Can One Hear the Shape of a Group?

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Abstract The iso-spectrum problem for marked lengnth spectrum for Riemannian manifolds of negative curvature has a rich history. We rephrased the problems for metrics on discrete groups, discussed its connection to a conjecture by Margulis, and proved some results for "total relatively hyperbolic groups" in Koji Fujiwara, Journal of Topology and Analysis, **7**(2), 345–359 (2015). This is a note from my talk on that paper and mainly discuss the connection between Riemannian geometry and group theory, and also some questions.

Keywords Marked length spectrum · Hyperbolic group · Relatively hyperbolic group · Coarsely equal metrics

1 Marked Length Spectrum

Let M be a closed Riemannian manifold of negative (or non-positive) sectional curvature, and $\mathscr{C}$ the set of free homotopy classes of loops (i.e., closed curves) in M. In negative curvature, each class $g \in \mathscr{C}$ is represented by a unique closed geodesic. The *marked length spectrum* is a function $\ell : \mathscr{C} \to \mathbb{R}$ that assigns the length of the closed geodesic, $\ell(g)$, to g.

Burns and Katok [6] conjectured that ℓ determines M up to isometry (the *marked length iso-spectrum problem*). The answer is known in dimension two.

Theorem 1 (Otal [19]) *The marked length spectrum determines a closed orientable surface of negative curvature up to isometry.*

Croke [7] generalized it to a setting of non-positive curvature in dimension two, but in higher dimension, not much is known. Building up on the work by Besson-Courtois-Gallot, Hamenstädt [15] proved

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Theorem 2 *A negatively curved closed manifold with the same marked length spectrum as a negatively curved closed locally symmetric space M is isometric to M.*

Let's look at the marked length spectrum from the view point of group action. We view the marked length spectrum as a function $\ell : \pi_1(M) \to \mathbb{R}$ that is constant on each conjugacy class.

Let $\tilde{M}$ be the universal cover of M, and $\pi_1(M)$ act on $\tilde{M}$ by isometries, preserving the distance d, as a Deck group. Each non-trivial element $g \in \pi_1(M)$ has a unique invariant (Riemannian) geodesic $\gamma(g) \subset \tilde{M}$ that maps to the closed geodesic in M for g. Pick a point $x_0 \in \gamma(g)$, then $d(x_0, g(x_0)) = \ell(g)$.

The *translation length* of g, denoted by $\tau(g)$ is defined by

$$\tau(g) = \lim_{n \to \infty} \frac{d(x, g^n(x))}{n}$$

for a point $x \in \tilde{M}$. $\tau(g)$ does not depend on the choice of x by the triangle inequality.

Now since M has negative curvature (non-positive curvature suffices), $\gamma(g)$ is a distance minimizing path in $\tilde{M}$, therefore $\tau(g) = \ell(g)$ for each g.

So, we rephrase the marked length iso-spectrum problem as "does the translation length function τ on $\pi_1(M)$ determine M up to isometry?"

2 Coarsely Isometric Metrics and Conjecture by Margulis

Let G be a group and d a left-invariant pseudo metric on G. We write $a =_C b$ if $|a - b| \le C$. Two pseudo metrics d_1, d_2 on a space X are *coarsely equal* if there exists $C > 0$ such that

$$d_1(x, y) =_C d_2(x, y), \ \forall x, y \in X \tag{1}$$

From now on we assume G is finitely generated. We say that two left invariant proper pseudo metrics d_1, d_2 on G are *asymptotically isometric* if

$$\lim_{g \to \infty} \frac{d_1(1, g)}{d_2(1, g)} = 1 \tag{2}$$

Here, by a proper metric, we mean that there are only finitely many elements $g \in G$ with $d(1, g) \le K$ for each $K > 0$. Then $d_1(1, g) \to \infty \Leftrightarrow d_2(1, g) \to \infty$, and by $g \to \infty$ we mean that $d_1(1, g) \to \infty$.

Clearly (1) implies (2). Margulis conjectured that (2) implies (1), therefore (2) is equivalent to (1), [18]. He verified the equivalence in a setting for reductive groups, [1]. A metric space (X, d) is *coarsely geodesic* if there exists $C > 0$ such that for any two points $x_0, x_1 \in X$ there is a parametrized path $x(t), 0 \le t \le a$ such that $d(x(t), x(s)) =_C |s - t|$ for all $s, t \in [0, a]$.

Theorem 3 *On the following groups, any two asymptotically isometric, proper, coarsely geodesic pseudo metrics are coarsely equal:*

1. $\mathbb{Z}^n$ *(Burago [5])*
2. $H_3(\mathbb{Z})$ *(Krat [16])*
3. *Hyperbolic groups (Krat [16])*

$H_3(\mathbb{Z})$ is the discrete Heisenberg group. Hyperbolic groups (in the sense of Gromov) form a wide class of groups that has been extensively studied in geometric group theory. We do not give a definition (see for example [4]) but list some examples.

Example 1 (Hyperbolic group) Examples of hyperbolic groups:

- Free groups
- The fundamental groups of closed Riemannian manifolds of negative sectional curvature.
- Uniform lattices in semi-simple Lie groups of rank-1, i.e., $SO(n, 1)$, $SU(n, 1)$, $Sp(n, 1)$, F_4.

Examples of groups that are *not* hyperbolic:

- $\mathbb{Z}^n, n > 1$. More generally a group that contains $\mathbb{Z}^2$ as a subgroup.
- Non-uniform lattices in $SO(n, 1), n > 2$; $SU(n, 1), n > 1$; $Sp(n, 1)$; F_4. For example, the fundamental group of a complete, non-compact Riemannian manifold of sectional curvature $= -1$, of finite volume, of dimension at least 3.

By now a counter example to the conjecture by Margulis is given by Breuillard.

Theorem 4 ([2]) *On $H_3(\mathbb{Z}) \times \mathbb{Z}$, there are two (word) metrics that are asymptotically isometric but not coarsely equal.*

Given a left invariant metric d on G, define

$$s\ell_d(g) = \lim_{n\to\infty} \frac{d(1, g^n)}{n}$$

$s\ell_d : G \to \mathbb{R}$ is called the *(stable) length* function. The limit always exists since d is left invariant. It is easy to see that if two left invariant proper metrics d_1, d_2 on G are asymptotically isometric, then

$$s\ell_{d_1} = s\ell_{d_2} \tag{3}$$

In [10] two metrics that satisfy (3) are called *weakly asymptotic*. To summarize the straightforward implication,

$$(1) \Rightarrow (2) \Rightarrow (3)$$

We ask a question that is analogous to the marked length iso-spectrum problem:

Question 1 If two left invariant, proper, coarsely geodesic, pseudo metrics on a (finitely generated) group have same length functions, are they coarsely equal?, i.e., (3) $\Rightarrow$ (1)?

The answer is yes for the following groups:

- $\mathbb{Z}^n$ (Burago [5]. It is implicit in the paper, see [10])
- Hyperbolic groups (Furman [12])

The main result of [10], Theorem 3.1, is

Theorem 5 *Let G be a toral relatively hyperbolic group, d_1 a proper geodesic metric, and d_2 a proper, coarsely geodesic metric on G. If they have the same stable length function, then they are coarsely equal.*

The theorem recovers the case of hyperbolic groups (our argument is different from [12]), but we use a variant of the theorem by Burago on $\mathbb{Z}^n$.

We do not give the definition of toral relatively hyperbolic groups, but discuss an example. In a way, it is a hybrid of hyperbolic groups and $\mathbb{Z}^n$. Let G be a lattice in the Lie group $SO(n, 1)$, $n > 1$. If G is a uniform lattice, then it is hyperbolic, while a non-uniform lattice is not hyperbolic if $n > 2$, but is a toral relatively hyperbolic group. So, given a proper geodesic metric d on a lattice in $SO(n, 1)$, the length function $s\ell_d$ determines d up to a constant (i.e., such metrics are coarsely equal to each other).

It is natural to ask

Question 2 If d_1, d_2 are proper, (coarsely) geodesic metric on a lattice G in $SU(n, 1)$ such that $s\ell_{d_1} = s\ell_{d_2}$, then are they coarsely equal?

If G is a uniform lattice, then it is hyperbolic and the answer is yes. If G is a non-uniform lattice with $n > 1$ then it contains (non-abelian) nilpotent subgroups. In particular G is not a toral relatively hyperbolic group. As we said the implication (3) $\Rightarrow$ (1) does not hold in general for nilpotent groups, but it is reasonable to expect the implication holds for a class of nilpotent groups (Heisenberg groups) that appears as subgroups in lattices of $SU(n, 1)$. We can ask the same question for $Sp(n, 1)$, F_4.

We mention another setting where the length function determines the group action. An $\mathbb{R}$-*tree* is a metric space in which any two points are joined by a unique arc and this arc is a geodesic. A group action is *minimal* if there is no proper invariant subtree.

Theorem 6 (Culler-Morgan [8]) *Let T_1, T_2 be $\mathbb{R}$-trees. Assume a group G acts on each of them by isometries such that actions are minimal and semi-simple. If they have the same (translation/stable) length function on G then there is a G-equivariant isomerty from T_1 to T_2.*

The assumption that actions are *semi-simple* is not so restrictive, see [8] for the definition. On a tree (T, d), we have $\tau(g) = s\ell_d(g)$ for each g.

3 Marked Length Iso-spectrum and (1, C)-Quasi-isometry

Let's go back to the marked length iso-spectrum problem. Let M be a closed Riemannian manifold, $\pi_1(M)$ its fundamental group and $\tilde{M}$ its universal cover with a metric d defined by the Riemannian metric.

Fix a point $x \in \tilde{M}$ and define a metric d_x on $\pi_1(M)$ by $d_x(g, h) = d(g(x), h(x))$. d_x is a proper, coarsely geodesic metric. For any another point $y \in \tilde{M}$, d_x and d_y are coarsely equal. Indeed for $C = 2d(x, y)$, we have $d_x =_C d_y$. It follows that $s\ell_{d_x} = s\ell_{d_y}$. So we suppress the point x and write the length function on $\pi_1(M)$ by $s\ell_d$.

Then as a function on $\pi_1(M)$,

$$\tau = s\ell_d$$

To see it, fix a point $x \in X$ then

$$\tau(g) = \lim_{n\to\infty} \frac{d(x, g^n(x))}{n} = \lim_{n\to\infty} \frac{d_x(1, g^n)}{n} = s\ell_{d_x}(g) = s\ell_d(g)$$

Now assume that M has negative curvature. Then we also know $\ell = \tau$. (In general we only know $\tau \le \ell$ since maybe $\gamma(g)$ is not distance minimizing on $\tilde{M}$) In other words, in this setting, the assumption in the marked length iso-spectrum problem and the assumption in Question 1 are equivalent.

Let $(X_1, d_1), (X_2, d_2)$ be two metric spaces such that G acts on by isometries. A G-equivariant map $f : X_1 \to X_2$ is a $(1, C)$-*quasi-isometry* for a constant $C \ge 0$ if for any $x, y \in X_1$, we have $d(x, y) =_C d(f(x), f(y))$. Using this terminology, that two metrics d_1, d_2 on X are coarsely equal is rephrased as that the identity map is a $(1, C)$-quasi-isometry (for some $C > 0$).

Remark 1 A stronger conclusion of Theorem 4 is known. On $H_3(\mathbb{Z}) \times \mathbb{Z}$, there are two (word) metrics that are asymptotically isometric but not $(1, C)$-quasi-isometric for any C, [3].

Here is a consequence of Theorem 5 that is most relevant to this paper.

Corollary 1 ([10, Corollary4.2]) *Let $(M_1, d_1), (M_2, d_2)$ be closed Riemannian manifolds of non-positive curvature with the isomorphic fundamental group G that is toral relatively hyperbolic. Assume they have the same marked length spectrum. Then there is a G-equivariant* $(1, C)$- *quasi-isometry map $f : \tilde{M}_1 \to \tilde{M}_2$.*

Notice that if $C = 0$ then M_1 and M_2 are isometric, that would solve the marked length iso-spectrum problem. As we said the length function determines a metric up to a constant on hyperbolic groups, so we can rephrase the marked length iso-spectrum problem as follows (cf. [12]):

Question 3 Let M be a closed manifold and d_1, d_2 Riemannian metrics of negative curvature (or, more generally, d_1, d_2 have non-positive curvature and $\pi_1(M)$ is toral relatively hyperbolic). Assume that there is a $\pi_1(M)$-equivariant $(1, C)$-quasi-isometry map between the universal covers $(\tilde{M}, d_1)$, $(\tilde{M}, d_2)$. Then are (M, d_1), (M, d_2) isometric?

Here are two classes of examples of closed Riemannian manifolds of non-positive curvature whose fundamental groups are toral relatively hyperbolic.

Example 2 (Dehn filling)

Let M be a 4-dimensional, non-compact, complete hyperbolic (i.e., sectional curvature $= -1$) manifold of finite volume. M has finitely many cusps and assume that the cusp subgroups $H_1, \ldots, H_n < \pi_1(M)$ are isomorphic to $\mathbb{Z}^3$. Remove disjoint open neighborhoods of the cusps from M and obtain a compact manifold M' with boundary. Each boundary component is a 3-dimensional torus. To each boundary, we glue a solid 3-dimensional torus along its boundary and obtain a closed manifold X. It is known that by choosing a gluing map carefully, we can put various Riemannian metrics of non-positive sectional curvature on X (see [Theorem 2.7, Remark 2.10] [11]). This is called a *Dehn filling* of M. $\pi_1(X)$ is a quotient of $\pi_1(M)$ (killing an infinite cyclic subgroup in each H_i) and a toral relatively hyperbolic group. $\pi_1(X)$ contains $\mathbb{Z}^2$ from each cusp.

Example 3 (Graph manifolds) Let M be a 3-dimensional, orientable, complete, non-compact, hyperbolic manifold of finite volume. As in the previous example, remove disjoint open neighborhoods of the cusps and obtain a compact manifold M' with boundary. Now prepare a copy of M', denoted by M'', make the boundary tori of M', M'' into pairs, then glue two tori in each pair by a homeomorphism, that gives a connected closed 3-manifold X. We can put various Riemannian metrics of non-positive curvature on X (see [17]. In fact, the construction applies to a closed, irreducible 3-manifold such that each piece of its JSJ-decomposition is atoroidal, i.e., hyperbolic). Then $\pi_1(X)$ is a toral relatively hyperbolic group.

In the above examples, if two metrics d_1, d_2 on X have same marked length spectrum, then by Corollary 1 there is a $\pi_1(X)$-equivariant $(1, C)$-quasi-isometry between the universal covers of X with respect to the two metrics. It would be very interesting to know if (X, d_1), (X, d_2) are isometric.

4 Heisenberg Groups

As we said there is a counter example to the conjecture by Margulis using nilpotent groups. Nilpotent groups are rich source of examples for the study of spectral geometry.

Let H_n denote the n-dimensional Heisenberg group ($n = 3, 5, 7, \ldots$). A *Heisenberg manifold* is of the form $(G\backslash H_n, g)$ where G is a (uniform) lattice in H_n and g is a Riemannian metric that lifts to a left invariant metric on H_n.

Theorem 7 (Eberlein [9], cf. [13]) *Heisenberg manifolds with the same marked length spectrum are isometric.*

For the free homotopy class of a loop, maybe there is more than one closed geodesic, so there is an issue to define the marked length spectrum ℓ on $\mathscr{C}$. See [9]. The function ℓ is different from the stable length and the translation length in general.

Let G be a simply connected nilpotent Lie group. G is *strictly nonsingular* if for all $z \in Z(G)$ and for all noncentral $x \in G$ there exists $a \in G$ such that $axa^{-1}x^{-1} = z$.

For example, the Heisenberg group H_n is strictly nonsingular. Conversely, a simply connected, strictly nonsingular, two-step nilpotent group with a 1-dimensional center is H_n for some n. $\mathbb{R} \times H_3$ is not strictly non-singular. Gornet [13, Example V in §4] found a first example of a pair of Riemannian manifolds with the same marked length spectrum, but not the same Laplace spectrum on one-forms (but the same Laplace spectrum on functions), in particular, they are not isometric. The examples are quotient by lattices G_1, G_2 in a simply connected, strictly nonsingular, three-step nilpotent group.

In connection to Question 2 we ask

Question 4 Let N be a simply connected, strictly nonsingular, nilpotent Lie group and G a lattice. Let d_1, d_2 be proper, coarsely-geodesic, G-left invariant pseudo metrics on G. If d_1, d_2 are asymptotically isometric (or with the same stable length function), then are they coarsely equal?

In view of Theorem 4,

Question 5 Does the example V (or some other examples) in [13] give a counter example to the conjecture by Margulis?

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