The ultracontractivity of a non-symmetric
Markovian semigroup and its applications

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1. Convergence of the transition probability

**Killing at the boundary**

- $M$: a compact connected Riemannian manifold with a boundary $\partial M$.
- $m$: normalized Riemannian volume
- $\triangle$: the Laplace-Beltrami operator
- $b$: a vector field
- $X_t, Y_t$: diffusion processes generated by $\triangle$ and $\triangle + b$ respectively:

  generator    fundamental solution
  \[
  \begin{align*}
  \triangle & \quad p(t, x, y) \\
  \triangle + b & \quad q(t, x, y)
  \end{align*}
  \]

We assume $\text{div} \ b = 0$ and we impose the **Dirichlet** boundary condition.
Probabilistic point of view

\[ P_x(X_t \in dy) = p(t, x, y)m(dy) \]

\((X_t)\) dies when it reaches the boundary.
• Differential equation point of view

\[ u(t, x) = \int_M p(t, x, y) f(y) m(\,dy) \] satisfies the following differential equation:

\[
\begin{cases}
\frac{\partial u}{\partial t} = \Delta u \\
u(0, x) = f(x) \\
u(t, x) = 0, \quad x \in \partial M.
\end{cases}
\]

\[ p(t, x, y) \to 0, \]
\[ q(t, x, y) \to 0. \]

How fast?

\[
\tilde{\lambda}_{1 \to \infty} = - \lim_{t \to \infty} \frac{1}{t} \log \sup_{x,y \in M} p(t, x, y),
\]
\[
\lambda_{1 \to \infty} = - \lim_{t \to \infty} \frac{1}{t} \log \sup_{x,y \in M} q(t, x, y).
\]
Our aim is to show that
\[ \tilde{\lambda}_{1 \to \infty} \leq \lambda_{1 \to \infty}. \]

A Non-symmetric diffusion dies quicker than the symmetric diffusion.

**Convergence to an invariant measure**

- \( M \): a compact connected Riemannian manifold without boundary.
- \( X_t, Y_t \): diffusion processes generated by \( \triangle \) and \( \triangle + b \) respectively:

<table>
<thead>
<tr>
<th>generator</th>
<th>fundamental solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \triangle )</td>
<td>( p(t, x, y) )</td>
</tr>
<tr>
<td>( \triangle + b )</td>
<td>( q(t, x, y) )</td>
</tr>
</tbody>
</table>

We assume \( \text{div} \ b = 0 \).

\[ p(t, x, y) \to 1, \]
\[ q(t, x, y) \to 1. \]
How fast?

\[ \tilde{\gamma}_{1 \to \infty} = - \lim_{t \to \infty} \frac{1}{t} \log \sup_{x,y \in M} |p(t, x, y) - 1|, \]

\[ \gamma_{1 \to \infty} = - \lim_{t \to \infty} \frac{1}{t} \log \sup_{x,y \in M} |q(t, x, y) - 1|. \]

Our aim is to show that

\[ \tilde{\gamma}_{1 \to \infty} \leq \gamma_{1 \to \infty}. \]

A Non-symmetric diffusion converges to the invariant measure quicker than the symmetric diffusion.
2. Ultracontractivity

A semigroup \( \{T_t\} \) is called ultracontractive if \( T_t : L^1 \to L^\infty \) is bounded for all \( t > 0 \).

It is well-known that the following three conditions are equivalent for a symmetric Markovian semigroup. Let \( \mu > 0 \) be given.

(i) \( \exists c_1 > 0, \forall f \in L^1: \)

\[
\|T_t f\|_\infty \leq c_1 t^{-\mu/2} \|f\|_1, \quad \forall t > 0.
\]

(ii) \( \exists c_2 > 0, \forall f \in \text{Dom}(\mathcal{E}) \cap L^\infty: \)

\[
\|f\|_2^{2+4/\mu} \leq c_2 \mathcal{E}(f,f) \|f\|_1^{4/\mu}.
\]

(iii) \( \mu > 2, \exists c_3 > 0, \forall f \in \text{Dom}(\mathcal{E}): \)

\[
\|f\|_2^{2_{\mu/(\mu-2)}} \leq c_3 \mathcal{E}(f,f).
\]

We extend this result for non-symmetric Markovian semigroups.
Non-symmetric Markovian semigroups

We give a framework in general Hilbert space scheme.

- $H$: a Hilbert space
- $\{T_t\}$: a contraction $C_0$ semigroup
- $\{T^*_t\}$: the dual semigroup
- $\mathfrak{A}, \mathfrak{A}^*$: the generators of $\{T_t\}$ and $\{T^*_t\}$

A natural bilinear form $\mathcal{E}$ is defined by

$$\mathcal{E}(u, v) = -(\mathfrak{A}u, v).$$

We do not assume the sector condition and so we can not use this bilinear form.
We introduce a symmetric bilinear form. For this, we assume the following condition:

(A.1) $\text{Dom}(A) \cap \text{Dom}(A^*)$ is dense in $\text{Dom}(A)$ and $\text{Dom}(A^*)$.

Under this condition, we define a symmetric bilinear form $\tilde{E}$ by

$$\tilde{E}(u, v) = -\frac{1}{2} \{ (Au, v) + (u, Av) \}, \quad u, v \in \text{Dom}(A) \cap \text{Dom}(A^*).$$

**Proposition 1.** Under the condition (A.1), $\tilde{E}$ is closable and its closure contains $\text{Dom}(A)$ and $\text{Dom}(A^*)$. 
Covex set preserving property

- \( C \): a convex set of \( H \).
- \( Pu \): the shortest point from \( u \) to \( C \)

\[(u - Pu, v - Pu) \leq 0, \quad \forall v \in C.\]

**Theorem 2.** If \( \{T_t\} \) and \( \{T^*_t\} \) preserve a convex set \( C \), then \( Pu \in \text{Dom}(\tilde{\mathcal{E}}) \) for any \( u \in \text{Dom}(\tilde{\mathcal{E}}) \) and we have

\[\tilde{\mathcal{E}}(Pu, u - Pu) \geq 0.\]
Markovian semigroup

- \((M, m)\): a measure space
- \(H = L^2(m)\): a Hilbert space
- \(\{T_t\}\): a Markovian semigroup

We assume that \(\{T_t^*\}\) is also a Markovian semigroup.

Under the assumption (A.1), we can define a symmetric bilinear form \(\tilde{E}\) and \(\tilde{E}\) is a Dirichlet form.
We have the following implications. For $\mu > 0$,

\[
\|T_t f\|_\infty \leq c_1 t^{-\mu/2} \|f\|_1, \quad \forall t > 0
\]

\[
\uparrow \quad \downarrow \quad \text{under (1)}
\]

\[
\|f\|_2^{2+4/\mu} \leq c_2 \tilde{E}(f, f) \|f\|_1^{4/\mu}
\]

\[
\updownarrow
\]

\[
\|f\|_2^2 \leq c_3 \tilde{E}(f, f) \quad (\mu > 2)
\]

(1)

\[
(\mathbf{A}^2 f, f)_2 + (\mathbf{A} f, \mathbf{A} f)_2 \geq 0.
\]

(1) holds if $\mathbf{A}$ is normal, i.e. $\mathbf{A} \mathbf{A}^* = \mathbf{A}^* \mathbf{A}$. 
Moreover

\[ \|T_t f\|_\infty \leq c_1 t^{-\mu/2} \|f\|_1, \quad \forall t \in (0, 1] \]

\[ \uparrow \downarrow \text{under (2)} \]

\[ \|f\|_2^{2+4/\mu} \leq c_2 (\tilde{E}(f, f) + \|f\|_2^2) \|f\|_1^{4/\mu} \]

\[ \Downarrow \]

\[ \|f\|_2^{2\mu/(\mu-2)} \leq c_3 (\tilde{E}(f, f) + \|f\|_2^2) \quad (\mu > 2) \]

There there exists a constant \( M > 0 \) so that for all \( f \in \text{Dom}(\mathcal{A}^2) \)

\[ ((\mathcal{A} - M)^2 f, f)_2 + ((\mathcal{A} - M) f, (\mathcal{A} - M) f)_2 \geq 0. \]

(2)
\textbf{\(L^2\) theory}

We introduce three indices.

\begin{equation}
\lambda_P = \inf \left\{ \frac{\tilde{E}(f, f)}{\|f\|_2^2}; \; f \neq 0 \right\} \quad \text{i.e.,} \quad \lambda_P \|f\|_2^2 \leq \tilde{E}(f, f).
\end{equation}

\begin{equation}
\lambda_{2\to2} = - \lim_{t \to \infty} \frac{1}{t} \log \|T_t\|_{2\to2}.
\end{equation}

\begin{equation}
\lambda_B = \inf \Re(\sigma(-A)).
\end{equation}

(3) is equivalent to

\begin{equation}
\|T_t f\|_2^2 \leq e^{-2\lambda t} \|f\|_2^2, \quad \forall t > 0.
\end{equation}
Theorem 3. We have the following inequalities:

\[(7) \quad \lambda_P \leq \lambda_{2 \rightarrow 2} \leq \lambda_B\]

Theorem 4. If \( \mathcal{A} \) is normal, then we have

\[(8) \quad \lambda_P = \lambda_{2 \rightarrow 2} = \lambda_B.\]

From these theorems, we have \( \tilde{\lambda}_{2 \rightarrow 2} \leq \lambda_{2 \rightarrow 2} \).
Ultracontractivity

We introduce the following index:

\[ \lambda_{1\to\infty} = - \lim_{t\to\infty} \frac{1}{t} \log \|T_t\|_{1\to\infty}. \]  

\( (9) \)

**Theorem 5.** Let \( \mu > 0 \) be given. Assume that there exists a constant \( c_2 > 0 \) such that

\[ \|f\|_2^{2+(4/\mu)} \leq c_2 \tilde{\mathcal{E}}(f,f) \|f\|_1^{4/\mu}, \quad \forall f \in \text{Dom}(\tilde{\mathcal{E}}) \cap L^1. \]

\( (10) \)

Then we have \( \lambda_{1\to\infty} = \lambda_{2\to2} \). Therefore

\[ \lambda_{1\to\infty} \geq \tilde{\lambda}_{1\to\infty}. \]

\( (11) \)
3. Dirichlet forms having invariant measure

We continue to assume the sector condition. In addition, we assume

- $m$ is an invariant probability measure.

\[ \int_M T_t f \, dm = \int_M f \, dm \]

- $T_t 1 = 1$ and $\mathcal{A} 1 = 0$ and 1 is the unique eigenvalue.

- $m(f) = \int_M f(x) \, m(dx)$. 
We have the following implications. For $\mu > 0$,

\[
\|T_t f - m(f)\|_\infty \leq c_1 t^{-\mu/2} \|f\|_1, \quad \forall t > 0
\]

\[
\uparrow \quad \downarrow \quad \text{under (12)}
\]

\[
\|f - m(f)\|_2^{2+4/\mu} \leq c_2 \tilde{E}(f, f) \|f - m(f)\|_1^{4/\mu}
\]

\[
\uparrow
\]

\[
\|f - m(f)\|_2^{2/\mu/(\mu-2)} \leq c_3 \tilde{E}(f, f) \quad (\mu > 2)
\]

(12)

\[
(\mathcal{A}^2 f, f)_2 + (\mathcal{A} f, \mathcal{A} f)_2 \geq 0.
\]
Moreover

\[ \| T_t f - m(f) \|_\infty \leq c_1 t^{-\mu/2} \| f \|_1, \quad \forall t \in (0, 1] \]

\[ \uparrow \quad \downarrow \text{under (13)} \]

\[ \| f - m(f) \|_2^{2+4/\mu} \leq c_2 (\tilde{E}(f, f) + \| f \|_2^2) \| f - m(f) \|_1^{4/\mu} \]

\[ \uparrow \]

\[ \| f - m(f) \|_2^{2\mu/(\mu-2)} \leq c_3 (\tilde{E}(f, f) + \| f \|_2^2) \quad (\mu > 2) \]

There there exists a constant \( M > 0 \) so that for all \( f \in \text{Dom}(\mathcal{A}^2) \)

(13) \[ ((\mathcal{A} - M)^2 f, f)_2 + ((\mathcal{A} - M)f, (\mathcal{A} - M)f)_2 \geq 0. \]
**L^2** theory

We introduce the following three indices:

\[
\gamma_P = \inf \left\{ \frac{\tilde{E}(f,f)}{\| f - m(f) \|_2^2} ; f \neq m(f) \right\} \quad \text{i.e., } \gamma_P \| f - m(f) \|_2^2 \leq \tilde{E}(f,f). \tag{14}
\]

\[
\gamma_{2\to2} = -\lim_{t\to0} \frac{1}{t} \log \| T_t - m \|_{2\to2} \tag{15}
\]

\[
-\gamma_{SG} = \sup \Re(\sigma(A) \setminus \{0\}). \tag{16}
\]

\(\gamma_P\) is called a Poincaré constant. (14) is equivalent to

\[
\| T_t f - m(f) \|_2^2 \leq e^{-2\lambda t} \| f - m(f) \|_2^2, \quad \forall t > 0. \tag{17}
\]
We have the following theorems.

**Theorem 6.** We have the following inequalities:

\[ \gamma_P \leq \gamma_{2 \to 2} \leq \gamma_{SG}. \]  

(18)

**Theorem 7.** If \( \mathfrak{A} \) is normal, then we have

\[ \gamma_P = \gamma_{2 \to 2} = \gamma_{SG}. \]  

(19)

From these theorem, we have

\[ \tilde{\gamma}_{2 \to 2} \leq \gamma_{2 \to 2}. \]
**Ultracontractivity**

We introduce another index $\gamma_{1 \to \infty}$ as follows:

\[(20)\]

\[
\gamma_{1 \to \infty} = - \lim_{t \to \infty} \frac{1}{t} \log \|T_t - m\|_{1 \to \infty}
\]

**Proposition 8.** We have

\[(21)\]

\[
\gamma_{1 \to \infty} \leq \gamma_{2 \to 2}.
\]

Moreover, if $\gamma_{1 \to \infty} > -\infty$, then the identity holds.
**Theorem 9.** Let $\mu > 0$. Assume the following Nash inequality: there exists a constant $c_2 > 0$ such that

\begin{equation}
\| f - m(f) \|_2^{2+(4/\mu)} \leq c_2 \tilde{E}(f, f) \| f - m(f) \|_1^{4/\mu}, \quad \forall f \in \text{Dom}(\tilde{E}) \cap L^1.
\end{equation}

Then $\gamma_{1 \to \infty} > 0$ and so $\gamma_{2 \to 2} = \gamma_{1 \to \infty}$. Therefore we have

\begin{equation}
\tilde{\gamma}_{1 \to \infty} \leq \gamma_{1 \to \infty}.
\end{equation}
4. Compact Riemannian manifold with a boundary

- $M$: $d$-dimensional compact Riemannian manifold with a boundary $\partial M$.
- $m$: normalized Riemannian volume.
- The generator is given by
  \begin{equation}
  \mathcal{A} = \triangle + b. \tag{24}
  \end{equation}
  
  We assume that $\text{div} \, b \geq 0$ and we impose the Dirichlet boundary condition:
  \begin{equation}
  f = 0 \quad \text{on} \, \partial M. \tag{25}
  \end{equation}
- The dual operator is
  \begin{equation}
  \mathcal{A}^* = \triangle - \nabla b - \text{div} \, b. \tag{26}
  \end{equation}
- Associated symmetric form is
  \begin{equation}
  \tilde{\mathcal{E}}(u, v) = \int_M (\nabla u, \nabla v) \, dm + \frac{1}{2} \int_M uv \, \text{div} \, b \, dm. \tag{27}
  \end{equation}
**Theorem 10.** We have

\[ \tilde{\lambda}_{2 \to 2} \leq \lambda_{2 \to 2}. \]

If \( \mathfrak{A} \) is normal, then \( \tilde{\lambda}_{2 \to 2} = \lambda_{2 \to 2} \).

Since \( \mathcal{M} \) is compact, the following Nash inequality holds:

\[ \| f \|_2^{2+(4/d)} \leq c_2 \bar{\mathcal{E}}(f, f) \| f \|_1^{4/d}. \]

**Theorem 11.** We have

\[ \tilde{\lambda}_{1 \to \infty} \leq \lambda_{1 \to \infty}. \]

If \( \mathfrak{A} \) is normal, then \( \tilde{\lambda}_{1 \to \infty} = \lambda_{1 \to \infty} \).
The semigroup $T_t$ has a transition density $q(t, x, y)$ w.r.t. $\nu$. $q(t, x, y)$ is $C^\infty$ from the hypoellipticity. From the definition,

$$\lambda_{1 \to \infty} = - \lim_{t \to \infty} \frac{1}{t} \log \sup_{x, y \in M} q(t, x, y).$$

Similary for $\tilde{E}$, there exists a transition density $p(t, x, y)$ w.r.t. $\nu$ and

$$\tilde{\lambda}_{1 \to \infty} = - \lim_{t \to \infty} \frac{1}{t} \log \sup_{x, y \in M} p(t, x, y).$$

We have

$$\tilde{\lambda}_{1 \to \infty} \leq \lambda_{1 \to \infty}.$$
Theorem 12. We have

\[ \tilde{\lambda}_{2 \rightarrow 2} \leq \lambda_{2 \rightarrow 2} = \lambda_B. \]  

If \( \tilde{\lambda}_{2 \rightarrow 2} = \lambda_{2 \rightarrow 2} \), then \( \mathcal{A} \) has an eigenvalue \( -\tilde{\gamma}_{2 \rightarrow 2} \) and its eigenfunction coincides with the eigenfunction \( \varphi \) of \( \frac{1}{2}(\mathcal{A} + \mathcal{A}^*) \) for the eigenvalue \( -\tilde{\gamma}_{2 \rightarrow 2} \). The vector fields \( b \) satisfies

\[ b\varphi = -\frac{1}{2}(\text{div } b)\varphi. \]
Example: the unit disc

- \( M = \{ x \in \mathbb{R}^2; |x| \leq 1 \} \)
- \( \text{div } b = 0. \)
- \( r = (x_1^2 + x_2^2)^{1/2}. \)

\[ br \neq 0 \quad \Rightarrow \quad \tilde{\lambda}_{2 \rightarrow 2} < \lambda_{2 \rightarrow 2} \]
5. Compact Riemannian manifold without boundary

Let us return to the diffusion on a Riemannian manifold $M$ generated by

$$\mathcal{A}f = \Delta f + bf = \Delta f + (\nabla f, \omega_b).$$

If $M$ is compact, then there exists an invariant probability measure.

- $\nu$: an invariant probability measure: $\nu = e^{-U} m$

We use the following notations

- $\nabla$: the Levi-Civita covariant derivative
- $\nabla^*$: the dual operator of $\nabla$ w.r.t. $m$
- $\nabla^*_\nu$: the dual operator of $\nabla$ w.r.t. $\nu$
- $\omega_b$: 1-form corresponding to $b$
We now change the reference measure to $\nu$. So our Hilbert space changes to $L^2(\nu)$.

Set

$$\mathcal{G}_\nu = \{ \mathcal{A} ; \mathcal{A} \text{ has an invariant measure } \nu \}$$

We set

$$\tilde{b} = \frac{1}{2} (\nabla U)^\# + b,$$

$$\omega_{\tilde{b}} = \frac{1}{2} \nabla U + \omega_b.$$
Theorem 13. $\mathfrak{A} \in \mathcal{G}_\nu$ if and only if $\nabla^*_\nu \omega \tilde{b} = 0$. In this case, 

$$\mathfrak{A} f = -\nabla^*_\nu \nabla f + (\omega \tilde{b}, \nabla f)$$

and 

$$\mathfrak{A}^* f = -\nabla^*_\nu \nabla f - (\omega \tilde{b}, \nabla f).$$

Further the associated symmetric Dirichlet form is given by 

$$\tilde{\mathcal{E}}(f, h) = \int_M (\nabla f, \nabla h) d\nu.$$
\( T_t \) has a density \( q(t, x, y) \) with respect to \( \nu \). Define
\[
\gamma_{1 \to \infty} = - \lim_{t \to \infty} \frac{1}{t} \log \sup_{x, y \in M} |q(t, x, y) - 1|.
\]

Let \( p(t, x, y) \) be a transition density for \( \tilde{E} \). Define
\[
\tilde{\gamma}_{1 \to \infty} = - \lim_{t \to \infty} \frac{1}{t} \log \sup_{x, y \in M} |p(t, x, y) - 1|.
\]

**Theorem 14.** We have
\[
\tilde{\gamma}_{1 \to \infty} \leq \gamma_{1 \to \infty}.
\]

The equality holds if \( \mathcal{A} \) is normal.
Recall that

\[ \gamma_{SG} = \inf \{ \Re \eta; \eta \in \sigma(-A) \setminus \{0\} \}. \]

We have \( \gamma_{1 \to \infty} = \gamma_{SG} \).

**Theorem 15.** If \( \tilde{\gamma}_{1 \to \infty} = \gamma_{1 \to \infty} \), then \(-A\) has an eigenvalue \( \xi \) so that \( \Re \xi = \tilde{\gamma}_{1 \to \infty} \) and its eigenfunctions is also an eigenfunction of \( \nabla^* \nabla \) for an eigenvalue \( \tilde{\gamma}_{1 \to \infty} \).
Example: 2-dimensional torus

- $M = T^2$
- $(x, y)$: the standard local coordinate
- $b = f(x) \frac{\partial}{\partial y} + g(y) \frac{\partial}{\partial x}$

Then

$$f = \text{constant}, \ g = \text{constant} \quad \Rightarrow \quad \tilde{\gamma}_{1\rightarrow \infty} = \gamma_{1\rightarrow \infty}$$

$$f = 0 \quad \Rightarrow \quad \tilde{\gamma}_{1\rightarrow \infty} = \gamma_{1\rightarrow \infty}$$

$$f \neq \text{constant}, \ g \neq \text{constant} \quad \Rightarrow \quad \tilde{\gamma}_{1\rightarrow \infty} > \gamma_{1\rightarrow \infty}.$$
Thanks a lot!