Sendai Symposium, August 17, 2011

Introduction to the Conley index theory

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Plan of the talk:

(1) Dynamical dichotomy: Gradient-like vs Recurrent

- Conley's fundamental theorem of dynamics
- Morse decomposition

(2) Conley index for ODEs and iterated maps

- isolated invariant set, index pair
- shift equivalence

(3) Conley-Morse theory for dynamics

- connection matrix, transition matrix

Dynamical system   ODE / iterated map

dynamical system of continuous time (ODE)

$$\begin{aligned} \frac{dx}{dt} &= f(x) \\ x(0) &= \xi \end{aligned} \quad \Longrightarrow \quad x(t) = \varphi(t;\xi)$$

Existence/Uniqueness of ODE

dynamical system of discrete time (iterated map)

$$\begin{aligned} x_{n+1} &= f(x_n) \\ x_0 &= \xi \end{aligned} \quad \Longrightarrow \quad x_n = f^n(\xi) \left(= \overbrace{f \circ \cdots \circ f}^{n \text{ times}} (\xi) \right)$$

a flow on X with *continuous/discrete time*

$$\varphi : \mathbb{T} \times X \to X \quad (\mathbb{T} = \mathbb{R} \text{ or } \mathbb{Z})$$

$$\varphi^t(x) = \varphi(t,\xi) \ (t \in \mathbb{T}, \xi \in X)$$

Given a dynamical system,
how can one understand its global dynamics?

Dynamical dichotomy: Gradient-like vs. Recurrent

Dynamics of gradient systems is simple (Morse theory):

The dynamics of a gradient system
can be described by its equilibria
and connecting orbits between them

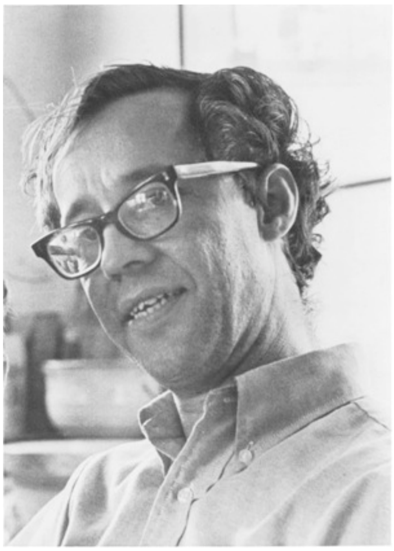
For *smooth hyperbolic* dynamical systems:

Spectral decomposition theorem [Smale 1960's]

gradient-like dynamics = filtration (lattice)

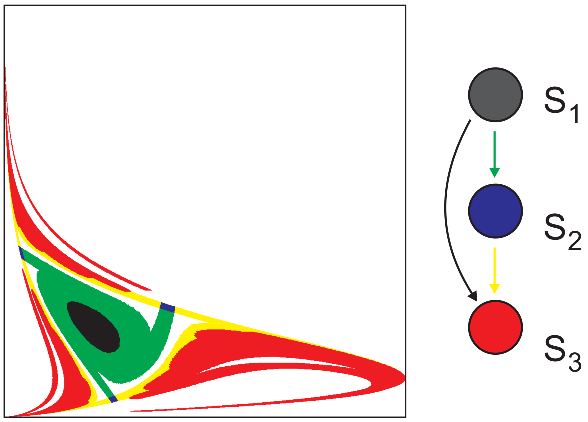
recurrent dynamics = hyperbolic basic set

For more general *topological* dynamics


Charles C. Conley 1933-1984

Fundamental Theorem of Dynamics
[Conley 1978?]

Any continuous dynamical system
on a compact metric space
is *gradient-like*
off of its *chain-recurrent set.*



(X, d) : metric space $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$

$\varphi : X \times \mathbb{T} \to X$
: a continuous flow on X with *continuous/discrete time*

Notation $\varphi^t(x) = \varphi(x, t)$ $(x \in X, t \in \mathbb{T})$

Definition

(1) $\{x = x_0, x_1, x_2, \ldots, x_n = y \ ; t_1, t_2, \ldots, t_n\}$
: an ε-chain from x to y for the flow $\varphi = \{\varphi^t\}_{t \in \mathbb{T}}$
$\overset{\text{def}}{\Leftrightarrow}$ $\forall i = 1, \ldots, n;\ t_i \geq 1$ and $d(\varphi^{t_i}(x_{i-1}), x_i) < \varepsilon$

(2) x : a chain-recurrent point for φ
$\overset{\text{def}}{\Leftrightarrow}$ $\forall \varepsilon > 0$; there is an ε-chain from x to itself

(3) chain-recurrent set for φ
$\mathcal{R}(\varphi)$ = the set of all chain-recurrent points for φ

Proposition

The chain-recurrent set is closed and flow-invariant

Fundamental Theorem of Dynamical Systems [Conley]

For a flow $\varphi = \{\varphi^t\}_{t\in\mathbb{T}}$ on a compact metric space X,
there exists a continuous function $L : X \to \mathbb{R}$
which is strictly decreasing on $X \setminus \mathcal{R}(\varphi)$,
and moreover, $L(\mathcal{R}(\varphi))$ is totally disconnected.

In particular, the chain-recurrent set is
the *largest* non-trivial invariant set of a dynamical system.

Possibly *infinitely many* components in chain-recurrent set

- Not easy to obtain the decomposition in practice
- Decomposition is *NOT* robust under perturbation

Morse decomposition:
a *finite analogue* of the Conley-type decomposition

- Morse decomposition can be defined for an isolated invariant set of a flow
- Morse decomposition is not unique in general
- Morse decomposition is robust under perturbation

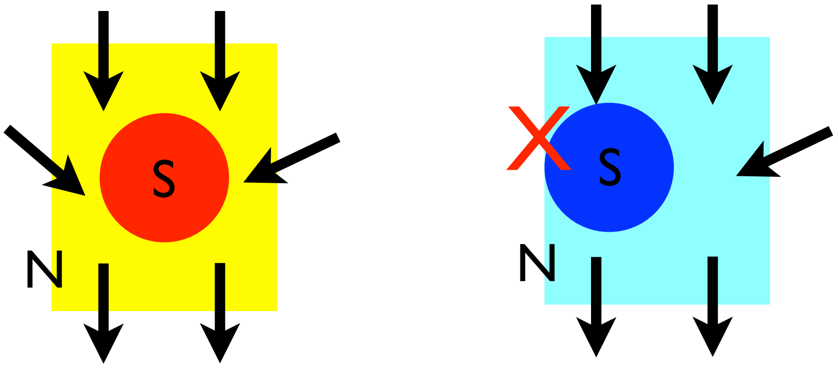
Isolated invariant set and its Morse decomposition

X : a locally compact metric space
$\varphi = \{\varphi^t\}_{t \in \mathbb{T}}$: a cont/discrete time flow on X
S : a compact invariant set of φ

Definition

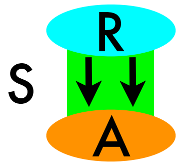
(1) An invariant set S is isolated

$$\overset{\text{def}}{\Leftrightarrow} \exists N \text{: cpt nbd s.t. } S = \mathrm{Inv}(N) \subset \mathrm{int}(N)$$



N : an isolating neighborhood of S

(2) (A, R) : AR decomposition of S



$$\overset{\text{def}}{\Leftrightarrow} \exists U \text{: nbd of } A \text{ s.t. } A = \omega(U \cap S)$$
$$R = \{x \in S \mid \omega(x) \cap A = \emptyset\}$$

In this case, we have: $S = R \cup \mathrm{Connect}(R, A) \cup A$

$$\mathrm{Connect}(R, A) = \{x \in S \mid \omega(x) \subset A \;\&\; \alpha(x) \subset A\}$$

the set of *connecting orbits* from R to A

Remark

If S is not an (absolute) attractor,

the "attractor" $A (\subset S)$ of an AR decomposition of S may not be an absolute attractor,

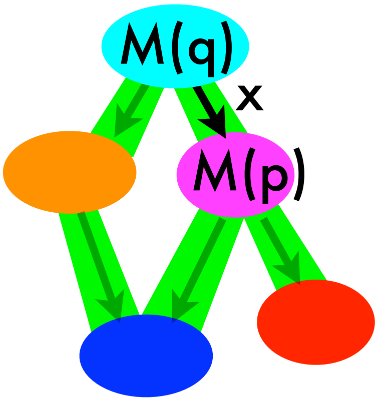
hence, A should be considered as a relative attractor.

(3) a <u>Morse decomposition</u> of an isolated inv set S is
a finite collection $\{M(p) \mid p \in P\}$ of
disjoint compact invariant subsets of S
and a strict partial order $<$ on P satisfying:

$$\forall x \in S \setminus (\cup_{p \in P} M(p)) \ \exists p, q \in P \text{ with } p < q$$

s.t. $x \in \mathrm{Connect}(M(q), M(p))$

Minimal such partial order is called
the <u>flow-defined order</u>



<u>Remark</u>

Most interesting (and non-trivial) dynamics is in a recurrent behavior

Yet, there are reasons for studying non-recurrent dynamics

Morse decomposition describes how recurrent invariant sets are connected

Change of connecting orbits (under perturbation of the flow) often leads to change of global structure of dynamics

In application, connecting orbits sometimes represent specific solutions of interest
(e.g. traveling waves of PDEs)

(1) Given a dynamical system,
how can one obtain its Morse decomposition?

(2) Once a Morse decomposition is obtained,
how can one understand its recurrent dynamics?

Possible answers:

(1) Computer-assisted Morse decomposition

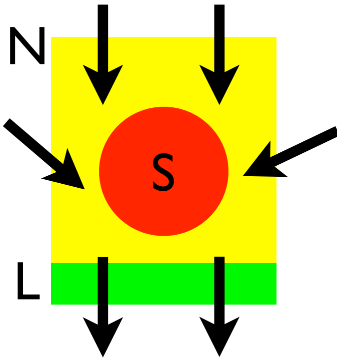
see the second talk

(2) Conley index

an extension of the Morse index

Conley index for continuous time flows (ODEs)

X : a locally compact metric space
$\varphi = \{\varphi^t\}_{t\in\mathbb{R}}$: a continuous time flow on X
S : an isolated invariant set of φ
N : an isolating neighborhood of S



Definition

$L(\subset N)$ is an exit set of N, if

[isolation] $N\setminus L$ isolates S

[pos inv] orbits from L must stay in L
as long as they remain in N

[exit] orbits leaving N must go through L

(N, L) : an index pair of S

Theorem [Conley], [Salamon]

(N_i, L_i) $(i = 1, 2)$: index pairs of an isol inv set S

$\Rightarrow$ $N_1/L_1 \sim N_2/L_2$ (homotopically equivalent)

Definition

homotopy Conley index

$h(S)$ = homotopy type of the quotient space N/L

homology Conley index

$$CH_*(S) \;=\; H_*(N/L, [L]) \;\cong H_*(N, L)$$

Remark:

(1) $h(S)$ and hence $CH_*(S)$ independent of choice of index pairs

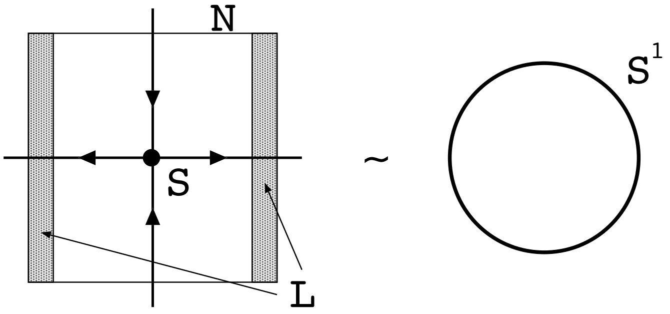
(2) Use field coefficient (very often $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$)

i.e. $CH_*(S)$ is a graded vector space

(3) Conley index is a generalization of Morse index

$$h(\text{crit pt of Morse index } k) = [S^k]$$

$$CH_*(\text{crit pt of Morse index } k) = \begin{cases} \mathbb{Z}_2 & (* = k) \\ 0 & (\text{else}) \end{cases}$$



Conley index for discrete time flows (iterated maps)

$f : X \to X$: a continuous map (not necessary a homeo)

$N \subset X$: a compact domain

$$\mathrm{Inv}_f(N) = \left\{ x \in N \;\middle|\; \begin{array}{l} \exists \sigma : \mathbb{Z} \to X \text{ s.t.} \\ \quad \sigma(0) = x \\ \quad \forall n \in \mathbb{Z}\ f(\sigma(n)) = \sigma(n+1) \\ \quad \sigma(\mathbb{Z}) \subset N \end{array} \right\}$$

Definition of isolated invariant set and isolating neighborhood remain the same as the continuous time case.

An invariant set S is isolated

$$\overset{\text{def}}{\Leftrightarrow} \exists N\text{: cpt nbd s.t. } S = \mathrm{Inv}(N) \subset \mathrm{int}(N)$$

N : an isolating neighborhood of S

Definition of an index pair also remains the same.

$L(\subset N)$ is an exit set of N, if

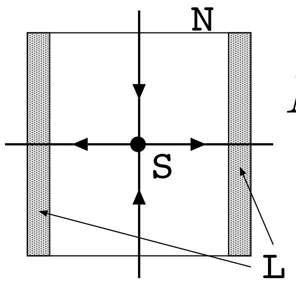
[isolation] $N \setminus L$ isolates S

[pos inv] $f(L) \cap \overline{N \setminus L} = \emptyset$

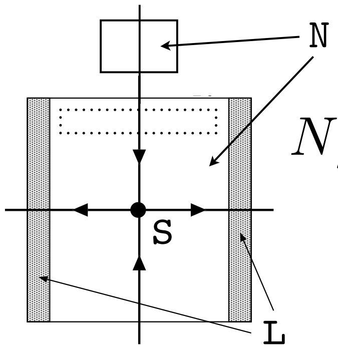
[exit] L is a nbd of $\{x \in N \mid f(x) \notin N\}$ in N

(N, L) : an index pair (filtration pair) of S

However, the homotopy type of the quotient space N/L does depend on the choice of index pairs.



$N/L \sim S^1$



$N/L \sim S^1 \sqcup \mathrm{pt}$

Shift equivalence

Two maps $a : A \to A$ and $b : B \to B$ are <u>shift equivalent</u>, if $\exists r : A \to B,\ \exists s : B \to A$ s.t.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, shift right, "r"'] & A \arrow[d, shift right, "r"'] \\ B \arrow[r, "b"'] \arrow[u, shift right, "s"'] \arrow[ru, phantom, "\circlearrowleft"] & B \arrow[u, shift right, "s"'] \end{tikzcd} \quad \text{and } \exists m \in \mathbb{N}, \quad \begin{array}{l} r \circ s = b^m \\ s \circ r = a^m \end{array}$$

For a linear map $e : V \to V$ on a fin-dim vector space,

$$\mathrm{gker}(e) = \{v \in V \mid \exists n\ e^n(x) = 0\}$$

<u>Prop</u>

(1) $e : V \to V$ and $\bar{e} : \bar{V} = V/\mathrm{gker}(e) \to \bar{V}$ are shift equiv

(2) $e_i : V_i \to V_i\ (i = 1, 2)$ are shift equiv $\Leftrightarrow$ $\bar{e}_i : \bar{V}_i \to \bar{V}_i\ (i = 1, 2)$ are linearly conjugate

<u>Cor</u> $e_i : V_i \to V_i\ (i = 1, 2)$ are shift equiv $\Rightarrow$ $\bar{e}_1$ and $\bar{e}_2$ have the same non-zero eigenvalues

For an index pair $Q = (N, L)$,
define the induced map $f_Q : (N/L, [L]) \circlearrowleft$ by

$$f_Q(x) = \begin{cases} [f(x)] & x \neq [L] \\ [L] & x = [L] \end{cases}$$

<u>Theorem</u> ([Mrozek],[Szymczak],[Franks-Richeson])

$Q_i = (N_i, L_i)\ (i = 1, 2)$: index pairs of an isol inv set S

⇨ $f_{Q_i} : (N_i/L_i, [L_i]) \circlearrowleft$ are shift equivalent

<u>Definition</u>

homotopy Conley index for iterated maps

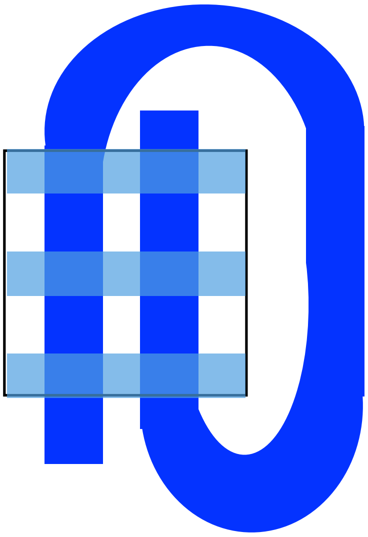
$$h(S, f) = [f_Q : (N/L, [L]) \circlearrowleft]_{\text{shift equiv}}$$

homology Conley index for iterated maps

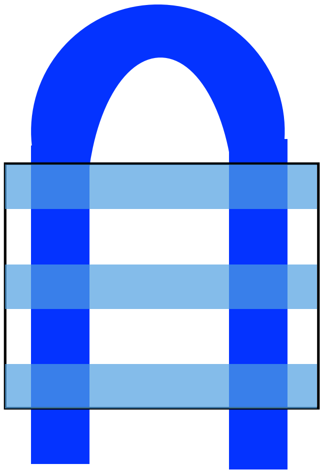
$$CH_*(S, f) = [(f_Q)_* : H_*(N, L) \circlearrowleft]_{\text{shift equiv}}$$

Example

G-horseshoe



U-horseshoe



$$N/L \sim S^1 \vee S^1$$

$$(f_Q)_* : H_*(N, L) \circlearrowright$$

$$(f_Q)_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \overset{\text{shift}}{\sim} 2$$

eigenvalue=2, 0

$$(f_Q)_1 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \overset{\text{shift}}{\sim} 0$$

eigenvalue=0 (double)

Remark

Shift-equivalence is very hard to compute in general.

In practice, can use non-zero eigenvalues of the homology maps of $(f_Q)_* : H_*(N, L) \circlearrowright$ in place of the homology Conley index

How can one understand dynamics via Conley index?

want to see how much the analogy of (classical) Morse theory holds true for Morse decomposition of general dynamical systems

Morse theory	Morse decomposition
finitely many non-degenerate critical pts	finitely many Morse components
entire dynamics = crit pts & connecting orbits	entire dynamics = Morse components & connecting orbits
Morse index = dim of unstable mfd	Conley index
total order on crit pts by Morse function	flow-defined partial order on Morse components
filtration given by level sets of Morse function	index filtration by partially ordered set [Franzosa]

"Conley-Morse theory for dynamics"

For a continuous time flow,

suppose a Morse decomp $\{M(p)\}_{p\in P}$ is given;

$\alpha \subset P$: *lower set* $\overset{\text{def}}{\iff} [p < q \ (\exists q \in \alpha) \Rightarrow p \in \alpha]$

$\mathcal{L}(P)$:= set of lower sets in P

$I \subset P$: *interval* $\overset{\text{def}}{\iff} [p < r < q \ (\exists p, q \in I) \Rightarrow r \in I]$

$M(I)$:= $\cup_{p\in I} M(p) \cup$ (connecting orbits between $M(p)$ $(p \in I)$)

Theorem:[Franzosa 1986]

$\exists$ collection of compact subsets $\{N_\alpha\}_{\alpha\in\mathcal{L}(P)}$ of N, s.t.

(1) $N_\alpha \cap N_\beta = N_{\alpha\cap\beta}$, $N_\alpha \cup N_\beta = N_{\alpha\cup\beta}$

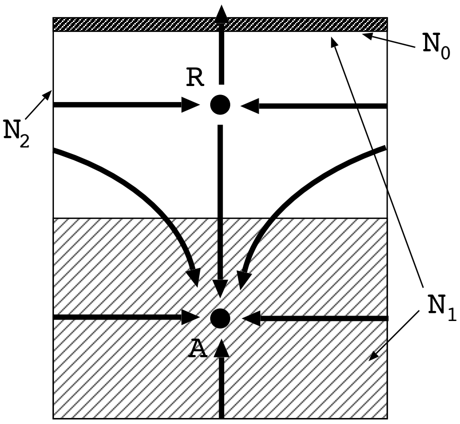
(2) $\forall$ interval $I \subset P$ $\exists$ $\alpha, \beta \in \mathcal{L}(P)$ with $\alpha \subset \beta$, $I = \beta \setminus \alpha$
s.t. $CH_*(M(I)) = H_*(N_\beta, N_\alpha)$

$\{N_\alpha\}_{\alpha\in\mathcal{L}(P)}$: index filtration of the Morse decomposition

Index filtration for an attractor-repeller decomposition

$N_0 \subset N_1 \subset N_2$

s.t. (N_2, N_0) is an index pair of S
(N_2, N_1) is an index pair of R
(N_1, N_0) is an index pair of A



Hence,

$$CH_*(S) = H_*(N_2, N_0),$$
$$CH_*(R) = H_*(N_2, N_1),$$
$$CH_*(A) = H_*(N_1, N_0)$$

Homology long exact sequence for the index triple:

$$\begin{array}{ccccccc} \to & H_{*+1}(N_2,N_1) & \xrightarrow{\partial} & H_*(N_1,N_0) & \to & H_*(N_2,N_0) & \to \\ & CH_{*+1}(R) & & CH_*(A) & & CH_*(S) & \end{array}$$

$$\begin{array}{ccccc} \to & H_*(N_2,N_1) & \xrightarrow{\partial} & H_{*-1}(N_1,N_0) & \to \\ & CH_*(R) & & CH_{*-1}(A) & \end{array}$$

Existence of a connecting orbit for AR decomposition

$$\begin{array}{ccccccc} \to & H_{*+1}(N_2,N_1) & \xrightarrow{\partial} & H_*(N_1,N_0) & \to & H_*(N_2,N_0) & \to \\ & CH_{*+1}(R) & & CH_*(A) & & CH_*(S) & \end{array}$$

$$\begin{array}{ccccc} \to & H_*(N_2,N_1) & \xrightarrow{\partial} & H_{*-1}(N_1,N_0) & \to \\ & CH_*(R) & & CH_{*-1}(A) & \end{array}$$

If $S = R \sqcup A$ (i.e. no connecting orbit $R \to A$),
then $CH_*(S) \cong CH_*(R) \oplus CH_*(A)$,
$\therefore$ connecting hom $\partial : CH_*(R) \to CH_{*-1}(A)$ must be zero

contrapositive: $\partial \neq 0 \;\Rightarrow\; \exists$ conn orbit $R \to A$

$$\Delta = \begin{pmatrix} 0 & \partial \\ 0 & 0 \end{pmatrix} : CH_*(R) \oplus CH_*(A) \circlearrowright$$

deg -1 hom, upper triangular, $\Delta^2 = 0$ (boundary map)

connection complex $\mathcal{C}_\Delta$

Theorem:[Conley] [Franzosa 1989],[Robbin-Salamon 1992]

$\exists\Delta : \oplus_{p\in P} CH(M(p)) \circlearrowleft$ deg -1 hom s.t.

(1) Δ strictly upper triangular

i.e. $\Delta(p,q) \neq 0 \Rightarrow p < q$

(2) $\Delta^2 = 0$ (boundary map)

hence $\mathcal{C}_\Delta = \{\oplus CH(M(p)), \Delta\}$ chain complex

(3) $H_*(\mathcal{C}_\Delta) \cong CH_*(S)$

Δ is called a connection matrix

Morse theory	Morse decomposition
Morse complex $\mathcal{M}_f = \{\oplus_{k\in\mathbb{Z}} \mathrm{Crit}_k, \partial\}$ [Smale, Witten]	connection complex $\mathcal{C}_\Delta$ with connection matrix $\Delta : \oplus_{p\in P} CH_*(M(p)) \circlearrowleft$
$H_*(\mathcal{M}_f) \cong H_*(M)$	$H_*(\mathcal{C}_\Delta) \cong CH_*(S)$

Remark

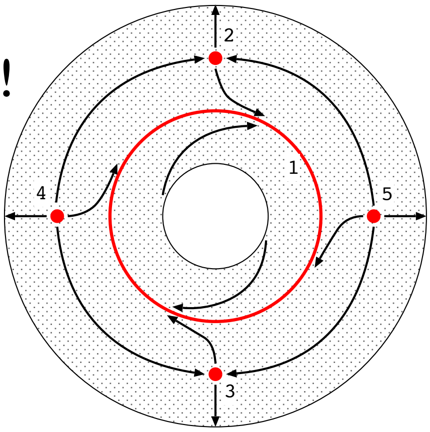
(1) Connection matrix is NOT unique!

Reineck's example:

non-uniqueness of connection matrix

$$\Delta = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Here $(a,b) = (1,0), (0,1)$ both possible



$CH_*(M(p)) = \mathbb{Z}_2$ if

$p = 1$ and $* = 0, 1$, or

$p = 2, 3$ and $* = 1$, or

$p = 4, 5$ and $* = 2$,

else $CH_*(M(p)) = 0$

(2) $\Delta(p,q) \neq 0 \Rightarrow p < q$, hence

existence of a connecting orbit $M(q) \to M(p)$

Can detect connecting orbits by linear algebra

How to represent dynamics? : semi-conjugacy

Definition: A flow $\Phi = \{\varphi^t\}$ on S is semi-conjugate to another flow $\Psi = \{\psi^t\}$ on Σ, if

$\exists\ \rho : S \to \Sigma$ continuous surjection, s.t.

$$\begin{tikzcd} S \arrow[r, "\varphi^t"] \arrow[d, "\rho"'] \arrow[dr, phantom, "\circlearrowright"] & S \arrow[d, "\rho"] \\ \Sigma \arrow[r, "\psi^t"] & \Sigma \end{tikzcd} \qquad (\forall t \in \mathbb{R})$$

several results for $\exists$ of semi-conjugacies in restrictive situations, e.g.

- semi-conj onto gradient flow [McCord-Mischaikow]
- semi-conj onto symbolic dynamics [Carbinatto-Kwapisz-Mischaikow]
- semi-conj onto simplicial model flow [McCord]

Can obtain more general representation by semi-conj?

References

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