_1) \mid t \in \mathbb{R}\}$ which has a second intersection point with $\mathrm{Fix}\, R_0$, namely $p_2 := \phi_{T_0/2}(p_1)$. Now $R_0 f(p_1) = -f(R_0 p_1) = -f(p_1)$, and since R_0 is semisimple it follows that $X = \mathbb{R} f(p_1) \oplus V$ for some R_0-invariant subspace V which must then necessarily contain $\mathrm{Fix}\, R_0$. This implies $p_1 \in V$, and we can use V as a transversal section to κ and construct a Poincaré-map $P : V \to V$ which is well-defined near p_1 and such that $P(p_1) = p_1$. Moreover, from $R_0(V) = V$ and $\phi_t(R_0 x) = R_0 \phi_{-t}(x)$ we get $R_0 \circ P \circ R_0 = P^{-1}$. Let $R := R_0\big|_V$, and define $\Phi : V \to V$ by $\Phi(x) := P(p_1 + x) - p_1$ (for all $x \in V$); then $\dim V = 2n - 1$, $\dim(\mathrm{Fix}\, R) = n$, Φ is R-reversible and $\Phi(0) = 0$.

Next we have to consider those eigenvalues of $A_0 = D\Phi(0) = DP(p_1)$ which are roots of unity. It is a standard result that the eigenvalues of A_0 can be obtained from those of the monodromy matrix $M = D\phi_{T_0}(p_1)$; indeed, we have

$$A_0 x = Mx + \gamma(x) f(p_1), \qquad \forall x \in V, \tag{37}$$

where $\gamma \in \mathcal{L}(V; \mathbb{R})$ is such that the right hand side of (37) belongs to V for all $x \in V$. Clearly $R_0 M R_0 = M^{-1}$, and therefore if $\mu \in \mathbb{C}$ is an eigenvalue of

M then so are $\bar{\mu}$, μ^{-1} and $\bar{\mu}^{-1}$. A further consequence is that $(\det M)^2 = 1$, and since M belongs to the connected component of the identity in $\mathrm{GL}(X;\mathbb{R})$ we conclude that $\det M = 1$. Taking all this into account it follows that if $\mu = -1$ is an eigenvalue of M (i.e. a Floquet multiplier) then its multiplicity must be even. Also, $\mu = 1$ is always a multiplier, since $Mf(p_1) = f(p_1)$; the (algebraic) multiplicity of $\mu = 1$ must necessarily be even. Generically, $\mu = 1$ will be a non-semisimple multiplier with multiplicity equal to 2; in such case both $\mathrm{Ker}\,(M - I)$ and $\mathrm{Ker}\,((M - I)^2)$ are R_0-invariant, and since R_0 is semisimple we can find an R_0-invariant and one-dimensional complement U_0 of $\mathrm{Ker}\,(M - I)$ in $\mathrm{Ker}\,((M - I)^2)$. Then there exists a unique vector $u_0 \in U_0$ such that $Mu_0 = u_0 + f(p_1)$. Using the R_0-invariance of U_0, the R_0-reversibility of M and the fact that $R_0 f(p_1) = -f(R_0 p_1) = -f(p_1)$, it follows that $R_0 u_0 \in U_0$ and $MR_0 u_0 = R_0 u_0 + f(p_1)$; the uniqueness of u_0 then implies $R_0 u_0 = u_0$. Using (37) this gives the following possibilities for the eigenvalues of A_0 which are q-th roots of unity:

- $q = 1$: the eigenvalue $\mu = 1$ has always odd multiplicity, and there is at least one eigenvector u_0 which belongs to $\mathrm{Fix}\,R$; typically $\mu = 1$ will be a simple eigenvalue, with $\mathrm{Ker}\,(A_0 - I) = \mathbb{R}u_0$;
- $q = 2$: if $\mu = -1$ is an eigenvalue then it must have even multiplicity; generically this multiplicity will be equal to 2, and the eigenvalue will be non-semisimple;
- $q \geq 3$: eigenvalues of the form $\mu = \exp(\pm 2\pi ip/q)$, with $q \geq 3$, $0 < p < q$ and $\gcd(p, q) = 1$ can have any multiplicity but will typically be simple eigenvalues.

We next consider the bifurcation of q-periodic points from the fixed point $x = 0$ for different choices of q, each time assuming the simplest possible hypotheses for the resonant eigenvalues. To obtain the bifurcating solution branches we will concentrate on the normal form part of the equations, neglecting the higher order terms; a more careful analysis shows that the results obtained in this way persist when the neglected terms are taken into account.

6.1 The primary branch

We start with the bifurcation of fixed points, i.e. we take $q = 1$ in our reduction scheme. According to the foregoing discussion we can generically assume that $\mu = 1$ is a simple eigenvalue of A_0, with an eigenvector $u_0 \in V$ such that $Su_0 = u_0$, $\mathcal{N}_0 u_0 = 0$ and $Ru_0 = u_0$. Then $U = \{\alpha u_0 \mid \alpha \in \mathbb{R}\}$, and the normal form vectorfield Z restricted to U takes the form $Z(\alpha u_0) = g(\alpha) u_0$; the reversibility of Z implies that $g(\alpha) = -g(\alpha)$, i.e. $g(\alpha) \equiv 0$. So the equation (32) is satisfied for all $u = \alpha u_0 \in U$, meaning that the fixed point $x = 0$ of Φ belongs to a one-parameter family of symmetric fixed points, given by $\{x^*(\alpha u_0) \mid \alpha \in \mathbb{R}\}$. For the original reversible system (36) the conclusion is that under the generic assumption for the multiplier $\mu = 1$ the symmetric periodic orbit κ belongs to a one-parameter family of such symmetric periodic orbits. In view of what follows we will call this the *primary branch* of symmetric periodic orbits.

6.2 Period-doubling bifurcations

Next we consider period doubling, corresponding to $q = 2$. We assume again that $\mu = 1$ is a simple eigenvalue of A_0 (with eigenvector u_0 such that $Ru_0 = u_0$); we also assume that $\mu = -1$ is a non-semisimple eigenvalue with algebraic multiplicity equal to 2 and with eigenvector v_0. Then $S_0 v_0 = -v_0$, $\mathcal{N}_0 v_0 = 0$, and since $\mathrm{Ker}\,(A_0 + I)$ is invariant under R (by the reversibility) we must have either $Rv_0 = v_0$ or $Rv_0 = -v_0$; we can without loss of generality assume that $Rv_0 = v_0$, since in the other case ($Rv_0 = -v_0$) we have $RS_0 v_0 = v_0$, and replacing R by RS_0 in what follows the same analysis goes through. Both $\mathrm{Ker}\,(A_0 + I) = \mathbb{R}v_0$ and $\mathrm{Ker}\,((A_0 + I)^2)$ are R-invariant, and we can find a one-dimensional and R-invariant complement W_0 of $\mathbb{R}v_0$ in $\mathrm{Ker}\,((A_0 + I)^2)$. There exists a unique element $w_0 \in W_0$ such that $A_0 w_0 = -w_0 - v_0$; from this one deduces that $A_0 R w_0 = -Rw_0 + v_0$, and hence $Rw_0 = -w_0$. The reduced phase space U is then given by

$$U = \{\alpha u_0 + \xi v_0 + \eta w_0 \mid \alpha, \xi, \eta \in \mathbb{R}\},$$

and the restrictions of S_0, $\mathcal{N}_0$ and R to U by

$$\begin{cases} S_0(\alpha u_0 + \xi v_0 + \eta w_0) &= \alpha u_0 - \xi v_0 - \eta w_0, \\ \mathcal{N}_0(\alpha u_0 + \xi v_0 + \eta w_0) &= \eta v_0, \\ R(\alpha u_0 + \xi v_0 + \eta w_0) &= \alpha u_0 + \xi v_0 - \eta w_0. \end{cases}$$

The normal form vectorfield Z (restricted to U) can be written as

$$Z(\alpha u_0 + \xi v_0 + \eta w_0) = g(\alpha, \xi, \eta) u_0 + h_1(\alpha, \xi, \eta) v_0 + h_2(\alpha, \xi, \eta) w_0,$$

with functions $g(\alpha, \xi, \eta)$, $h_1(\alpha, \xi, \eta)$ and $h_2(\alpha, \xi, \eta)$ which are of second order in the origin (since $Z(0) = 0$ and $DZ(0) = 0$), and such that

$$\begin{cases} g(\alpha, -\xi, -\eta) = g(\alpha, \xi, \eta), \\ h_1(\alpha, -\xi, -\eta) = -h_1(\alpha, \xi, \eta), \\ h_2(\alpha, -\xi, -\eta) = -h_2(\alpha, \xi, \eta), \end{cases} \quad \text{and} \quad \begin{cases} g(\alpha, \xi, -\eta) = -g(\alpha, \xi, \eta), \\ h_1(\alpha, \xi, -\eta) = -h_1(\alpha, \xi, \eta), \\ h_2(\alpha, \xi, -\eta) = h_2(\alpha, \xi, \eta). \end{cases}$$

(These correspond to the conditions that Z should commute with S_0 and anti-commute with R). In order to impose the condition (25) we can use a scalar product such that the basis $\{u_0, v_0, w_0\}$ of U is orthonormal (compare with Lemma 4.1 and the condition $R^T R = I$). Then $\mathcal{N}_0^T(\alpha u_0 + \xi v_0 + \eta w_0) = \xi w_0$, and (25) takes the form

$$\xi \frac{\partial g}{\partial \eta}(\alpha, \xi, \eta) = 0, \quad \xi \frac{\partial h_1}{\partial \eta}(\alpha, \xi, \eta) = 0 \quad \text{and} \quad \xi \frac{\partial h_2}{\partial \eta}(\alpha, \xi, \eta) = h_1(\alpha, \xi, \eta).$$

Imposing all these conditions leads to

$$g(\alpha,\xi,\eta)\equiv 0,\qquad h_1(\alpha,\xi,\eta)\equiv 0\qquad\text{and}\qquad h_2(\alpha,\xi,\eta)=\xi\,\varphi(\alpha,\xi^2),$$

with $\varphi:\mathbb{R}^2\to\mathbb{R}$ a smooth function such that $\varphi(0,0)=0$. The solutions of (32) are given by either $(\alpha,\xi,\eta)=(\alpha,0,0)$, with $\alpha\in\mathbb{R}$ small and arbitrary (since $S_0(\alpha u_0)=u_0$ these correspond to the primary branch), or by $(\alpha,\xi,\eta)=(\alpha,\xi,0)$ with $(\alpha,\xi)\in\mathbb{R}^2$ such that

$$\varphi(\alpha,\xi^2)=0. \tag{38}$$

We have $\varphi(0,0)=0$, and we assume that also the *transversality condition*

$$\frac{\partial\varphi}{\partial\alpha}(0,0)\neq 0 \tag{39}$$

is satisfied. One can easily verify that the linear operator $\mathcal{N}_0+DZ(\alpha u_0)$ restricted to U has eigenvalues $\mu=0$ and $\mu=\pm\sqrt{\varphi(\alpha,0)}$, corresponding to respectively the multiplier $\mu=1$ and the approximate multipliers $\mu=-\exp\left(\pm\sqrt{\varphi(\alpha,0)}\right)$ along the primary branch; the condition (39) means that as we move along the primary branch two complex conjugate multipliers move along the unit circle and with non-zero speed towards -1, and after colliding split off the unit circle into a pair of real multipliers moving away from -1 along the real axis, one inside and the other one outside the unit circle. Assuming (39) we can solve (38) for $\alpha=\alpha^*(\xi^2)$, giving us the solution branch $\{(\alpha^*(\xi^2),\xi,0)\mid\xi\in\mathbb{R}\}$ of the determining equation. For fixed $\xi\neq 0$ the two solutions $(\alpha^*(\xi^2),\pm\xi,0)$ correspond to the two points of a symmetric 2-periodic orbit of Φ. For the original reversible system (36) this means that a single branch of symmetric periodic orbits bifurcates from the primary branch; the limiting period along this branch is $2T_0$, and so we have period-doubling.

Using the approach of Section 5 one can also determine the stability of these bifurcating periodic solutions. Writing $\varphi(\alpha,\xi^2)=C(\alpha)+D(\alpha)\xi^2+\mathrm{O}(\xi^4)$ one finds that the eigenvalues of $\mathcal{N}_0+DZ(\alpha^*(\xi^2)u_0+\xi v_0)$ (restricted to U) are given by $\mu=0$ and $\mu=\pm|\xi|\sqrt{2D(0)}+\mathrm{O}(|\xi|^2)$. Taking the exponential one obtains along the bifurcating branch two critical multipliers: these multipliers are real and off the unit circle (i.e. we have instability) if $D(0)>0$, and they lie on the unit circle (stability) if $D(0)<0$. So the stability is determined by the sign of a third order coefficient in the normal form.

6.3 Subharmonic bifurcation with $q\geq 3$

In this final subsection we look at the case $q\geq 3$. Let $\theta_0:=2\pi p/q$, with $q\geq 3$, $0<p<q$ and $\gcd(p,q)=1$. We assume that next to the simple eigenvalue $\mu=1$ the operator $A_0\in\mathcal{L}(V)$ has the pair $\mu=\exp(\pm i\theta_0)$ as simple eigenvalues, and that there are no further eigenvalues which are q-th roots of unity. The 2-dimensional subspace $U_q:=\mathrm{Ker}\,((A_0-(\cos\theta_0)I)^2+(\sin\theta_0)^2I)$

is invariant under R; let $v_0 \in U_q$ be an eigenvector of R, i.e. $Rv_0 = \epsilon v_0$, with $\epsilon = \pm 1$. Setting $w_0 := (\sin\theta_0)^{-1}(A_0 - (\cos\theta_0)I)v_0$ we find

$$Rw_0 = \frac{1}{\sin\theta_0}\left(A_0^{-1} - (\cos\theta_0)I\right)Rv_0 = \frac{-\epsilon}{\sin\theta_0}\left(A_0 - (\cos\theta_0)I\right)v_0 = -\epsilon w_0.$$

So we can assume that $\epsilon = 1$ (just interchange v_0 and w_0 in the other case), resulting in a basis $\{u_0, v_0, w_0\}$ of $U = U_0 \oplus U_q$ such that $Ru_0 = u_0$, $Rv_0 = v_0$, $Rw_0 = -w_0$, $\mathcal{N}_0 u_0 = 0$, $\mathcal{N}_0 v_0 = 0$, $\mathcal{N}_0 w_0 = 0$, $S_0 u_0 = u_0$ and

$$S_0 v_0 = (\cos\theta_0)v_0 + (\sin\theta_0)w_0, \qquad S_0 w_0 = -(\sin\theta_0)v_0 + (\cos\theta_0)w_0. \quad (40)$$

Bifurcating q-periodic points can be approximated by determining the equilibria of the normal form system $\dot{u} = Z(u)$; the vectorfield $Z(u)$ must commute with S_0 and anti-commute with R. To find the form of $Z(u)$ we identify U with $\mathbb{R} \times \mathbb{C}$, via the isomorphism

$$\zeta : \mathbb{R} \times \mathbb{C} \to U, \quad (\alpha, z) \longmapsto \zeta(\alpha, z) := \alpha u_0 + \Re(z(v_0 - iw_0));$$

then $S_0(\alpha, z) = (\alpha, \exp(i\theta_0)z)$ and $R(\alpha, z) = (\alpha, \bar{z})$. Some elementary analysis shows that the system $\dot{u} = Z(u)$ must have the form

$$\begin{array}{rcl} \dot{\alpha} & = & g(\alpha, z)\,\Im(z^q), \\ \dot{z} & = & ih_1(\alpha, z)z + ih_2(\alpha, z)\bar{z}^{q-1}, \end{array} \quad (41)$$

where the functions $g : \mathbb{R} \times \mathbb{C} \to \mathbb{R}$ and $h_i : \mathbb{R} \times \mathbb{C} \to \mathbb{R}$ $(i = 1, 2)$ are invariant under S_0 and R; also $h_1(0, 0) = 0$, and we will assume that $\delta := h_2(0, 0) \neq 0$. Setting $z = r\exp(i\theta)$ we can write (41) in the equivalent form

$$\begin{array}{rcl} \dot{\alpha} & = & r^q g(\alpha, r\exp(i\theta))\sin(q\theta), \\ \dot{r} & = & r^{q-1}h_2(\alpha, r\exp(i\theta))\sin(q\theta), \\ \dot{\theta} & = & h_1(\alpha, r\exp(i\theta)) + r^{q-2}h_2(\alpha, r\exp(i\theta))\cos(q\theta). \end{array} \quad (42)$$

The α-axis forms a line of equilibria, corresponding to the primary branch (see subsection 6.1). Using (41) one finds that $DZ(\alpha u_0)$ (restricted to U) has the eigenvalues $\mu = 0$ and $\mu = \pm ih_1(\alpha, 0)$, corresponding to the multipliers $\mu = 1$ and $\mu = \exp(\pm i(\theta_0 + h_1(\alpha, 0)))$ along the primary branch. We have already observed that $h_1(0, 0) = 0$, and we will assume that

$$\tau := \frac{\partial h_1}{\partial \alpha}(0, 0) \neq 0; \quad (43)$$

this *transversality condition* means that as we move along the primary branch a pair of simple multipliers moves with non-zero speed along the unit circle, passing through $\exp(\pm i\theta_0)$ for $\alpha = 0$.

From (42) and our assumption $h_2(0, 0) \neq 0$ it follows that nontrivial equilibria (with $r \neq 0$) must be such that $\sin(q\theta) = 0$, i.e. $\theta = 0$ or $\theta = \pi/q$ modulo θ_0. For $\theta = 0 \pmod{\theta_0}$ the bifurcation equation reduces to

$$h_1(\alpha, r) + r^{q-2}h_2(\alpha, r) = 0, \quad (44)$$

and for $\theta = \pi/q \pmod{\theta_0}$ to

$$h_1(\alpha, r\exp(i\pi/q)) - r^{q-2}h_2(\alpha, r\exp(i\pi/q)) = 0. \tag{45}$$

Under the transversality condition (43) both these equations can be solved for α as a function of r, giving respectively $\alpha = \alpha_+^*(r)$ for (44) and $\alpha = \alpha_-^*(r)$ for (45). The full set of nontrivial equilibria is then given by the union of

$$B_q^+ = \{(\alpha, z) = (\alpha_+^*(r), r\exp(ik\theta_0) \mid r > 0,\ 0 \le k \le q-1\}$$

and

$$B_q^- = \{(\alpha, z) = (\alpha_-^*(r), r\exp(i(\pi/q + k\theta_0)) \mid r > 0,\ 0 \le k \le q-1\}.$$

Observe that for fixed $r_0 > 0$ the intersection of B_q^+ with $r = r_0$ is invariant under S_0 and R; it therefore corresponds to a symmetric q-periodic orbit of Φ. The same holds for B_q^-, and since $\alpha_+^*(0) = \alpha_-^*(0) = 0$ we have found two branches of symmetric subharmonics bifurcating from the primary branch at the orbit κ of (36); the limiting period along these branches is equal to qT_0.

To determine the stability of these subharmonics we linearize (42) at the points of B_q^+ and B_q^- and calculate the eigenvalues of this linearization; after somewhat lengthy but straightforward calculations we find next to the trivial eigenvalue $\mu = 0$ a pair of nontrivial eigenvalues $\mu = \pm\sqrt{\Lambda_+(r)}$ along B_q^+, and another pair $\mu = \pm\sqrt{\Lambda_-(r)}$ along B_q^-. To simplify the expressions for $\Lambda_+(r)$ and $\Lambda_-(r)$ we introduce next to the (non-zero) constants δ and τ defined before also the constant $\gamma := g(0,0)$; moreover, we can expand $h_1(\alpha, z)$ as $h_1(\alpha, z) = h_1(\alpha, 0) + \tilde{h}_1(\alpha)r^2 + \mathrm{O}(r^3)$, and we set $\nu := \tilde{h}_1(0)$. One finds then that

$$\alpha_\pm^*(r) = -\frac{\nu}{\tau}r^2 \mp \frac{\delta}{\tau}r^{q-2} + \mathrm{O}(r^3) \tag{46}$$

and

$$\Lambda_\pm(r) = \pm q(\gamma\tau + 2\nu\delta)r^q + q(q-2)\delta^2 r^{2q-4} + \mathrm{O}(r^{q+1}). \tag{47}$$

For $q = 3$ we have $\alpha_\pm^*(r) = \mp\delta\tau^{-1}r + \mathrm{O}(r^2)$ and $\Lambda_\pm(r) = 3\delta^2 r^2 + \mathrm{O}(r^3)$; therefore both bifurcating branches will be unstable (remember that we have assumed that $\delta \ne 0$). For $q = 4$ the expressions are

$$\alpha_\pm^*(r) = -\frac{\nu \pm \delta}{\tau}r^2 + \mathrm{O}(r^3) \quad \text{and} \quad \Lambda_\pm(r) = 4(\pm\gamma\tau \pm 2\nu\delta + 2\delta^2)r^4 + \mathrm{O}(r^5).$$

The sign of $\Lambda_\pm(r)$ (and the corresponding stability properties along the branches $B_q^\pm$) depends on all the constants involved, and a detailed analysis becomes rather messy; we leave it to the interested reader. Finally, for $q \ge 5$ we find

$$\alpha_\pm^*(r) = -\frac{\nu}{\tau}r^2 + \mathrm{O}(r^3) \qquad \text{and} \qquad \Lambda_\pm(r) = \pm q(\gamma\tau + 2\nu\delta)r^q + \mathrm{O}(r^{q+1});$$

assuming that $\gamma\tau + 2\nu\delta \ne 0$ it follows that the branches B_q^+ and B_q^- have opposite stability properties: one is stable, the other unstable. Comparing the equations (44) and (45) it is also easily seen that the two branches will be very close to each other for high values of q; more precisely, we have $\alpha_-^*(r) = \alpha_+^*(r) + \mathrm{O}(r^{q-2})$ as $r \to 0$.

We conclude with the remark that along the stable branch of subharmonics we will have a pair of simple multipliers moving along the unit circle, starting at $\mu = 1$; these multipliers will necessarily cross roots of unity, thereby causing further subharmonic bifurcations. Repeating this scheme leads to a *cascade* of subharmonic branches, a phenomenon which is not yet fully understood.

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