ON NON BOUNDED GENERATION OF DISCRETE SUBGROUPS IN RANK-1 LIE GROUP

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Abstract. A discrete subgroup G in a rank-1 simple Lie group is not boundedly generated if G does not contain a nilpotent subgroup of finite index.

1. Introduction

The purpose of this note is to record some remarks on the non bounded generation of some discrete groups. We point out that a theorem by Kotschick in [K] has more examples than mapping class groups.

A group G is said *boundedly generated* if there exist finitely many elements $g_1, \cdots, g_k \in G$ such that for any $g \in G$ there exist $n_i \in \mathbb{Z}$ with

$$g = g_1^{n_1} \cdots g_k^{n_k}.$$

One may say G is boundedly generated by $g_1, \cdots, g_k$.

Most non-uniform lattices in a Lie group of rank at least two are known to be boundedly generated (cf.[T]). For example $SL(n, \mathbb{Z}), n > 2$ and $SL(2, \mathbb{Z}[1/p])$ such that p is a prime number are boundedly generated.

On the other hand a uniform lattice G in a rank-1 simple Lie group is not boundedly generated. It is because G is a non virtually cyclic, word-hyperbolic group, which is known to have an infinite quotient which is a torsion group (cf.Prop 3.6 [FLM]).

Margulis and Vinberg [MV] showed that many discrete subgroups in a rank-1 simple Lie group are virtually mapped by homomorphisms to non-abelian free groups, so that they are not boundedly generated. A group is said to *virtually* have some property if some subgroup of finite index in the group has this property.

It seems the following result is new in this generality.

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Theorem 1. *Let G be a discrete subgroup in a rank-1 simple Lie group. If G does not contain a nilpotent subgroup of finite index then it is not boundedly generated.*

The author dedicates this paper to the memory of Robert Brooks. He is the first one who constructed many quasi-homomorphisms on non-abelian free groups, [Br]. He thanks Alex Lubotzky for comments.

2. Proof and examples

Let G be a discrete group. A function $f : G \to \mathbb{R}$ is called a *quasi-homomorphism* if there exists constant C such that for any $g, h \in G$

$$|f(gh) - f(g) - f(h)| \leq C.$$

A quasi-homomorphism f is called *homogeneous* if for any $g \in G$ and $n \in \mathbb{Z}$

$$f(g^n) = nf(g).$$

If f is a quasi-homomorphism then the function, ϕ, defined by the following is a homogeneous quasi-homomorphism:

$$\phi(g) = \lim_{n\to\infty} \frac{f(g^n)}{n}.$$

The set of all homogeneous quasi-homomorphisms on G is a vector space over $\mathbb{R}$, denoted by $HQH(G)$.

We quote a result by Kotschick relating the existence of many quasi-homomorphisms on G to the non bounded generation of G.

Proposition 2 (Prop 5 in [K]). *If G is boundedly generated by $g_1, \cdots, g_k$ then the dimension of $HQH(G)$ as a vector space is at most k.*

We remark that he introduced a weaker notion of bounded generation, which is called "weak bounded generation", and showed that it suffices to assume it in the proposition. It had been known that the dimension of HQH is infinite for mapping class groups ([BF]. cf.Theorem 4 and Example 1), so that he applied Prop 2 to mapping class groups, and gave a new proof to the following theorem by Farb-Lubotzky-Minsky.

Theorem 3 ([FLM]). *The mapping class group MCG of a closed orientable surface of genus at least one is not boundedly generated.*

In fact the following theorem implies that any subgroup G in MCG is not boundedly generated if G is not virtually abelian (see Example 1).

Theorem 4 ([BF]). *Suppose X is a Gromov-hyperbolic space. Assume a group G acts on X by isometries, weakly properly discontinuously. Then the dimension of $HQH(G)$ is infinite.*

We remark that they [BF] showed that the dimension of the vector space of quasi-homomorphisms on G divided by the subspace of bounded functions, $QH(G)$, is infinite by constructing an infinite sequence of quasi-homomorphisms $f_1, f_2, \cdots$ on G which are linearly independent in $QH(G)$. It is easy to see that the homogeneous quasi-homomorphisms, ϕ_i, for f_i are linearly independent in $HQH(G)$.

A geodesic space X is called *Gromov-hyperbolic*, [G], if there is a constant $\delta \geq 0$ such that every geodesic triangle in X satisfies the following property: any side is in the δ-neighborhood of the union of the other two sides. In general, a simply connected complete Riemannian manifold is a Gromov-hyperbolic space if there is a negative constant such that the sectional curvature is bounded from above by the constant. A rank-1 symmetric space is a standard example.

An isometric action of G on a Gromov-hyperbolic space X is *weakly properly discontinuous*, [BF], if

1. G is not virtually cyclic.
2. G contains an element which acts by a hyperbolic isometry on X.
3. For every hyperbolic element $g \in G$, every $x \in X$, and every $C > 0$, there exists $N > 0$ such that the set

$$\{\gamma \in G | d(x, \gamma x) \leq C, d(g^N x, \gamma g^N x) \leq C)\}$$

is finite.

Clearly the property 3 is satisfied by any proper discontinuous action.

As an example, let G be a discrete subgroup in a rank-1 simple Lie group, and X the rank-1 symmetric space associated to the Lie group. Then the isometric action of G on X is properly discontinuous. If G is not virtually nilpotent then the action is weakly properly discontinuous, so that $\dim HQH(G) = \infty$ by Theorem 4 (cf.Cor 1.2 in [F]).

The following is immediate from Prop 2.

Corollary 5. *Let G be as in Theorem 4. Then G is not boundedly generated.*

Example. By Theorem 4, the following groups have infinite dimensional HQH, so that they are not boundedly generated:

1. a non virtually abelian subgroup in the mapping class group of a compact orientable surface, ([BF]).
2. a non virtually cyclic subgroup of a word-hyperbolic group, ([EF]).

3. a discrete subgroup G of a rank-1 simple Lie group such that G is not virtually nilpotent, (Th 1.1, Cor 1.2 in [F]).

The last example implies Theorem 1.

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