CAT(0) 空間の等長変換
- CAT(0) SPACES FOR RIEMANNIAN GEOMETERS -

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This is a note for my talk at Geometry Symposium in August 2004. I will talk about CAT(0) spaces from the view point of isometries or isometric group actions.

CAT(0) spaces are generalizations of Hadamard manifolds, which are simply connected, complete Riemannian manifolds such that the sectional curvature is non positive. CAT(0) spaces are not manifolds in general. It can be a tree, for example. However, many of the results on Hadamard manifolds remain true for CAT(0) spaces. Some are easy, or even simpler. For example, the Cartan-Hadamard theorem, which says that Hadamard manifolds/CAT(0) spaces are contractible. That CAT(0) spaces are contractible is straightforward from the definition of CAT(0) spaces.

Some are not easy, but true. Typically, if one shows something on Hadamard manifolds using calculus, then it may require some extra effort to show the same thing for CAT(0) spaces because they are not differentiable manifolds. For example, we show a first variation formula for convex functions on CAT(0) spaces, and give an application in the geometry of CAT(0) spaces (Theorem 3.2).

I don't think I can cover all of the materials in this note during my talk. One of my goals of writing this note is to invite Riemannian geometers to work on CAT(0) spaces.

There are books I found useful. [BriH] is like an encyclopedia on CAT(0) geometry, and more. I don't give all of the precise definitions in this note, but you can find it in [BriH]. One finds a very readable account of symmetric spaces in 10, Part II. You may consult [E] for symmetric spaces as well. Ch I, II of [B] may serve as a quick introduction to the subject. [BGSc] is only about Hadamard manifolds, which can be read as a list of problems on CAT(0) spaces. We found Theorem 3.2 for Hadamard manifolds in this book.

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1. CAT(0) spaces

1.1. **Definitions.** Let X be a geodesic space and $\Delta(x,y,z)$ a geodesic triangle in X, which is a union of three geodesics. Let $[x,y]$ denote the geodesic side between x,y, etc. A *comparison triangle* for Δ is a triangle $\bar{\Delta}(\bar{x},\bar{y},\bar{z})$ in $\mathbf{E}^2$ with the same side lengths as Δ. The interior angle of $\bar{\Delta}$ at $\bar{x}$ is called the *comparison angle* between y and z at x, and is denoted $\angle_x(y,z)$. Let p be a point on a side of Δ, say, $[x,y]$. A *comparison point* in $\bar{\Delta}$ is a point $\bar{p} \in [\bar{x},\bar{y}]$ with $d(x,p) = d_{\mathbf{E}^2}(\bar{x},\bar{p})$.

$\bar{\Delta}$ satisfies the CAT(0) inequality if for any $p,q \in \Delta$ and their comparison points $\bar{p},\bar{q} \in \bar{\Delta}$,

$$d(p,q) \le d_{\mathbf{E}^2}(\bar{p},\bar{q}).$$

X is a *CAT*(0) *space* if all geodesic triangles in X satisfy the CAT(0) inequality.

Smilarly, one defines CAT(1) and CAT(-1) spaces by comparing geodesic triangles in X with the comparison triangles in the standard 2-sphere $\mathbf{S}^2$ and the hyperbolic plane $\mathbf{H}^2$, respectively. In the case of CAT(1) we only consider geodesic triangles of total perimeter length less than 2π.

1.2. **Examples.** Standard examples of CAT(0) spaces.

- Euclidean space, $\mathbf{E}^n$
- Hyperbolic spaces, $\mathbf{H}^n$.
- Symmetric spaces of non-compact type. For example, $SL(n,\mathbb{R})/SO(n)$.
- Hadamard manifolds, i.e., complete, simply connected Riemannian manifolds of non-positive sectional curvature.
- Trees.
- products of CAT(0) spaces.
- Gluing CAT(0) spaces in a certain way.
- Euclidean buildings.

2. Isometries

2.1. **Classification of isometries.** Let's take the upper plane model of the hyperbolic plane.

$$\mathbf{H}^2 = \{x+iy | x,y \in \mathbb{R}, y>0\}.$$

The hyperbolic metric is given by

$$ds^2 = \frac{dx^2+dy^2}{y^2}.$$

A matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ acts as an orientation preserving isometry as follows:

$$x + iy \mapsto \frac{a(x+iy)+b}{c(x+iy)+d}.$$

The kernel of the action is ± 1, so that we take the quotient group $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\pm 1$. It is a fundamental fact of the geometry of the hyperbolic plane that the group of orientation preserving isometries of $\mathbf{H}^2$, $\mathrm{Isom}_+(\mathbf{H}^2)$, is $PSL(2,\mathbb{R})$.

There is a classification of isometries of the hyperbolic plane. For an orientation preserving one, it is by linear algebra. Suppose $f \in SL(2,\mathbb{R})$. f is called

1. *elliptic* iff it is ± 1 or diagonalizable in $SL(2,\mathbb{C})$, but not in $SL(2,\mathbb{R})$.
2. *hyperbolic* iff it is not ± 1 and diagonalizable in $SL(2,\mathbb{R})$.
3. *parabolic* iff it is not diagonalizable in $SL(2,\mathbb{C})$.

Note that this classification applies to an element in $PSL(2,\mathbb{R})$.

This classification can be given in a geometric language. The advantage is that an isometry does not have to be orientation preserving, and more importantly, it is applied to an isometry of a complete CAT(0) space, X.

An isometry of X is

1. elliptic iff there is a fixed point in X.
2. hyperbolic iff there is no fixed point, but there is an invariant geodesic in X.
3. all others are parabolic.

After this classification, it is natural to introduce the following geometric objects to analyze isometries, f.

1. the set of fixed points, denoted $\mathrm{Fix}(f)$.
2. the set of invariant geodesics, denoted $\mathrm{Min}(f)$.

Unfortunately, they are useless for parabolic isometries because they are empty. It's time to talk about the ideal boundary of X.

2.2. **Ideal boundary.** It is always a good idea to compactify a noncompact object. Let's compactify a complete, locally compact CAT(0) space, X, putting the ideal boundary.

Two geodesics $\gamma(t), \gamma'(t)$ in a complete CAT(0) space X are *asymptotic* if there exists a constant C such that for all $t \geq 0$, $d(\gamma(t), \gamma'(t)) \leq C$. This defines an equivalence relation, $\sim$. We consider only unit speed geodesics. The *ideal boundary* of X, $X(\infty)$, is the set of equivalence classes of geodesics in X. The equivalence class of a geodesic $\gamma(t)$ is denoted by $\gamma(\infty)$. The equivalence class of a geodesic $\gamma(-t)$ is denoted by $\gamma(-\infty)$. It is a theorem that for any geodesic $\gamma(t)$ and a point $x \in X$, there exists a unique geodesic $\gamma'(t)$ such that $\gamma \sim \gamma'$ and $\gamma'(0) = x$.

There is a natural topology on $X(\infty)$ which is called the *cone topology*. $X \cup X(\infty)$ is compact if X is proper (equivalently, complete and locally compact for a geodesic space). A metric space is called *proper* if its all closed metric balls are compact.

In general a parabolic isometry may not have any fixed point in $X(\infty)$, but if X is proper then there is always at least one.

Proposition 2.1 (cf. [BriH], [FSY]). *Let X be a proper $CAT(0)$ space. Suppose f is a parabolic isometry on X. Then there is a point $p \in X(\infty)$ such that f fixes p and leaves each horosphere centered at p invariant.*

Any isometry g of X with $gf = fg$ has those two properties about p.

For an isometry, f, of a CAT(0) space X let's denote the set of fixed points by f in $X(\infty)$ as $X_f(\infty)$. This is the object we want to study.

3. Parabolic isometries

To study the geometry of $X_f(\infty)$ of a parabolic isometry f, we need a metric on $X(\infty)$.

3.1. **Tits metric.** Let X be a complete CAT(0) space. Let $p, p' \in X(\infty)$ and $x \in X$. Take the geodesics $\gamma(t), \gamma'(t)$ such that $\gamma(0) = \gamma'(0) = x$ and $\gamma(\infty) = p, \gamma'(\infty) = p'$. Define the angle between p, p' at x by $\angle_x(p, p') = \lim_{t \to 0} \angle_x(\gamma(t), \gamma'(t))$, where $\angle_x(\gamma(t), \gamma'(t))$ is the comparison angle. Define the *angle* between p, p' by $\angle(p, p') = \sup_{x \in X} \angle_x(p, p')$. This is a metric on $X(\infty)$. The path metric induced by $\angle$ on $X(\infty)$ is called *Tits metric* and denoted by d_T. If there is no path between p, q then define $d_T(p, q) = \infty$. We call the ideal boundary with the topology by the Tits metric the *Tits boundary.*

The following is a standard fact.

Theorem 3.1 (cf. [BriH] 9, ChII). *Let X be a complete CAT(0) space. Then, $(X(\infty), d_T)$ is a complete CAT(1) space. Any two points $p, q \in X(\infty)$ with $d_T(p, q) < \infty$ is joined by a Tits geodesic in $X(\infty)$.*

3.2. **Radius of $X_f(\infty)$.** For a bounded set B in a metric space S, the radius is defined by $\mathrm{rad}(B) = \inf_{x \in B} \sup_{y \in B} d(x, y)$.

Theorem 3.2 ([FNS]). *Let X be a proper CAT(0) space. Suppose f is a parabolic isometry of X. Then the radius of $X_f(\infty)$ is at most $\pi/2$, so that $X_f(\infty)$ is contractible in the Tits boundary.*

This result is known for Hadamard manifolds (Schroeder, [BGSc] App 3. cf. [E]). Although we followed his line of argument, there are two points we need extra ingredients, as I said in the introduction. To explain it, I recall that the definition of the *displacement function*, $d_f : X \to \mathbb{R}$, of an isometry, f, of a CAT(0) space X is defined by

$$d_f(x) = d(x, f(x)).$$

This is a convex, f-invariant function.

There is another, equivalent, classification of isometries in terms of d_f: f is hyperbolic if and only if $\inf d_f > 0$ and the infimum is attained. A point $x \in X$ is on an invariant geodesic, called *axis* of f, if and only if d_f attains its infimum at x. This explains why we write the union of invariant geodesics as $\mathrm{Min}(f)$. f is parabolic if and only if d_f does not attain its infimum. The infimum may be positive and sometimes f is called *strictly parabolic* if $\inf d_f = 0$.

Schroeder used the gradient curves of d_f of the parabolic isometry f. For this part, we used the theory on gradient curves of convex functions on CAT(0) spaces developed by Jost and Mayer. We also obtained a first variation formula for a convex function on a CAT(0) space, and applied it to d_f. It turns out that our formula is not equality as the manifold case, but an inequality (see [FNS] for detail).

In general, the cone topology and the Tits topology are different. The identity map from the Tits boundary to the ideal boundary with the cone topology is continuous, but not for the other way around in general.

Example 3.1.
1. $\mathbf{E}^n(\infty)$ is isometric to the unit $(n-1)$-sphere, $\mathbf{S}^{n-1}$ in terms of the cone topology and also the Tits topology.
2. $\mathbf{H}^n(\infty)$ is isometric to $\mathbf{S}^{n-1}$ in terms of the cone topology. The Tits boundary is the $(n-1)$-sphere, S^{n-1} with discrete topology.
3. $\mathbf{H}^2 \times \mathbf{R}$ is a CAT(0) space, whose ideal boundary is isometric to $\mathbf{S}^2$ with respect to the cone topology. The Tits boundary is the spherical suspension of the Tits boundary of $\mathbf{H}^2$, so that it is an infinite graph of diameter π. As a set, it is S^2, with the graph structure of meridians.

I suspect the answer is yes to the following question.

Question 3.1. In Theorem 3.2, is $X_f(\infty)$ contractible in terms of the cone topology ?

X is called *visible* if for any two distinct points, p, q, in the ideal boundary, there is a geodesic in X which joins the two points. Clearly, $\angle(p, q) = \pi$, so that $Td(p, q) = \infty$.

Corollary 3.1. *Let X be a proper CAT(0) space which is visible. Then $X_f(\infty)$ is one point for a parabolic isometry, f.*

3.3. **Centers of $X_f(\infty)$.** In general, $X_f(\infty)$ is not a point. So, we want to find a special point in $X_f(\infty)$. A *center* of a bounded set is a point where inf is achieved in the definition of the radius. In a CAT(0) space, a bounded set has a unique center. This is because the distance function is convex. Since $X(\infty)$ is only a CAT(1) space, one can not expect a unique center for a random bounded subset. However, if it is so small that the Tits metric is convex, then one gets a unique center. How small ? Less than $\pi/2$. Therefore, Theorem 3.2 is nearly enough, but not exactly. There is an example of a parabolic isometry on a CAT(0) space such that a center of $X_f(\infty)$ is not unique.

Example 3.2. Let g be a parabolic isometry of $\mathbf{H}^2$. Suppose $X = \mathbf{H}^2 \times \mathbf{H}^2 \times \mathbf{R}$. Then the product $f = g \times g \times \mathrm{id}$ is a parabolic isometry of X such that $X_f(\infty)$ is the spherical suspension of a segment of length $\pi/2$. In other words, $X_f(\infty)$ is isometric to the region on the unit sphere which are bounded by 0-meridian and $\pi/2$-meridian. Then the set of the centers is the segment of length $\pi/2$, which is on the equator.

Let's denote the set of the centers of a bounded set, B, as $C(B)$. $C(B)$ may be an empty set. In Example 3.2, $C(X_f(\infty))$ is a segment, so that it has a unique center, i.e., $C^2(X_f(\infty)) = C(C(X_f(\infty)))$ is one point. It turns out this is the case in generalities. There is a notion of the *space of directions* at each point p in a CAT(0) space X. It is a generalization of the space of the unit tangent vectors in Riemannian manifolds.

Theorem 3.3. *Let X be a proper CAT(0) space. Suppose $\dim X < \infty$, and $\Sigma_p X$ is compact for all points $p \in X$. Then $C^2(X_f(\infty))$ is one point for a parabolic isometry, f.*

This result is also known for Hadamard manifolds by Schroeder ([BGSc] App. 3). By a clever idea, he reduced the argument to the geometry of the unit tangent sphere of an Hadamard manifold, which is isometric to a unit sphere. We had to argue differently.

In Thereom 3.3, it seems the condition $\dim X < \infty$ is essential, although we don't know a counter example. The dimension means the covering dimension. A warning; there is a subset, S, in some CAT(1) space of infinite dimension such that $C(S) = S$!

Example 3.3. Let Δ^n be the spherical simplex of dimension n. One constructs it inductively as a subset in the unit n-sphere, $\mathbf{S}^n$. Start with two points, n, s, which is the 0-sphere, $\mathbf{S}^0$. Take the spherical join of two $\mathbf{S}^0$'s. It is isometric to $\mathbf{S}^1$. In $\mathbf{S}^1$, there is the spherical join of two n's, which is a segment of length $\pi/2$. This is Δ^1. Take the spherical join of $\mathbf{S}^0$ and $\mathbf{S}^1$, which is isometric to $\mathbf{S}^2$. The spherical join of n and Δ^1 is contained in there, which is Δ^2. In this way, we obtain $\Delta^n \subset \mathbf{S}^n$. Note that $\mathbf{S}^n$ is a CAT(1) space, the diameter of Δ_n is $\pi/2$, $\mathrm{rad}(\Delta^n) < \pi/2$, and $\lim_{n\to\infty} \mathrm{rad}(\Delta^n) = \pi/2$. Define $\Delta^\infty = \cup_n \Delta^n$, which is a subset in $\mathbf{S}^\infty = \cup_n \mathbf{S}^n$. $\mathbf{S}^\infty$ is a CAT(1) space of infinite dimension. The diameter of Δ^∞ is $\pi/2$, $\mathrm{rad}(\Delta^\infty) = \pi/2$, and $C(\Delta^\infty) = \Delta^\infty$.

It is important that the special point $p = C^2(X_f(\infty))$ is obtained geometrically. It follows that if g is an isometry of X which commutes with f, then g also fixes p. For example, if f is an element of $G < \mathrm{Isom}(X)$ such that G is abelian, then p is a common fixed point of G.

4. Symmetric spaces

Non-compact symmetric spaces are nice examples of CAT(0) spaces. They are Riemannian manifolds, and easy enough to do computation and complicated enough to be interesting. Since I am interested in CAT(0) spaces which are not hyperbolic or CAT(−1), I choose $SL(3,\mathbb{R})/SO(3)$.

Let's denote $X = SL(3,\mathbb{R})/SO(3)$. It is identified with the space, $P(3)$, of all positive definite, symmetric (3×3)-matrices with real coefficients, of determinant 1. It is a differential manifold of dimension 5. There is a natural way to put a Riemannian metric on X such that X is an irreducible symmetric space of non-compact type of rank 2.

Let's denote the Tits boundary $(X(\infty), Td)$ by $X(\infty)$ for simplicity. As a set, $X(\infty)$ is the 4-sphere, but as a metric/topological space, it is an infinite, but bounded, graph. $X(\infty)$ is so called a "thick spherical building" of dimension 1 such that each apartment is isometric to $\mathbf{S}^1$ and each Weyl chamber at infinity is an edge of length $\pi/3$. $\mathrm{diam} X(\infty) = \pi$. We have seen something similar, but simpler, in Ex 3.1.3.

The isometry group of X, $I(X)$, has two connected components, and the one which contains the identity map, $I_0(X)$, is $SL(3,\mathbb{R})$. The action is given by matrix multiplication as follows: for $p \in P(3)$ and $f \in SL(3,\mathbb{R})$,

$$f(p) = f p\, {}^t f,$$

where ${}^t f$ is the transpose of f. One notices that the stabilizer of the identity matrix, id, is $SO(3)$, and this is how we obtain the identification $X = P(3)$.

Let σ be the involution of X at id, which is an orientation reversing isometry. It is given by $\sigma(f) = {}^t f^{-1}$. It is known that $I(X) = I_0(X) \cup \sigma I_0(X)$.

I want to calculate $X_f(\infty)$ for a parabolic element $f \in SL(3, \mathbb{R})$. It's done mostly by linear algebra. First of all, which f is parabolic ? Recall the classification by linear algebra in the case of the hyperbolic plane. An isometry is called *semi-simple* if it is either hyperbolic or elliptic. It is known that $f \in SL(3, \mathbb{R})$ is semi-simple as an isometry of X if and only if it is semi-simple as a matrix, i.e., diagonalizable in $GL(3, \mathbb{C})$. Therefore, f is parabolic iff it is conjugate to one of the following matrices, g, by an element, h, in $SL(3, \mathbb{R})$; $hfh^{-1} = g$. It is special about $f \in SL(n, \mathbb{R}), n \leq 3$ that if f has a complex number as an eigenvalue, then it is diagonalizable, so that not parabolic.

List of parabolic elements.

1. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.
2. $\begin{pmatrix} 1/a^2 & 0 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix}$, where $0, 1 \neq a \in \mathbb{R}$.
3. $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

Since $h \in I(X)$ and $hfh^{-1} = g$, we have $X_f(\infty) = hX_g(\infty)$. Since h is an isometry on $X(\infty)$ as well, we discuss g instead of f.

Recall that $X(\infty)$ is an infinite graph such that each edge is isometric to a segment of length $\pi/3$. The following result is by linear algebra and some CAT(0) geometry.

Theorem 4.1 ([FNS])**.** *Let g be one of the matrix in the above list.*

1. *$X_g(\infty)$ is the union of all edges incident to one edge. $X_g(\infty)$ is not compact in $(X(\infty), Td)$, with uncountably many edges.*
2. *$X_g(\infty)$ is isometric to a segment of length π, consisting three edges.*
3. *$X_g(\infty)$ is a segment of length $\pi/3$, having one edge.*

Theorem 4.1 covers only one half of $I(X)$. We don't know about the other half; $\sigma SL(3, \mathbb{R})$.

Problem 4.1. Compute $X_f(\infty)$ for a parabolic isometry, $f \in \sigma SL(3, \mathbb{R})$.

First of all, I even don't know if there is a parabolic isometry in there.

Problem 4.2. Compute $X_f(\infty)$ for a parabolic isometry, f, when X is other symmetric space, for example, $X = SL(n, \mathbb{R})/SO(n), n \geq 4$.

I am sure one can do it for $f \in SL(n, \mathbb{R})$.

5. CAT(0) dimension of groups

Now I talk about not individual isometries of CAT(0) spaces, but isometric group actions on CAT(0) spaces. Let G be a discrete group. Does it act on some CAT(0) space, X, by isometries ? If it does, what is the minimal dimension of such X ? I want the actions to be "nice" such that, for example, the quotient space of the action is Hausdorff. Therefore, let's consider *proper* actions in the sense that for any point $x \in X$, there exists $r > 0$ such that $\{g \in G | B(x, r) \cap gB(x, r) \neq \emptyset\}$ is finite. $B(x, r)$ is a metric ball.

If X is a Hadamard manifold, and if we further assume that the quotient space is compact, in other words, X/G is a closed, non-positively curved, Riemannian manifold, then the following restriction on the group G is known.

Theorem 5.1 (S.T.Yau. 1971, Ann.Math.)**.** *Let M be a closed Riemannian manifold with non-positive sectional curvature. Suppose $G = \pi_1(M)$ is solvable, then M is at and G contains a subgroup of finite index which is a free abelian group of finite rank.*

In other words, the theorem says that if a solvable group G is acting on a Hadamard manifold, X, properly and co-compactly by isometries, then X is flat and G is "almost" abelian (in the sense that it contains an abelian subgroup of finite index).

One readily generalizes this to CAT(0) spaces; if G is acting on a CAT(0) space properly and co-compactly by isometries, then G is almost abelian. A key result is the following well known fact in CAT(0) geometry. This is easily shown by a compactness argument.

Proposition 5.1. *Suppose a group G is acting on a CAT(0) space by isometries, properly, and co-compactly. Then each $g \in G$ is a semi-simple isometry.*

5.1. **Baumslag-Solitar groups.**

As an example of a solvable group, let's consider a solvable Baumslag-Solitar group. It is defined, for each integer m, as follows;

$$BS(1, m) = \{a, b | aba^{-1} = b^m\}.$$

Suppose $|m| \geq 2$ in the following discussion. Then, $BS(1, m)$ is a solvable, but not virtually nilpotent group. It is torsion free. There is a finite CW-complex of dimension 2 which is a $K(\pi, 1)$ space for each $BS(1,m)$. We already know $BS(1,m)$ does not act properly and co-compactly on a CAT(0) space by isometries. Therefore, only non-co-compact actions are interesting.

Proposition 5.2 (cf. [FSY]). *Let $G = BS(1,m)$ such that $|m| \geq 2$.*

1. *If G acts freely on a $CAT(0)$ space by isometries then the element b is parabolic. In particular, G does not act freely, hence not properly, on any $CAT(0)$ space by semi-simple isometries.*
2. *G acts freely on $\mathbf{H}^2$ by isometries. But G does not act properly on $\mathbf{H}^n$ for any $n \geq 1$ by isometries.*
3. *G does not act on any Hadamard manifold of dimension 2 by isometries, properly.*
4. *G acts properly on $\mathbf{H}^2 \times T$ by isometries where T is a regular tree of index $m+1$. $\mathbf{H}^2 \times T$ is a proper CAT(0) space of dimension 3.*

If we are interested in the least dimension of a CAT(0) space for each group G, it is natural to ask the following question.

Question 5.1. Does $BS(1,m)$ act properly on a CAT(0) space X of dimension 2 ?

5.2. **Torus bundle over a circle.** I discuss another solvable group. Let S be the group given by the following presentation.

$$S = \{a, b, c | ab = ba, cac^{-1} = a^2 b, cbc^{-1} = ab\}.$$

S is solvable (but not virtually nilpotent), torsion free, and of cohomological dimension 3 because it is the fundamental group of a closed three-manifold, M, which is a torus bundle over a circle. M is a $K(\pi, 1)$ space of S. The subgroup generated by a, b, which is isomorphic to $\mathbb{Z}^2$, is the fundamental group of the torus fiber, and the base circle gives the element c.

S is an HNN extension as $\mathbb{Z}^2 *_{\mathbb{Z}^2 f}$ such that the self monomorphism f of $\mathbb{Z}^2$ is the linear map given by a matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in SL(2, \mathbb{Z})$. The other monomorphism to define the HNN extension is the identity.

S is also a semi-direct product with the action of $\mathbb{Z}$ on $\mathbb{Z}^2$ given by A:

$$0 \to \mathbb{Z}^2 \to S \to \mathbb{Z} \to 0.$$

Proposition 5.3 (cf. [FSY]).

1. *S acts properly, hence freely, by isometries on $X = \mathbf{H}^2 \times \mathbf{H}^2$ such that X/S is homeomorphic to $M \times \mathbf{R}$.*
2. *S does not act properly on any Hadamard manifold of dimension 3 by isometries.*

In [FSY], we construct an action of S on X concretely. There is a geodesic $\gamma \subset \mathbf{H}^2$ such that the point, $p \in X(\infty)$, defined by $(\gamma \times \gamma)(\infty)$ is fixed by S and each horosphere, H_t, centered at p is invariant by S, so that H_t/S is homeomorphic to M. Indeed X/S is foliated by $H_t/S, t \in \mathbb{R}$ as a product. While we seek for a CAT(0) space of less dimension for S to act, as a by-product, we have geometrized the manifold M as H_t/S.

The following question is natural.

Question 5.2. Does S act properly on some 3-dimensional CAT(0) space by isometries ?

By Prop 5.3 (2), the answer is no for Hadamard manifolds of dimension 3. The reason is if an action did exist, then it would have to be co-compact, since otherwise, the cohomological dimension of S would be less than 3. But we know that a solvable group does not have a co-compact action by Theorem 5.1 unless it's virtually abelian.

References

[B] Werner Ballmann, "Lectures on spaces of nonpositive curvature". DMV Seminar, 25. Birkhauser, 1995

[BGSc] Werner Ballmann, Mikhael Gromov, Victor Schroeder, Manifolds of nonpositive curvature. Progress in Mathematics, 61. Birkhauser (1985).

[BriH] Martin R. Bridson, Andre Hae iger, "Metric spaces of non-positive curvature". Grundlehren der Mathematischen Wissenschaften 319. Springer, 1999.

[E] Patrick B. Eberlein, Geometry of nonpositively curved manifolds. Chicago Lectures in Mathematics. University of Chicago Press, 1996.

[FNS] Koji Fujiwara, Koichi Nagano, Takashi Shioya. Fixed point sets of parabolic isometries of CAT(0)-spaces. preprint, 2004.

[FSY] Koji Fujiwara, Takashi Shioya, Saeko Yamagata. Parabolic isometries of CAT(0) spaces and CAT(0) dimensions. preprint, 2003.

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