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Stability of Cavity Soliton for the Lugiato-Lefever Equation

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• **Aim**: Stability of stationary solutions for the nonlinear Schrödinger equation with damping and spatially homogeneous forcing terms:
  1. Introduction
  2. Existence of Stationary Solution and Linear Stability
  3. Nonlinear Stability and Strichartz Estimate

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Introduction

- Lugiato-Lefever (1987), optical cavity

$$\frac{\partial}{\partial t} A = -(1 + i\theta)A + ib^2 \frac{\partial}{\partial x^2} A$$

$$+ i|A|^2 A + F, \quad t > 0, \quad x \in T = \mathbb{R}/2\pi\mathbb{Z}.$$ (1)

- $A$; slowly varying envelope of electric field,
  $\theta > 0$; detuning parameter,
  $b > 0$; diffraction parameter,
  $F > 0$; spatially homogeneous input field,
• Physical Background

Figure 1. Ring and Fabry–Pérot cavities.
2 Existence of Stationary Solution and Linear Stability

- Homogeneous Stationary Solutions $A_S$

\[ A_S = \frac{F}{1 + i(\theta - \alpha)}, \quad \alpha = |A_S|^2, \quad (2) \]

where $\alpha$ is uniquely determined by

\[ F^2 = \alpha\{1 + (\alpha - \theta)^2\}, \quad \theta < \sqrt{3}. \quad (3) \]
• Stationary Lugiato-Lefever Equation

Change of unknown function $A$ to $B$:

Set $A = A_S(1 + B)$. Then, $B$ satisfies

$$0 = -(1 + i\theta)B + i b^2 \partial_x^2 B \quad (4)$$
$$+ i\alpha(2B + \bar{B} + B^2 + 2|B|^2 + |B|^2 B), \quad x \in T.$$ 

**Remark 1** Instead of $F$, we regard $\alpha$ as a bifurcation parameter. In that case, $A_S$ and $F$ are determined for given $\alpha$ through (2), (3).
Bifurcation

**Theorem 1 (Miyaji-Ohnishi-Y.T, 2010)**

There exist $b > 0, \sqrt{3} > \theta > 0, \eta > 0, n \in \mathbb{N}, B_0 \in \mathbb{C}$ such that equation (4) has a family of solutions

$$\{(\alpha(s), B(s)) \in \mathbb{R} \times H^2; -\eta < s < \eta\}$$

satisfying

$$B(s) = sB_0 \cos(2\pi nx) + r(s), \quad s \in (-\eta, \eta),$$
\[ \| r(s) \|_{H^2} = o(s) \quad (s \to 0), \]
\[ r(s) \perp \text{span}\{\cos(2\pi nx)\}, \]
\[ r(\cdot, x) = r(\cdot, -x), \quad x \in \text{T}, \]
\[ \alpha(s) = 1 - \frac{30\theta - 41}{9(2 - \theta)^2} s^2 + o(s^2) \quad (s \to 0). \]

Proof: “Equivariant Branching Lemma”

Zero eigenvalue of linearized operator for (4) has multiplicity 2. \( \Gamma \) is an isotropy subgroup of \( \text{O}(2) \), which consists of the identity and the reflection and leaves (4) invariant. Zero is
simple in a space of fixed points of $\Gamma$ (i.e., space consisting of even functions).


- Spectrum of Linearized Operator and Stability

$$A_s(1 + B(s)) = w(s) + iz(s), \quad s \in (-\eta, \eta),$$

where $w$ and $z$ are real-valued functions.
$L$ ; Linearized Operator around $(w, z)$ for (1):

$$
\begin{pmatrix}
-1 - 2wz & -\Delta_{b,\theta} - 2V_+ + V_- \\
\Delta_{b,\theta} + 2V_+ + V_- & -1 + 2wz
\end{pmatrix},
$$

(5)

where

$$\Delta_{b,\theta} = b^2 \partial_x^2 - \theta, \quad V_\pm = w^2 \pm z^2.$$
Theorem 2 (Miyaji-Ohnishi-Y.T, 2010)
Assume $0 < \theta < 41/30$. Then,
\[ \exists \eta' > 0, \exists \gamma \in C((\eta', -\eta'); \mathbb{R}) \text{ such that} \]
\[ \gamma(s) > 0 \ (0 < |s| < \eta'), \quad \gamma(0) = 0 \text{ and} \]
\[ \sigma(L) \subset \{ z \in \mathbb{C}; \, \text{Re} z \leq -\gamma \} \cup \{0\} \]
\[ (0 < |s| < \eta'). \]

In addition, when $\eta' > |s| > 0$, the eigenspace belonging to the zero eigenvalue of $L$ consists of the derivative of stationary solution $(w, z)$, which is an odd function.
Remark 2 A stationary solution is said to be *linearly stable* and *nonlinearly stable*, respectively, if the linearized operator around it has no spectrum on the right-half plane of $\mathbb{C}$ and if all the solutions starting from a neighborhood of it remains close to it.

Theorem 2 $\iff$ linear stability of the stationary solution given by Theorem 1 for $\theta < 41/30$.

Generally speaking, it is more difficult to
study how solutions behave near equilibrium points which are embedded in a continuum of equilibria than isolated equilibria. In the ODE case, it is well known that when all eigenvalues of the linearized operator have negative real part except for the zero eigenvalue and every orbit starting from a neighborhood of a stationary solution has a non-empty $\omega$-limit set, the stationary solution is nonlinearly stable. See, e.g., the following lecture note.

Spatial translation invariance of (1) produces the continuum of equilibria: Decomposition of effective dynamical components + a priori estimates in nonlinear PDEs

**Remark 3** It can be proved that when \( \theta > 41/30 \), the stationary solution given by Theorem 1 is nonlinearly unstable.
3 Nonlinear Stability and Strichartz Estimate

- Nonlinear Stability Theorem

**Theorem 3 (Miyaji-Ohnishi-Y.T.)** Assume $D$ is a stationary solution given by Theorem 1. Let $A_0 \in L^2$ and let $\varepsilon > 0$. For $c \in \mathbb{R}$, we put

$$D_c(x) = D(x + c).$$
Then, \( \exists \delta > 0, \, 0 \leq \exists c_0 < 2\pi \) such that

\[
\inf_{0 \leq c < 2\pi} \| A_0 - D_c \|_{L^2} < \delta
\]

\[
\implies \sup_{t \geq 0} \left[ \inf_{0 \leq c < 2\pi} \| A(t) - D_c \|_{L^2} \right] < \varepsilon,
\]

\[
\| A(t) - D_{c_0} \|_{L^2} \to 0 \quad (t \to \infty),
\]

where \( A \) is a solution of (1) with \( A(0) = A_0 \).

**Remark 4** Theorem 2 and Strichartz estimate \( \implies \) Theorem 3.
3.1 Strichartz Estimate

\[ i \partial_t u = -\partial_x^2 u + Vu, \quad t > 0, \quad x \in T \]  \hspace{1cm} (6)

\[ u(0, x) = u_0(x). \]  \hspace{1cm} (7)

(A1) \quad V \in L^\infty(T), \text{ complex valued,}

(A2) \quad \exists \gamma > 0 \text{ such that}

\[ \sigma(i(\partial_x^2 - V)) \subset \{ z \in \mathbb{C} \mid \text{Re} \ z \leq -\gamma \}. \]

Remark 5 (i) (A2) \iff \exists \gamma > 0;

\[ \text{Re}(-i(\partial_x^2 - V)v, v) \geq \gamma \|v\|_{L^2}^2, \quad v \in H^1. \]
(ii) The linearized equation of (1) has the form: \( i\partial_t u = -\partial^2_{xx} u + V_1 u + V_2 \overline{u} \). But the Strichartz estimate of this equation can be proved in the same way.

\[
U(t) = e^{it(\partial^2_{xx} - V)}, \quad U_0(t) = e^{it\partial^2_x}, \quad \mathbb{R}_+ = (0, \infty).
\]

**Theorem 4 (Miyaji-Ohnishi-Y.T.)** Assume (A1) and (A2). Let \( 0 < \gamma' < \gamma \). Then,

\[
\|e^{\gamma' t}U(\cdot)u_0\|_{L^4(\mathbb{R}_+ \times \mathbb{T})} \leq C\|u_0\|_{L^2(\mathbb{T})}.
\]
Remark 6 We note that Theorem 4 also holds for the linear Schrödinger equation with shift term:

\[ i \partial_t u = -\partial_x^2 u - i\dot{c}(t)\partial_x u + Vu, \]

\[ t > 0, \ x \in T \]

\[ u(0, x) = u_0(x), \]

where \( c(t) \) is a continuously differentiable real-valued function.

• Proof of Theorem 4
Lemma 1 (Zygmund 1974, Bourgain 1993)

\[ \| U_0(\cdot)u_0 \|_{L^4((0,2\pi) \times T)} \leq C \| u_0 \|_{L^2(T)}. \]

Proof.

\[ u_0 = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} \hat{u}_0(m) e^{imx}, \]

\[ \hat{u}_0(m) = \frac{1}{\sqrt{2\pi}} \int_{0}^{2\pi} u_0(x) e^{-imx} \, dx. \]
Then,

$$U_0(t)u_0 = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} \hat{u}_0(m)e^{-i(m^2t-mx)}.$$ 

$$\int_T \int_0^{2\pi} \left|U_0(t)u_0\right|^4 \, dt \, dx = \frac{1}{(2\pi)^2} \times$$

$$\int_T \int_0^{2\pi} \sum_{m_1,m_2,m_3,m_4} \hat{u}_0(m_1)\hat{u}_0(m_2)\hat{u}_0(m_3)\hat{u}_0(m_4)$$

$$\times e^{-i(m_1^2+m_2^2-m_3^2-m_4^2)t} e^{i(m_1+m_2-m_3-m_4)x} \, dt \, dx.$$
On the right hand side of (8), the integrals remain if

\[ m_1^2 + m_2^2 - m_3^2 - m_4^2 = 0, \quad m_1 + m_2 + m_3 + m_4 = 0 \]

\[ \iff \begin{cases} m_1 = m_3, \\ m_2 = m_4, \end{cases} \quad \text{or} \quad \begin{cases} m_1 = m_4, \\ m_2 = m_3. \end{cases} \]

Thus, the right hand side of (8) is equal to:

\[
2 \left( \sum_{m_1} \left| \hat{u}_0(m_1) \right|^2 \right) \left( \sum_{m_2} \left| \hat{u}_0(m_2) \right|^2 \right) = 2 \| u_0 \|_{L^2(T)}^4.
\]
Proof of Theorem 4 (Part 2)
For the proof of Strichartz Estimate of $U(t)$, we consider the Cauchy problem of (6) with initial data prescribed at $t = t_0$, where $t_0 \geq 0$. By Duhamel’s principle,

$$u(t) = U_0(t - t_0)u(t_0) - iF(t), \quad t \geq t_0,$$

$$F(t) = \int_{t_0}^{t} U_0(t - s)Vu(s)ds.$$
If we can prove

\[ \| F \|_{L^4((t_0, t_0+2\pi) \times T)} \leq C \| u \|_{L^\infty((t_0, t_0+2\pi); L^2(T))}, \tag{9} \]

then, for any \( n \in \mathbb{N} \cup \{0\} \), we have by Lemma 1

\[ \| u \|_{L^4((2\pi n, 2\pi (n+1)) \times T)} \leq C \| u(2\pi n) \|_{L^2(T)} + C \| u \|_{L^\infty((2\pi n, 2\pi (n+1)); L^2(T))}. \tag{10} \]
On the other hand, for $0 < \gamma' < \gamma$, (A2) yields

$$
\|U(t)u_0\|_{L^2(T)}
\leq C \exp\left(-\frac{\gamma + \gamma'}{2} t \right) \|u_0\|_{L^2(T)}, \quad t \geq 0.
$$

Use this inequality to bound the two terms on the right hand side of (10) by

$$
C \exp\left(-\pi(\gamma + \gamma')n \right) \|u_0\|_{L^2(T)}.
$$
Accordingly, we conclude that for $0 < \gamma' < \gamma$,

$$\|e^{\gamma't}u\|_{L^4(\mathbb{R}_+ \times T)} \leq \sum_{n=0}^{\infty} e^{2\pi \gamma' n} \|u\|_{L^4((2\pi n, 2\pi(n+1)) \times T)}$$

$$\leq C \sum_{n=0}^{\infty} e^{-\pi(\gamma-\gamma') n} \|u_0\|_{L^2(T)}$$

$$\leq C \|u_0\|_{L^2(T)} \implies \text{Theorem 4}$$

Estimate (9) only remains to be proved!
• Proof of Theorem 4 (Part 3)
  
  We easily see by Lemma 1 that

\[
\left\| \int_{t_0}^{t_0+2\pi} U_0(t-s)f(s) \, ds \right\|_{L^4((t_0,t_0+2\pi) \times \mathbf{T})} \\
= \left\| U_0(t) \int_{t_0}^{t_0+2\pi} U_0(-s)f(s) \, ds \right\|_{L^4((t_0,t_0+2\pi) \times \mathbf{T})} \\
\leq C \left\| \int_{t_0}^{t_0+2\pi} U_0(-s)f(s) \, ds \right\|_{L^2(\mathbf{T})}
\]
\[
C \left\| f \right\|_{L^1((t_0, t_0+2\pi); L^2(T))}.
\]

The Christ-Kiselev lemma ensures that the integral operator

\[
\int_{t_0}^{t} U_0(t - s)f(s) \, ds
\]

has the same estimate as above.
Therefore, we obtain

\[ \| F \|_{L^4((t_0, t_0 + 2\pi) \times T)} \leq C \| Vu \|_{L^1((t_0, t_0 + 2\pi); L^2(T))} \leq C \| u \|_{L^\infty((t_0, t_0 + 2\pi); L^2(T))} \]

\[ \implies \text{Inequality (9)} \]

**Lemma 2 (Christ-Kiselev, 2001)** Let \( X, Y \) be Banach spaces and let \( T > 0 \). Assume \( K(t, s) \) is continuous from \([0, T] \times [0, T]\) to
\( B(X, Y) \) and that \( 1 \leq p < q \leq \infty \). We put

\[
S f(t) = \int_0^T K(t, s) f(s) \, ds,
\]

\[
\tilde{S} f(t) = \int_0^t K(t, s) f(s) \, ds.
\]

\[
\| S f \|_{L^q((0,T);Y)} \leq C \| f \|_{L^p((0,T);X)}
\]

\[
\implies \| \tilde{S} f \|_{L^q((0,T);Y)} \leq \tilde{C} \| f \|_{L^p((0,T);X)}
\]
Remark 7 For Lemma 2 of Christ and Kiselev, see the following paper (Lemma 3.1 on page 2179):

3.2 Proof of Theorem 3

$L$; linearized operator around $D(x)$ defined as in (5).
\[ L^2_0 = \text{span}\{ \partial_x D(x)) \}, \]

\[ L^2_; \text{ complementary subspace in } L^2(T) \text{ of } L^2_0, \]

**Remark 8** We choose \( L^2_; \) such that \( L^2_; \) is an invariant subspace of \( L. \)

\[ \partial_x D; \text{ normalization in } L^2(T) \text{ of } \partial_x D, \]

\[ Q; \text{ projection from } L^2(T) \text{ to } L^2_; \]

\[ P; \text{ projection from } L^2(T) \text{ to } L^2_0, \]

**Remark 9** The projections \( Q \) and \( P \) are
explicitly expressed as follows.

\[ Qf = f - (f, E)\partial_x D, \quad Pf = (f, E)\partial_x D, \]

where \( E \) is the normalized eigenfunction belonging to the zero eigenvalue of the adjoint operator of \( L \).

Choose \( 2\pi > c_1 \geq 0 \) such that

\[
\| A_0(\cdot + c_1) - D(\cdot) \|_{L^2} = \min_{2\pi > c \geq 0} \| A_0(\cdot + c) - D(\cdot) \|_{L^2}.
\]
Remark 10 Without loss of generality, we may change the initial data $A_0(x)$ to $A_0(x + c_1)$, since equation (1) is invariant under the spatial translation. We denote $A_0(x + c_1)$ by $A_0(x)$ again and we have
\[
\text{Re}(A_0, \partial_x D) = 0. \quad (11)
\]

• Setting of Problem
It is expedient to work with the real and the imaginary parts of complex-valued function and to regard the space $L^2(T)$ as a real
Hilbert space with scalar product $\text{Re}(\cdot, \cdot)$. In that case, (11) implies that $A_0$ is orthogonal to the subspace spanned by $\partial_x D$.

$w, z$; real and imaginary parts of $D(x)$

(Decomposition of Solution into Effective Dynamical Components)


$A$; solution of (1),

**Ansatz:**

\[ A(t, x) = D(x + c(t)) \quad (12) \]

\[ + u(t, x + c(t)) + iv(t, x + c(t)), \]

$u, v \in L^2_-, \text{ real} - \text{valued},$
c(t); continuously differentiable function with c(0) = 0 (given by Implicit Function Theorem)

Insert the ansatz (12) into (1) and remove the spatial translation c(t) by the change of variables  \( \Longrightarrow \) (1) is rewritten as in the following form.

\[
\partial_t T - LT + \dot{c}(t)Q\partial_x T = Q\mathcal{F}(x, T),
\]

\( t > 0, \ x \in T, \)

\[
\dot{c}(t) = \frac{(\mathcal{F}(x, T), E)}{a - (T, \partial_x E)}, \quad t > 0,
\]
\[ a = (\partial_x D, E), \]
\[ T(0, x) = T_0(x) \quad (x \in \mathbf{T}), \]
\[ c(0) = 0, \]

where

\[ T(t, x) = \begin{pmatrix} u(t, x) \\ v(t, x) \end{pmatrix} \in L^2, \]
\[ T_0(x) = \begin{pmatrix} \text{Re}(A_0(x) - D(x)) \\ \text{Im}(A_0(x) - D(x)) \end{pmatrix}, \]
\[ \mathcal{F} \in C^2(\mathbf{T} \times \mathbf{R}^2; \mathbf{R}^4), \]
\[
\left| \mathcal{F}(x, T_1) - \mathcal{F}(x, T_2) \right| \\
\leq C \left( |T_1| + |T_2| + |T_1|^2 + |T_2|^2 \right) |T_1 - T_2| \\
(x \in \mathbb{T}, \quad T_1, T_2 \in \mathbb{R}^2),
\]
\[
\left| \partial_T \mathcal{F}(x, T_1) - \partial_T \mathcal{F}(x, T_2) \right| \\
\leq C \left( 1 + |T_1| + |T_2| \right) |T_1 - T_2| \\
(x \in \mathbb{T}, \quad T_1, T_2 \in \mathbb{R}^2),
\]

Here, \(|T| = \sqrt{u^2 + v^2}\) for \(T = (u, v) \in \mathbb{R}^2\).

Let \(U_c(t, s) \ (t \geq s \geq 0)\) denote the evolution operator associated with the infinitesimal
generator $L - \dot{c}(t)\partial_x$ for each $c(t)$.

**Remark 12** The Strichartz estimate such as Theorem 4 is applicable to the first and the second components of the solution $T$ of (13) (see Remark 5 (ii) and Remark 6). Because

$$\dot{c}(t)Q\partial_x T = \dot{c}(t)\partial_x T - \dot{c}(t)P\partial_x T,$$

and the term $\dot{c}(t)P\partial_x T$ can be regarded as a small regular perturbation as long as $\dot{c}(t)$ is small.
Application of Theorem 4 to (13)

⇒ A Priori Estimates for \((T, \dot{c})\) with Exponential Decay in Time

⇒

\[
\|T(t)\|_{L^2(T)} \to 0,
\]

\[
\exists c_0 \in \mathbb{R}; \quad c(t) \to c_0 \quad (t \to \infty).
\]