The Hilbert scheme of two point of Enriques surface

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Introduction

Throughout this paper, we work over \( \mathbb{C} \). We denote the Hilbert scheme of \( n \) points of a surface \( S \) by \( S^{[n]} = \text{Hilb}^n S \). Let \( E \) be an Enriques surface. \( E^{[n]} \) has a Calabi-Yau manifold \( X_n \) as the universal covering space of degree 2. In [8, Theorem1.3], the author showed that for Enriques surfaces \( S, T \), and \( n \geq 3 \), if the universal covering spaces of \( S^{[n]} \) and \( T^{[n]} \) are isomorphic, then \( S^{[n]} \) and \( T^{[n]} \) are isomorphic by checking the action to cohomology ring of the automorphisms of them. However, in general, \( S \not\sim T \) even if their universal covering spaces are isomorphic by a result of Ohashi [10]. For \( n = 2 \), since the second cohomology of \( X_2 \) is bigger than that of \( E^{[2]} \) [8, Theorem5.1], the automorphisms of \( E^{[2]} \) and \( X_2 \) were not studied enough in [8]. The author does not know \( E^{[2]} \) is uniquely determined by \( X_2 \). In this paper, we study the automorphisms of the Hilbert scheme of two points of Enriques surfaces by using that the second cohomology of \( X_2 \) is bigger than that of \( E^{[2]} \). There are two main results (Theorem 0.3 and 0.5).

Let \( S \) be a smooth projective surface. First we study whether \( S \) could be restored from \( S^{[2]} \), i.e. for an two projective surfaces \( S \) and \( S' \), if \( S^{[2]} \cong S'^{[2]} \), then are \( S \) and \( S' \) isomorphic ? For \( K3 \) surfaces, this problem is fully studied. In [11, Example 7.2], Yoshioka showed that there exist two \( K3 \) surfaces \( K \) and \( K' \) such that \( K \not\cong K' \) and \( K^{[2]} \cong K'^{[2]} \). The following two theorems is very useful:

**Theorem 0.1.** For a smooth projective surface \( S \), we put

\[
\begin{align*}
    h^{p,q}(S) := \dim_{\mathbb{C}} H^q(S, \Omega^p_S) \quad \text{and} \\
    h(S, x, y) := \sum_{p,q} h^{p,q}(S)x^p y^q.
\end{align*}
\]

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Theorem 0.2. [7] Let $X$ be smooth projective variety with $n = \dim X \geq 1$. Then there is an isomorphism $\mathcal{S}_d \mathcal{H}^0(X, \omega_X^{\otimes mn}) \cong \mathcal{H}^0(X^n, \omega_X^{\otimes mn})$ and the Kodaira dimension $\kappa(X^n) = \dim \mathcal{E}(X)$, whenever $mn$ is even.

Therefore, for a smooth projective surface $S$ the Hodge number and Kodaira dimension can be restored from $S^{[n]}$. However, we may not necessarily restore $S$ from $S^{[n]}$ because there is the example of Yoshioka. In addition, the relationship of the deformation of $S$ and $S^{[n]}$ is known. Our first main result (Theorem 0.3) shows that this never happened to Enriques surfaces:

Theorem 0.3. Let $E$ be an Enriques surface and $S$ a smooth projective surface. If there is an isomorphism $\varphi : E^{[2]} \cong S^{[2]}$, then $S$ is an Enriques surface, and there is an isomorphism $\psi : E \cong S$ such that $\varphi$ is induced by $\psi$.

We also notice that for the universal covering $K3$ surfaces $X$ and $Y$ of Enriques surfaces $E$ and $F$, Sosna [4] showed if $X^{[n]} \cong Y^{[n]}$ for some $n \geq 2$, then $X \cong Y$.

Our second main result (Theorem 1.3) is on the naturality problem of automorphisms of $S^{[2]}$. First we recall the definition of the natural automorphism [3].

Definition. Let $S$ be a smooth compact surface. For $n \geq 2$, an automorphism $g \in \text{Aut}(S^{[n]})$ is called natural if there is an automorphism $f \in \text{Aut}(S)$ such that $g = f^{[n]}$. Here $f^{[n]}$ is the automorphism of $S^{[n]}$ that is naturally induced by $f \in \text{Aut}(S)$.

Theorem 0.4. For $n \geq 2$, let $S$ be a $K3$ surface or an Enriques surface, and $D$ the exceptional divisor of the Hilbert-Chow morphism $\pi : S^{[n]} \to S^{(n)}$. An automorphism $f$ of $S^{[n]}$ is natural if and only if $f(D) = D$.

When $S$ is not a $K3$ surface and an Enriques surface, this theorem does not hold good, i.e. there exist a smooth projective surface $S$ which has an automorphism $f$ of $S^{[n]}$ such that $f(D) = D$ but $f$ is not natural. Our second main result is the following theorem:

Theorem 0.5. Let $E$ be an Enriques surface. Then $\text{Aut}(E^{[2]}) \cong \text{Aut}(E)$, i.e. all automorphisms of $\text{Aut}(E^{[2]})$ are natural.

For a smooth quartic surface $Z$ of $\mathbb{P}^3$ which is a $K3$ surface, generic line $L$ on $\mathbb{P}^3$ meets $Z$ along 4 distinct points. By alternating them, Beauville showed $Z^{[2]}$ has an automorphism which is not natural [1]. Further, Oguiso showed the fact there exists a $K3$ surface $Y$ such that $[\text{Aut}(Y^{[2]}) : \text{Aut}(Y)] = \infty$ under the natural inclusion $[6, \text{Theroem 1.2 (1)}]$, which is completely different from Theorem 1.3.

The author does not know whether Theorem 1.1 and 1.3 are true or no for $n \geq 3$. By [7, Theorem 2] and [6, page 204], we have the equation (1):

$$
\sum_{n=0}^{\infty} \sum_{p,q} h^{p,q}(S^{[n]}) x^p y^q t^n = \prod_{k=1}^{\infty} \left( \frac{1}{1 - (-1)^{p+q} x^p y^{q+k-1} t^k} \right)^{(-1)^{p+q} h^{p,q}(S)}.
$$
Preliminaries

It is well known that $S^2 \cong \text{Blow}_{S_2} S^2 / S_2$, where

$$\Delta S := \{(x, y) \in S^2 : x = y\},$$

and $S_2$ is the symmetric group of degree 2, which acts by interchanging the two factors of the product.

Let $E$ be an Enriques surface, and $\mu : K \to E$ its universal covering space. Let $\pi : X \to E^{[2]}$ be the universal covering space of $E^{[2]}$.

From $\text{Blow}_{\Delta E} E^{[2]} / S_2$ and $\mu : K \to E$, we will construct $X$. Let $\sigma$ be the covering involution of $\mu$, $H$ the finite subgroup of $\text{Aut}(K^2)$ which is generated by $S_2$ and $\sigma \times \sigma$, and $G$ the finite subgroup of $\text{Aut}(K^2)$ which is generated by $S_2$ and $\text{id}_K \times \sigma$. Since $K^2 / G = E^{[2]} / S_2$, and $H$ is a normal subgroup of $G$, the covering space $\mu^2 : K^2 \to E^2$ induces the covering spaces $\mu^2 : K^2 \to E^{[2]}$.

Since $|G/H| = 2$, and $E^{[2]} \cong \text{Blow}_{\Delta E} E^{[2]} / S_2$, we have $X \cong \text{Blow}_{\mu^2 \Delta E} K^2 / H$, and the automorphism $\text{id}_K \times \sigma$ of $K^2 / H$ induces the covering involution $\rho$ of $X \to E^{[2]}$. From here, we consider $X$ as $\text{Blow}_{\mu^2 \Delta E} K^2 / H$.

Let $\eta : \text{Blow}_{\mu^2 \Delta E} K^2 / H \to K^2 / H$ be the natural morphism. We put

$$T := \{(x, y) \in K^2 : \sigma(x) = y\}.$$

Then we have $\mu^2 \Delta E = \Delta_K \cup T$. Furthermore, we put

$$D_1 := \eta^{-1}(T), \quad D_2 := \eta^{-1}(\Delta_K),$$

and

$$h_i$$

the first chern class of $D_i$ for $i = 1, 2$.

Since $\pi^{-1}(D) = D_1 \cup D_2$, we get

$$H^2(X, \mathbb{C}) = \pi^*(H^2(E^{[2]}, \mathbb{C})) \oplus \mathbb{C}\langle h_1 - h_2 \rangle.$$

Thus $\dim H^2(X, \mathbb{C}) = 12 = \dim H^2(E^{[2]}, \mathbb{C}) + 1$. Pay attention that for $n \geq 3$

$$\dim H^2(X, \mathbb{C}) = \dim H^2(E^{[n]}, \mathbb{C}) = 11.$$

Furthermore since $(\text{id}_K \times \sigma)(T) = \Delta_K$, we get $\rho^* h_1 = h_2$,

the eigenspace for the eigenvalue $-1$ of $\rho^*$ is $\mathbb{C}\langle h_1 - h_2 \rangle$,

and

the eigenspace for the eigenvalue $1$ of $\rho^*$ is $\pi^*(H^2(E^{[2]}, \mathbb{C}))$.  

Main theorems

Theorem 0.6. Let $E$ and $E'$ be two Enriques surfaces. For an isomorphism $g : E^{[2]} \cong E'^{[2]}$, we get $g(D) = D'$.

Sketch 0.7. From the uniqueness of the universal covering space, there is an isomorphism $f : X \rightarrow X'$ such that $g \circ \pi = \pi \circ f$. From this, we have only to show $f(\pi^{-1}(D)) = \pi^{-1}(D')$. Since the each degree of $\pi$ and $\pi'$ is 2, we have $f^{-1} \circ \rho' \circ f = \rho$ and $\rho^* = f^* \circ \rho'^* \circ f^{-1*}$ as an automorphism of $H^2(X, \mathbb{C})$. Since the eigenspace for the eigenvalue $-1$ of $\rho^*$ is $\mathbb{C}(h_1 - h_2)$, we have

$$-(h_1 - h_2) = \rho^*(h_1 - h_2) = f^* \circ \rho'^* \circ f^{-1*}(h_1 - h_2) \text{ in } H^2(X, \mathbb{C}).$$

Thus for a linear isomorphism $f^*$ from $H^2(X', \mathbb{C})$ to $H^2(X, \mathbb{C})$, we obtain

$$\rho'^*(f^{-1*}(h_1 - h_2)) = -f^{-1*}(h_1 - h_2) \text{ in } H^2(X', \mathbb{C}).$$

Since the eigenspace for the eigenvalue $-1$ of the linear mapping $\rho'^*$ is $\mathbb{C}(h'_1 - h'_2)$, there is some $a \in \mathbb{C}$ such that

$$f^*(h'_1 - h'_2) = a(h_1 - h_2) \text{ in } H^2(X, \mathbb{C}).$$

Since $X$ and $X'$ are Calabi-Yau manifolds, $\text{Pic}(X)$ and $\text{Pic}(X')$ are torsion free and the natural maps $\text{Pic}(X) \rightarrow H^2(X, \mathbb{Z})$ and $\text{Pic}(X') \rightarrow H^2(X', \mathbb{Z})$ are isomorphic. Thus there are some non zero integer $t \in \mathbb{Z}_{>0}$ and $s \in \mathbb{Z}\setminus\{0\}$ such that $a = \frac{t}{s}$, i.e.

$$f^*(\mathcal{O}_X(t(D'_1 - D'_2))) \cong \mathcal{O}_X(s(D_1 - D_2)) \text{ as a line bundle.}$$

Since $D_1$ and $D_2$ are the exceptional divisors of $X \rightarrow \text{Blow}_{T \cup \Delta K} K^2/H$, we get that $f(D_1 \cup D_2) = D'_1 \cup D'_2$.

References


