Homotopy Nilpotency in Localized Lie Groups

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Outline

Definitions

Homotopy Nilpotency

X: topological group.

- iterated commutator map $\gamma_n : \bigwedge^{n+1} X \to X$.
  $\gamma_1 = \gamma : (x, y) \mapsto xyx^{-1}y^{-1}$,
  $\gamma_n = \gamma \circ (1 \wedge \gamma_{n-1})$.
- homotopy nilpotency of X [BG]
  $\operatorname{nil} X = \min\{n | \gamma_n \simeq *\}$.
- X: homotopy commutative $\overset{\text{def}}{\Longleftrightarrow} \operatorname{nil} X = 1$.
- X: homotopy nilpotent $\overset{\text{def}}{\Longleftrightarrow} \operatorname{nil} X < \infty$.
  - (Hopkins, Rao [Ho],[Ra]) G: compact Lie group
    $G_{(p)}$ is homotopy nilpotent $\Leftrightarrow \mathbf{H}_*(G; \mathbf{Z})$ has no p-torsion.

Definitions

Type of Topological Group

- X has type $(n_1, n_2, \ldots, n_l)$ with $n_1 \leq \cdots \leq n_l$.
  $\overset{\text{def}}{\iff} X \simeq_{(0)} S^{2n_1-1} \times \cdots \times S^{2n_l-1}$.
- X: p-regular $\overset{\text{def}}{\iff} X \simeq_{(p)} S^{2n_1-1} \times \cdots \times S^{2n_l-1}$.
  - (Kumpel [Ku]) $p \geq n_l - n_1 + 2 \Rightarrow X$: p-regular.
- Types of simple Lie groups:

$A_l = SU(l+1)$	$(2, 3, \ldots, l+1)$	G_2	$(2, 6)$
$B_l = Spin(2l+1)$	$(2, 4, \ldots, 2l)$	F_4	$(2, 6, 8, 12)$
$C_l = Sp(l)$	$(2, 4, \ldots, 2l)$	E_6	$(2, 5, 6, 8, 9, 12)$
$D_l = Spin(2l)$	$(2, 4, \ldots, 2l-2, l)$	E_7	$(2, 6, 8, 10, 12, 14, 18)$
		E_8	$(2, 8, 12, 14, 18, 20, 24, 30)$

McGibbon's result

Homotopy commutativity in localized groups, Amer. J. Math. **106** (1984)

Theorem (McGibbon [Mc])

Let G be a compact, simple Lie group of type $(n_1, \ldots, n_l)$ with $n_1 \leq \cdots \leq n_l$.

- If $p > 2n_l$, then $G_{(p)}$ is homotopy commutative.
- If $p < 2n_l$, then $G_{(p)}$ is not homotopy commutative except for the cases that $(G, p) = (Sp(2), 3)$, $(G_2, 5)$.

There are some generalizations of this work:

- Saumell, L. *Homotopy commutativity of finite loop spaces.* Math. Z. **207** (1991), no. 2, 319–334.
- Saumell, L. *Higher homotopy commutativity in localized groups.* Math. Z. **219** (1995), no. 2, 203–213.

Main Theorem
Notion of Theorem

Theorem

G: compact, simple Lie group of type $(n_1, \ldots, n_l)$ with $n_1 \leq \cdots \leq n_l$.
p: regular prime
$\Rightarrow G_{(p)}$: homotopy nilpotent with:

1. $\mathrm{nil}\,(G_{(p)}) = 1$ if $2n_l < p$.
2. $\mathrm{nil}\,(G_{(p)}) = 2$ if $\frac{3}{2}n_l < p < 2n_l$.
3. $\mathrm{nil}\,(G_{(p)}) = 2$ if $(G, p) = (SU(2), 2)$, $(F_4, 17)$, $(E_6, 17)$, $(E_8, 41)$, $(E_8, 43)$.
4. $\mathrm{nil}\,(G_{(p)}) = 3$ if $n_l \leq p \leq \frac{3}{2}n_l$ and (G, p) is not the case above.

Proof

Outline of Proof

- For the upper bounds of homotopy nilpotency:
  - Decompose the commutator by elementary group theory.
  - p-primary part of $\pi_*(S^{2i-1})$ [To].
  - and easy arithmetics.
- For the lower bounds:
  - Non-commutativity Theorem of James and Thomas [JT].
    - $n_l < p < 2n_l \Rightarrow \mathrm{nil}\, G_{(p)} > 1$.
  - Bott's calculation of Samelson product in $SU(n)$ [Bo].
  - Some facts on classical Lie groups.
  - Finding non-trivial Whitehead product in BG along the method of Hamanaka-Kono [HK].

Basic Tools

Elementary Group Theory

H : group generated by $x_1, \ldots, x_n$.

- $[a, b] = aba^{-1}b^{-1}$. $(a, b \in H)$
- $Z_0 = \{x_i^{\pm 1} | 1 \leq i \leq n\}$, $Z_k = \{[a, b] | a \in Z_0, b \in Z_{k-1}\}$.
- Lower central series
  $H = H_0 \supset H_1 = [H, H] \supset \cdots \supset H_i = [H, H_{i-1}] \supset \cdots$.

Lemma

H_k is generated by $\bigcup_{i=k}^{\infty} Z_i$.

$\Rightarrow$ we only have to care about commutators for generators.

Basic Tools
Samelson Product

Definition

A, B: space, X: topological group
(generalized) Samelson product of maps $\alpha : A \to G$ and $\beta : B \to G$, denoted by $\langle \alpha, \beta \rangle$, is the composition
$A \wedge B \xrightarrow{\alpha \wedge \beta} G \wedge G \xrightarrow{\gamma} G$.

By Definition, $\mathrm{nil}\, G_{(p)} < k \Leftrightarrow \underbrace{\langle 1_{G_{(p)}}, \langle 1_{G_{(p)}}, \cdots \langle 1_{G_{(p)}}, 1_{G_{(p)}} \rangle \cdots \rangle \rangle}_{k} \simeq *$

Basic Tools

Samelson Product

G : type $(n_1, \ldots, n_l)$ with $n_1 \leq \cdots \leq n_l$.
p : regular prime ($\Rightarrow$ we regard $G_{(p)} = S^{2n_1-1}_{(p)} \times \cdots \times S^{2n_l-1}_{(p)}$)

- $\epsilon_{n_i} : S^{2n_i-1} \longrightarrow G_{(p)}$: generator.
- $\epsilon'_{n_i} : G \rightarrow G$:
  $S^{2n_1-1}_{(p)} \times \cdots \times S^{2n_l-1}_{(p)} \xrightarrow{\pi_{n_i}} S^{2n_i-1}_{(p)} \xrightarrow{\epsilon_{n_i}} S^{2n_1-1}_{(p)} \times \cdots \times S^{2n_l-1}_{(p)}$

Using the elementary group theory recalled in previous Lemma,

Lemma

$\mathrm{nil}\, G_{(p)} < k \Leftrightarrow \epsilon'_{n_{j_1}} \langle \epsilon_{n_{i_1}}, \epsilon'_{n_{j_2}} \langle \epsilon_{n_{i_2}}, \cdots, \epsilon'_{n_{j_k}} \langle \epsilon_{n_{i_k}}, \epsilon_{n_{i_{k+1}}} \rangle \cdots \rangle\rangle = 0$,
$1 \leq \forall i_m, \forall j_m \leq l$

$\Rightarrow$ We have only to consider maps between odd spheres.

Upper Bound

p-primary components of homotopy groups of spheres

From now on, we assume $p > n_l$[1].
We recall some facts on p-primary components of homotopy groups of spheres [To].

- $\pi_{2n-1+k}(S^{2n-1}_{(p)}) = \begin{cases} \mathbf{Z}/p & k = 2p-3 \\ 0 & 0 < k < 4p-6, k \neq 2p-3 \end{cases}$
- $\alpha_1(3) \in \pi_{2p}(S^3_{(p)}) = \mathbf{Z}/p$: generator
- $\alpha_1(n) \stackrel{\text{def}}{=} \Sigma^{2n-4}\alpha_1(3)$
  $\Rightarrow \pi_{2n+2p-4}(S^{2n-1}_{(p)}) = \mathbf{Z}/p$ is generated by $\alpha_1(2n-1)$.
- $\alpha_1(3) \circ \alpha_1(2p) \neq 0$.
- $\alpha_1(2n-1) \circ \alpha_1(2n+2p-4) = 0$, $(n > 2)$

[1]For the case $p = n_l$ (this occurs only when $G = SU(p)$), we have to consider a little more facts on α_2.

Upper Bound

Samelson products of p-regular groups

Recall that ϵ_{n_i} and π_{n_j} are defined as

$$S_{(p)}^{2n_i-1} \xrightarrow{\epsilon_{n_i}} S_{(p)}^{2n_1-1} \times \cdots \times S_{(p)}^{2n_l-1} \xrightarrow{\pi_{n_j}} S_{(p)}^{2n_j-1}$$

- Since $G_{(0)}$ is homotopy commutative,
  - $\langle \epsilon_{n_i}, \epsilon_{n_j} \rangle \in \pi_{2(n_i+n_j-1)}(G)$: torsion
  - $\pi_{n_s} \circ \langle \epsilon_{n_i}, \epsilon_{n_j} \rangle = \begin{cases} N\alpha_1(2n_s-1) & n_i+n_j = n_s+p-1 \\ 0 & n_i+n_j \neq n_s+p-1 \end{cases}$.
  - $p > 2n_l \Rightarrow \mathrm{nil}\, G_{(p)} < 2$.
- Since $\langle \epsilon_{n_i}, \epsilon_{n_j} \circ \alpha_1(2n_j-1) \rangle = \langle \epsilon_{n_i}, \epsilon_{n_j} \rangle \circ \Sigma^{2n_i-1}\alpha_1(2n_j-1)$,
  - There is a non-trivial 3-fold commutator $\langle \epsilon_{n_k}, \langle \epsilon_{n_i}, \epsilon_{n_j} \rangle \rangle$.
  - $\Leftrightarrow$ There exist n_t and n_s such that $\pi_{n_t} \circ \langle \epsilon_{n_k}, \epsilon_{n_s} \circ \pi_{n_s} \circ \langle \epsilon_{n_i}, \epsilon_{n_j} \rangle \rangle \neq 0$
  - $\Rightarrow \; n_t = 2, \; n_i+n_j = n_s+p-1, \; n_i+n_j+n_k = 2p$.
  - Therefore $p > \frac{3}{2}n_l \Rightarrow \mathrm{nil}\, G_{(p)} < 3$.
- Trivially $\mathrm{nil}\, G_{(p)} < 4$.

Upper Bound

Case: $(G, p) = (SU(2), 2)$, $(F_4, 17)$, $(E_6, 17)$, $(E_8, 41)$

There are some exceptional cases in the Main Theorem.

For the case $G = SU(2) = S^3, p = 2$,

- The only possible 3-fold commutator is $\langle 1_{S^3}, \langle 1_{S^3}, 1_{S^3} \rangle \rangle \in \pi_9(S^3)$
- It is known that $\pi_9(S^3) = \mathbf{Z}/3$
- Therefore $\mathrm{nil}\, SU(2)_{(2)} = 2$

For other cases, we use the previous arithmetics.

For example, in the case $G = E_8, p = 43$:

- If non-trivial 3-fold commutator exists, there exists n_i, n_j, n_k such that
- $n_i + n_j = n_s + p - 1, n_i + n_j + n_k = 2p = 86$.
- However the largest entry $n_l = 30$ and second $n_{l-1} = 24$
- Impossible.

Lower Bound

Samelson products in $SU(n)$

We give a lower bound for $\mathrm{nil}\, SU(n)_{(p)}$.

- $\hat{\epsilon}_i \in \pi_{2i-1}(SU(n)) = \mathbf{Z},\ (i = 2, \ldots, n)$: generator
- (Bott [Bo]) The order of $\langle \hat{\epsilon}_i, \hat{\epsilon}_j \rangle$ is divisible by $\frac{(i+j-1)!}{(i-1)!(j-1)!}$, $(i + j > n)$

This leads to the following:

- $\langle \epsilon_n, \epsilon_{p-n} \rangle \neq 0$
- $\langle \epsilon_n, \epsilon_{2p-2n} \rangle \neq 0$ $(n < p < \frac{3}{2}n)$
  then, $\langle \epsilon_n, \langle \epsilon_n, \epsilon_{2p-2n} \rangle \rangle \neq 0$

Lower Bound

Facts on Lie Groups

For p: odd prime,
below inclusions induces monomorphism on π_* when localized at p.

$$Sp(n) \hookrightarrow SU(2n) \tag{1}$$
$$Spin(2n+1) \hookrightarrow Spin(2n+2). \tag{2}$$

and Theorem by Friedlander [Fr],

$$Spin(2n+1)_{(p)} \simeq Sp(n)_{(p)}$$

give us desired result for $Sp(n), Spin(n)$.

Lower Bound

$\mathcal{P}^1$ and Samelson Product

G: topological group, p: regular prime.

- $\bar{\epsilon}_{n_j}$: suspension of $\epsilon_{n_j} : S^{2n_j-1} \hookrightarrow G_{(p)}$.
- $x_j \stackrel{\text{def}}{=} \mathrm{hur}(\bar{\epsilon}_{n_j}) \Rightarrow \mathbf{H}^*(BG) = \mathbb{Z}/p[x_1, \ldots, x_l]$

$$\langle \epsilon_{n_i}, \epsilon_{n_j} \rangle = 0 \Leftrightarrow \begin{array}{ccc} S^{2n_i} \vee S^{2n_j} & \xrightarrow{\bar{\epsilon}_{n_i} \vee \bar{\epsilon}_{n_j}} & BG_{(p)} \vee BG_{(p)} \\ \downarrow & & \downarrow \\ S^{2n_i} \times S^{2n_j} & \xrightarrow{\exists \theta} & BG_{(p)} \end{array}$$

$\Rightarrow \forall x_k, \theta \mathcal{P}^1 x_k = \mathcal{P}^1 \theta x_k = 0$

$\Rightarrow \forall x_k$, the component $\alpha x_i \cdot x_j$ in $\mathcal{P}^1 x_k$ is 0

Lower Bound

Results from Hamanaka-Kono's paper

We cite the results from the paper by Hamanaka and Kono.
By calculating $\mathcal{P}^1$ for $\mathbf{H}^*(BG)$, they got:

Theorem ([HK])

When G is exceptional and $p = n_{l+1}$,

$$\mathrm{nil}\, G_{(p)} \geq 3.$$

Therefore, there are only two remaining cases $(E_7, 23)$ and $(E_8, 37)$.

Lower Bound

Calculation of $\mathcal{P}^1$ for $\mathbf{H}^*(BG)$

We need some tedious calculation is necessary.
So we just give some strategy for the calculation.

- For $G = E_7$, we use $Spin(10) \to E_6 \to E_7$.
- For $G = E_8$, we use $Spin(16) \to E_8$.
- First, write pull-backed generators of $\mathbf{H}^*(BG)$ by power-sum on the torus.
- Calculate $\mathcal{P}^1$.
- With aid of Girard's formula, express the result by symmetric functions.
- Then push forward them to $\mathbf{H}^*(BG)$.

End of the talk

Thank you for listening.