Group actions on quasi-trees

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1. Dichotomy among manifolds of non-positive curvature

This talk will be about negative curvature in spaces and groups. To be precise, the difference between negative curvature and non-positive curvature is important.

I start with the celebrated Rank rigidity theorem proved by Ballmann in 80's after a sequence of works by many people. In this note, I usually state theorems for closed manifolds, but many of the conclusions hold for manifolds of finite volume.

Let X be a simply connected, complete Riemannian manifold of non-positive sectional curvature, of dimension at least two. (From now on, I only say curvature instead of sectional curvature.) A geodesic γ is *rank-1* unless it is the boundary of a flat Euclidean half plane in X. An (hyperbolic) isometry of X is *rank-1* if it leaves some rank-1 geodesic invariant and acts on it by a non-trivial translation. A manifold of non-positive curvature M is *rank-1* if $\pi_1(M)$ contains a rank-1 element when it acts on $\widetilde{M}$ by a Deck transformation.

Theorem 1.1 (Rank-rigidity, Ballmann, cf [Ball]). *Let M be a closed Riemannian manifold of non-positive sectional curvature of dimension at least two. Suppose M is not a product (as a manifold). Then either M is a locally symmetric space of rank at least two, or M is a rank-1 manifold.*

Notice that if M is a locally symmetric space of rank at least two, then any geodesic in $\tilde{M}$ is contained in a flat, therefore M is not rank-1. The hard part is the converse. If we think locally symmetric spaces are something we understand well, the main study will be on rank-1 manifolds.

The class of rank-1 manifolds contains all locally symmetric space of rank-1 and one can produce examples by gluing some of them along cusps. Let M_1 and M_2 be non-compact, complete hyperbolic 3-manifolds of finite volume. For simplicity assume that each one has only one cusp, therefore M_i is diffeomorphic to the interior of a compact manifold N_i with one torus boundary. Glue N_1 and N_2 along the boundary tori by a diffeomorphism and obtain a closed manifold M. One can put a Riemannian metric of non-positive curvature on M then it becomes a rank-1 manifold.

In dimension 3, by the geometrization, this is more or less all the way to obtain rank-1 manifolds. But in dimension 4 and higher, we do not have a good outlook toward a classification of rank-1 manifolds. For example, there is a closed 4-manifold of non-positive curvature which is a homology sphere, of positive "simplicial volume" ([RaTs], [FuMa], [FuMa2]).

So, what kind of questions shall we ask about rank-1 manifolds? Here is a question I heard from Ballmann.

Question 1.2. *Is the fundamental group of a rank-1 manifold simple ?*

This work was supported by KAKENHI (23244005). The note is revised after the talk and the last section is added.
2010 Mathematics Subject Classification: 20F65, 53C24.
Keywords: quasi-trees, hyperbolic groups, mapping class groups, stable commutator length, rank-rigidity, quasi-morphism, first Betti number, lattice.
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The expected answer is no, and Gromov had an idea how to prove it using minimal surfaces. We soon answer this question.

One can see the rank rigidity theorem as a dichotomy among Riemannian manifolds of non-positive curvature. We do not try to classify rank-1 manifolds, but we search for properties shared by rank-1 manifolds, and aim to rephrase the dichotomy using other terms, say, algebraic properties of the fundamental groups. To this end, we recall a deep result in Lie groups.

Theorem 1.3 (Normal subgroup theorem, Margulis)**.** *Let G be a lattice in a semi-simple Lie group of rank at least two, and $N \trianglelefteq G$ a normal subgroup. Then, either N is finite or of finite index.*

Elementary examples of G are $SL(n,\mathbb{Z}) < SL(n,\mathbb{R}), n \geq 3$. What is important for us is that it applies to the fundamental group of a closed Riemannian manifold which is a locally symmetric space of rank at least two. Roughly speaking this result says that a normal subgroup is nearly trivial or everything, namely G is like a simple group. Combining it with the following result, which answers the question 1.2, we obtain an algebraic dichotomy among manifolds of non-positive curvature.

Theorem 1.4. *Let M be a closed rank-1 manifold, and G its fundamental group. Then G contains an infinite normal subgroup N which is of infinite index. In particular, G is not simple.*

I also want to discuss the following theorem of our ancestors, Matsushima [Matsu] and others. After this D. Kazhdan proved those groups have *property (T)*, which implies $H^1(G,\mathbb{R}) = 0$.

Theorem 1.5 (Vanishing of the first Betti number)**.** *Let G be a uniform lattice of a semi-simple Lie group of rank at least two. Then, $H^1(G,\mathbb{R})$ is trivial.*

Remark 1.6 (History for $H^1 = 0$)**.** *Matsushima proved this theorem (with an exception) using a result of Borel. His argument is by showing there is no non-trivial harmonic one-form on the quotient space. Non-compact case is probably due to Borel. The first cohomology with coefficients is important for the rigidity result. Around 1960, Selberg was the one who proved that uniform lattice in $SL(n,\mathbb{R}), n > 2$ can not be continuously deformed in the Lie group except for conjugation. Calabi proved it for other Lie groups. Weil proved that this is a consequence of the vanishing of the first cohomology of the lattice with coefficient in the lie algebra of of the lie group.*

In particular if M is a compact, locally symmetric space of rank at least two, then the first Betti number of the fundamental group is 0, in other words, any homomorphism from $\pi_1(M) \to \mathbb{R}$ is trivial.

Now, is it true that a lattice of rank-1 Lie group always have a non-trivial homomorphism to $\mathbb{R}$? No, for example $SL(2,\mathbb{Z}) < SL(2,\mathbb{R})$ is generated by two torsion elements, therefore, clearly any homomorphism to $\mathbb{R}$ is trivial. But, $SL(2,\mathbb{Z})$ contains a free group of finite index, which obviously has a non-trivial homomorphism to $\mathbb{R}$. On the other hand, Matsushima-vanishing applies to any subgroup in G of finite index. So, is it true that if M is a closed Riemannian manifold of non-positive curvature, then M is a locally symmetric space of rank at least two if and only if any subgroup $G < \pi_1(M)$ of finite index has only trivial homomorphism to $\mathbb{R}$? The answer is No. Lattices in the Lie groups $Sp(n,1)$, the quaternionic hyperbolic isometries, have property (T), therefore the first Betti number is 0, while they are rank 1. But in this talk, I will mention a modern modification of the vanishing and then use it to state the dichotomy. We are still curious if the first Betti number is 0 if we take a finite index subgroup in a lattice of $Isom(\mathbb{H}^n)$. A partial answer is known. If a real hyperbolic manifold is non-compact, of finite volume and arithmetic, then it is true that it has a finite cover with non-trivial (as large as you like) first Betti number (Millson [Mill]).

To finish the introduction, I'd like to mention a fascinating progress recently made for closed hyperbolic 3-manifolds.

Theorem 1.7 (virtual fibration theorem, Agol [Agol]). *Let M be a closed hyperbolic 3-manifold. Then it has a finite cover which is a surface bundle over a circle.*

In particular the finite index subgroup in $\pi_1(M)$ corresponding to the finite cover has a non-trivial homomorphism to $\mathbb{Z}$. This theorem has been a conjecture by Thurston. It is solved last year by Agol using lots of technology from geometric group theory. Certain singular spaces of non-positive curvature, CAT(0) cube complex, play a central role in the argument and much of the work has been done by Dani Wise. What is really interesting is that they have to use higher dimensional spaces in a very essential way to handle the three dimensional problem.

2. δ-hyperbolic spaces and word-hyperbolic groups

Let Δ be a geodesic triangle in the hyperbolic plane $\mathbb{H}^2$. The three sides a, b, c are geodesics. Gauss-Bonnet theorem says that the area of Δ is at most π. It then follows that there is a constant δ (say, 2), which does not depend on Δ such that each side is contained in the δ-neighborhood of the union of the the other two sides: $a \subset N_\delta(b \cup c)$. We say that Δ is *δ-thin*.

Gromov turned this property into a definition. A geodesic metric space X is (δ-)*hyperbolic* if there is a constant δ such that all geodesic triangle are δ-thin. The hyperbolic spaces $\mathbb{H}^n$ are hyperbolic, and more generally, a complete, simply connected Riemannian manifold of sectional curvature at most c for some constant $c < 0$ is hyperbolic, which includes all symmetric spaces of rank one. On the other hand Euclidean spaces of dimension at least two are not hyperbolic (there is no bound on the thinness of a triangle), and neither are symmetric spaces of rank at least two. The theory of hyperbolic spaces has been very successful, and useful to study rank-1 manifolds, but it is not always the case that the universal cover of a rank-1 manifold is hyperbolic (the 3-dimensional example we gave contains a flat).

Another merit of the theory is that it handles singular spaces. For example, trees are hyperbolic (a simplicial tree is a connected graph which is simply connected. There is a more general notion of $\mathbb{R}$-trees), and also one can build a simplicial complex of higher dimension which are hyperbolic.

The significance of the theory is that it applies to discrete groups. There is more than one way (they are all equivalent) to define hyperbolic groups, but one definition is by using action. Let G be a group. If G acts on a δ-hyperbolic space X by isometries and the action is properly discontinuous and co-compact, then we say G is *(word) -hyperbolic*. It is straightforward from the definition that the fundamental group of a closed Riemannian manifold of negative curvature is hyperbolic since its universal cover is hyperbolic and the action by the Deck transformations satisfies the assumption. By the same reason, a free group of finite rank is hyperbolic (use a tree as a space to act on). On the other hand it may look difficult to conclude some groups are not hyperbolic. One expects that $\mathbb{Z}^2$ is not hyperbolic since it is the fundamental group of a (flat) torus. In fact that is the case. It is a theorem that if a group G act on a geodesic space X which is not hyperbolic and the action is by isometries, properly discontinuous and co-compact, then G is not word-hyperbolic. Therefore, the fundamental group of a closed Riemannian manifold of non-positive curvature which is not a rank-1 manifold is not hyperbolic since its universal cover is a symmetric space of rank at least two (by Rank-rigidity), which are not hyperbolic.

An important idea behind the theory of hyperbolic groups and geometric group theory is quasi-isometry. Let X, Y be two metric spaces. We denote the distance between x, y by $|x-y|$. A map $f: X \to Y$ is a (K, L)-*quasi-isometric embedding* if for all points $x, y \in X$,

$$\frac{|x-y|}{K} - L \leq |f(x) - f(y)| \leq K|x-y| + L.$$

If additionally it satisfies that for all point $y \in Y$ there exists $x \in X$ such that $|y - f(x)| \leq L$, then we say f is a *quasi-isometry* and X and Y are *quasi-isometric*. In those definitions, only the existence of constants K, L is important, and we sometimes omit to indicate them. Quasi-isometry is an equivalence relation among metric spaces, and it is a non-trivial theorem that the hyperbolicity is invariant under quasi-isometry.

3. Mapping class groups

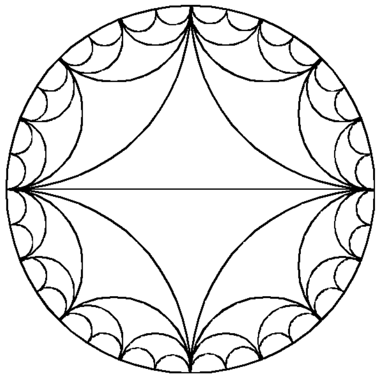
Let S be a compact orientable surface. Define the group $MCG(S)$, *the mapping class group* of S, as the quotient of the group of orientation preserving homeomorphisms of S divided by isotopies. It is a theorem that $MCG(S)$ is a finite generated, and more over, a finite presented group. The MCG of a sphere is trivial, and of a torus is isomorphic to $SL(2, \mathbb{Z})$. There is a long history of the study of the mapping class groups from various view points. For example see the book [Iva92]. Our view point is that it looks like a lattice in a Lie group or on its symmetric space. In this analogy, we may take the Teichmüller space of S for $MCG(S)$ to act on by isometries. The Teichmüller space is diffeomorphic to the Euclidean space of dimension $6g-6$ where g is the genus of S (assume $g > 0$), and there is the Teichmüller metric on it. It is not a Riemannian metric, but only a Finsler metric. It has been observed that there is a lot of negative curvature aspects there, but it is not δ-hyperbolic if $g > 0$. $MCG(S)$ acts on it by isometries and properly, but not co-compactly. $MCG(S)$ is not word-hyperbolic if $g > 0$ since it contains $\mathbb{Z}^2$, which is an algebraic obstruction for a group to be hyperbolic.

Our first question would be if the Teichmüller space is rank-1 in some sense. The answer is Yes, there is a "rank-1" isometry on the Teichmüller space. Indeed all pseudo-Anosov elements are rank-1 (Minsky [Min]). That raises a hope that we may be able to handle MCG in the same way as rank-1 manifolds, and that was one of the motivation of this project. Another approach to MCG is using the *curve complex* and Masur-Minsky succeeded to apply the theory of δ-hyperbolic spaces in this setting [MM99], [MM00].

4. Quasi-trees

Our method is letting a group G in concern to act on a space by isometries which is negatively curved. To use the full power of negative curvature, we first try to use trees. Bass-Serre systematically studied group actions on simplicial trees and built a theory, the Bass-Serre theory. It gives lots of information on a group once it acts on a simplicial tree by automorphisms without a common fixed point (then we say the action is *non-trivial*). At the same time, Serre observed that $SL(3, \mathbb{Z})$ does not act on any simplicial tree without a common fixed point, and by now it is known that any lattice in a semi-simple Lie group of rank at least two does not act non-trivially either. More importantly for us, $MCG(S)$ does not act on a simplicial tree without a fixed point either if the genus is at least two (Culler-Vogtmann).

Now we introduce a more flexible objects than trees. We say a graph Γ is a *quasi-tree* if it is quasi-isometric to some simplicial tree. We do not assume that Γ is locally finite. We consider a connected graph as a geodesic metric space by assigning length one to each edge. An example of a quasi-tree is the Farey graph, F. See the figure. You will notice that the dual graph to the tessellation is a binary tree, but the graph is not quasi-isometric to this tree. If you remove any edge from F, then it disconnects F. This is a result of the property called *the bottle neck property*, which characterizes a quasi-tree (J. Manning [Man05]).



The figure is the Farey tessellation of the hyperbolic plane by ideal triangles. You tessellate it forever toward the boundary. Make each ideal triangle into a usual triangle keeping the combinatorial structure of the graph. In this way we obtain a planer graph, which is called the *Farey graph*, F.

Clearly quasi-trees are hyperbolic. We will study groups by letting them act on quasi-trees. The merit is that more groups will act on them than on trees, and the geometry of quasi-trees is so special that we can say many thing on the groups.

The following is a list of groups G which acts on quasi-trees X *non-trivially* in the sense that for some (and any) point $x \in X$, the G-orbit is unbounded.

Theorem 4.1 (Bestvina-Bromberg-F [BBFb])**.** *The following groups act on some quasi-trees by isometries non-trivially.*

1. *Infinite hyperbolic groups.*
2. *$MCG(S)$ of a (closed) orientable surface S of genus at least one.*
3. *$Out(F_n)$ with $n > 1$.*
4. *$\pi_1(M)$ where M is a (closed) rank-1 manifold.*
5. *A group G which acts on a CAT(0) space by isometries properly and G contains at least one rank-1 element.*

It is known that certain infinite hyperbolic groups do not act on trees non-trivially (use property (T)). $Out(F_n)$ is the outer automorphisms group of the free group of rank n. It is a classical theorem that $MCG(S)$ is isomorphic to $Out(\pi_1(S))$ when S is a closed orientable surface. In view of that $MCG(S)$ and $Out(F_n)$ are somewhat similar.

A *CAT(0)* space is a geodesic space that has "non-positive curvature", which includes all simply connected, complete Riemannian manifolds of non-positive curvature. See the book [BH99]. A rank-1 element is defined in the same way as for manifolds using a flat half space. So, the last example is a generalization of rank-1 manifolds to a singular setting.

5. Applications

Now we list applications of group actions on quasi-trees. Some of the arguments are elaborate but use quasi-trees as a key.

Theorem 5.1 ([BBFb]). *The asymptotic dimension of $MCG(S)$ is finite.*

We don't give the definition of *asymptotic dimension*. It is a dimension defined by Gromov for metric spaces which is invariant under quasi-isometry. Using a non-trivial theorem by Yu, the Novikov conjecture for $MCG(S)$ follows, which was known before by different methods (Kida and Hamenstädt).

We recall the answer to the question 1.2.

Theorem 5.2. *The fundamental group of a rank-1 manifold of dimension at least two is not simple. Moreover, it contains an infinite normal subgroup of infinite index, which is a free group.*

This theorem relies on the following theorem.

Theorem 5.3 (Osin-Dahmani-Guirardel). *If a group G acts on a hyperbolic space by isometries and if $g \in G$ is a hyperbolic element which is weakly properly discontinuous, then the normal closure of g^N is a free group for sufficiently large N.*

The *weak proper discontinuity (WPD)* is introduced in [BF02], and I skip the definition. An essential ingredient is that the groups actions on quasi-trees the theorem 4.1 provides satisfy the assumption of the theorem 5.3. Therefore the fundamental group of a rank-1 manifold M always contains a free normal subgroup and it can not be of finite index since otherwise M has a finite cover whose fundamental group is a free group, which is impossible.

We go back to the vanishing of the first Betti number. To state a recent generalization, we need a definition. Let G be a group and $f : G \to \mathbb{R}$ be a map. f is a *quasi-(homo)morphism* if

$$\sup_{g,h\in G} |f(gh) - f(g) - f(h)| < \infty.$$

If it additionally satisfies the following, then we say it is *homogeneous*: for all $g \in G$ and n, $f(g^n) = nf(g)$. Clearly a homomorphism is a homogeneous quasi-morphism. But are there anything other than homomorphisms? We define vector spaces:
$QH(G) = \{$ all homogeneous quasi-morphisms on $G\}$,
$H^1(G) = \{$ all homomorphisms on G to $\mathbb{R}\}$, and $\widetilde{QH}(G) = QH(G)/H^1(G)$.

Theorem 5.4 (Burger-Monod [BM99]). *Let G be a lattice in a semi-simple Lie group of rank at least two. Then $\widetilde{QH}(G) = 0$.*

Since $H^1(G) = 0$ (this is true for non-uniform lattices), in fact $QH(G) = 0$.

In contrast, $\widetilde{QH}(G) \neq 0$ for non abelian free groups, [Bro81], and non-elementary hyperbolic groups, [EF97].

Theorem 5.5. *$\widetilde{QH} \neq 0$ for the fundamental group of a rank-1 manifold, [BF09], and moreover, for all group in the theorem 4.1 if it does not contain $\mathbb{Z}$ as a subgroup of finite index, [BBFa].*

Therefore $\widetilde{QH}(G) = 0$ or not is another (way to state the) dichotomy among manifolds of non-positive curvature.

One may think quasi-morphisms are artificial. We discuss a concrete application. Let G be a group, and $G' = [G, G]$ its commutator subgroup. For an element $g \in [G, G]$, let $cl(g)$ denote the *commutator length* of g, the least number of commutators whose product is equal to g, $g = [a_1, b_1] \cdots [a_n, b_n]$. We define $cl(g) = \infty$ for an element g not in $[G, G]$. For $g \in G$, the *stable commutator length*, $scl(g)$, is defined by

$$scl(g) = \liminf_{n \to \infty} \frac{cl(g^n)}{n} \leq \infty.$$

See the book [Cal09] for more information on scl. What is important for us is that a theorem by Bavard [Bav91] says that if $\widetilde{QH}(G) = 0$, then $scl = 0$ on G'.

Now the following is immediate from the theorem 1.5 and the theorem 5.4.

Theorem 5.6 ([BM99])**.** *Let G be a lattice in a semi-simple Lie group of rank at least two. Then $scl \equiv 0$ on G.*

We contrast this to the following.

Theorem 5.7.

1. *Let M be a closed hyperbolic manifold. Then there exists a constant $c > 0$ such that $scl(g) \geq c$ for all non-trivial $g \in \pi_1(M)$ (Calegari).*

2. *Let G be a word-hyperbolic group. Then there exists a constant $c > 0$ such that $scl(g) \geq c$ unless there exist $n > 0$ and $h \in G$ such that $g^n = hg^{-n}h^{-1}$ (Calegari-F [CF10]).*

3. *Let M be a rank-1 manifold. Then $scl(g) > 0$ for a rank-1 element $g \in \pi_1(M)$. In particular, $scl \not\equiv 0$ ([BF09]).*

Again, we see the dichotomy if $scl \equiv 0$ or not among manifolds of non-positive curvature.

We conclude this note by a discussion on scl of $MCG(S)$. First of all, it is known that $H^1(MCG(S), \mathbb{R}) = 0$, and in fact $MCG(S)' = MCG(S)$ (we say $MCG(S)$ is *perfect*) if the genus is at least two. Therefore $scl < \infty$ on MCG. It is one of the important questions left if $H^1(G, \mathbb{R}) = 0$ for all $G < MCG(S)$ of finite index or not. Regarding scl, a few partial results have been known:

Theorem 5.8.

1. *$scl(g) > 0$ for all Dehn twists (Endo-Kotschick [EK01])*

2. *If g is a pesudo-Anosov element, then $scl(g) > 0$ unless there exists $n > 0$ and $h \in MCG$ such that $g^n = hg^{-n}h^{-1}$ (Calegari-F [CF10]).*

Endo-Kotschick uses Seiberg-Witten theory. Using the quasi-tree technology, we now have

Theorem 5.9 ([BBFc])**.** *We have a precise description on which element $g \in MCG(S)$ has $scl(g) > 0$.*

There is a constant $c(S) > 0$ such that if $scl(g) > 0$, then $scl(g) \geq c$. On the other hand, if $scl(g) = 0$ then for each such g, $\{cl(g^n)\}_{n>0}$ is bounded.

We recover all known cases by a unified approach.

6. What is missing and can be expected

It has been 50 years since the vanishing theorem by Matsushima, and 30 years since the rank rigidity. It is already 10 years since the geometrization is proved, and the virtual fibration conjecture for hyperbolic 3-manifolds, which seemed too good to be true, is finally solved (it is still unbelievable that closed hyperbolic manifolds, which look most 3-dimensional, are in fact constructed from a circle and surfaces). I briefly discuss what I expect and hope in the future.

1. A big picture for manifolds of dimension 4 and higher is missing. The geometrization for 3-manifolds tells that most compact 3-manifolds have Riemannian metrics of non-positive curvature. Is that the case for dimension 4 and higher ? In view of the virtual fibration theorem, are there anything really higher dimensional going on in dimension 4 and higher ?

2. A rank rigidity for singular space of non-positive curvature (CAT(0) spaces) is missing (it is recently obtained for CAT(0) cube complexes). The notion of rank already exists (and same as manifolds). Probably it will be similar to the manifold case, namely, there is a list for the higher rank case, and the rank 1 case is difficult to classify.

3. We want to understand rank 1 manifolds better. To mention one example, the iso-spectral problem for marked length spectrum.

4. We are curious about $\beta_1 > 0$, in particular for π_1 of hyperbolic manifolds and MCG. Naively speaking $\widetilde{QH} \neq 0$ looks like a supporting evidence for virtual $\beta_1 > 0$, but as we mentioned lattices in $Sp(n, 1)$ are rank 1 and have property (T). Maybe such groups are very special.

5. The study on MCG and $Out(F_n)$ from the view point of geometric group theory looks steady and promising.

Acknowledgements

I learned the Rank-rigidity theorem from the first book I read in the graduate school. I'd like to thank my advisors and collaborators.

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