Homotopy Nilpotency in p-compact groups

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Outline

- Introduction
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- Our Theorems
  - Upper bound for homotopy nilpotency of H-groups
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- Future work

Previous work

- There was a basic question:
  Which compact Lie groups are homotopy commutative ?
- Hubbuck(1969) gave the complete answer:
  A finite homotopy commutative H-group is a torus.
- *How non-commutative are the rest ?*
- A candidate for measuring this is homotopy nilpotency.
- Next basic question:
  Determine the homotopy nilpotency for compact Lie groups.
  - However, a theorem of Hopkins(1989) says:
    A compact Lie group with no torsion is homotopy nilpotent.
  - Rao(1997) and Yagita(1993) proved the converse is true.
    $\Rightarrow$ Most of compact Lie groups are not homotopy nilpotent.
- *Determine the homotopy nilpotency for compact Lie groups when completed (or localized) at a prime.*

General goal

More generally,
Determine the homotopy nilpotency for all p-compact groups.

Definitions

We restrict ourselves to the following category $\mathcal{T}^*$:

- All spaces have the homotopy types of based, simply connected CW-complexes
- The base point is always denoted by "$*$"
- Denote by $[X, Y]$ the set of all based homotopy classes of based maps from X to Y

For a prime p, there is a *p-completion* functor:$\mathcal{T}^* \to \mathcal{T}^*$, $X \to X_p^\wedge$
This allows us to work with one prime at a time.
For example, $H^*(X_p^\wedge), \pi_*(X_p^\wedge)$ are the ordinary completions of the abelian groups $H^*(X), \pi_*(X)$ respectively, when they are finitely generated.

H-group

- A *H-group* X is a homotopical analogue of a topological group: X has an associative multiplication μ with unit $*$ and an inverse, all of which are up to homotopy.
- For an H-group X and $Y \in \mathcal{T}^*$, $[Y, X]$ is a group by $Y \xrightarrow{\Delta} Y \times Y \xrightarrow{f \times g} X \times X \xrightarrow{\mu} X$.
- G is a *loop space* if G has a *classifying space* BG, where $\Omega BG \simeq G$ as H-groups.
- A loop space G is called a *p-compact group* for a prime p if $BG_p^\wedge \cong BG$ and $H^*(G, \mathbb{Z}/p)$ is finite.
  (Note: the p-completion of a compact Lie group is a p-compact group for any prime p).
- at a prime p
  $\{$ H-groups $\} \supset \{$ loop spaces $\} \supset \{$ p-compact groups $\}$
  $\supset \{$ compact Lie groups $\}$.

Homotopy Nilpotency

The homotopy nilpotency of an H-group X has two equivalent definitions:

(1) $\mathrm{nil}(X) = \sup_{Y \in \mathcal{T}^*} \mathrm{nil}[Y, X]$

(2) $\mathrm{nil}(X) = \min\{n | \gamma_n \simeq *\}$,
where the *iterated commutator* $\gamma_n : \Pi^{n+1} X \to X$ is defined by
$\gamma_1 = \gamma : (x, y) \mapsto xyx^{-1}y^{-1}$,
$\gamma_n = \gamma \circ (1 \times \gamma_{n-1})$.

Given this definition, we have

- X is *homotopy nilpotent* $\overset{\mathrm{def}}{\Longleftrightarrow} \mathrm{nil}(X) < \infty$.
- X is *homotopy commutative* $\overset{\mathrm{def}}{\Longleftrightarrow} \mathrm{nil}(X) = 1$.

Known Facts on Nilpotency

- There are few examples where $\mathrm{nil}(X)$ are explicitly determined.
- (Hubbuck(1969)) Finite simply connected H-groups are not homotopy commutative.
- Finite H-groups localized at 0 are homotopy commutative.
- $\mathrm{nil}(X) - 1 \leq \sup_p \mathrm{nil}(X_p^\wedge) \leq \mathrm{nil}(X)$.
  $\Rightarrow$ Therefore we can focus only on the p-completed information.
- If $X_3 = X_1 \times X_2$ as H-groups, $\mathrm{nil}(X_3) = \max(\mathrm{nil}(X_1), \mathrm{nil}(X_2))$.
  $\Rightarrow$ We only have to consider irreducible H-groups.
- (Hopkins(1989), Yagita(1993), Rao(1997)) G: compact Lie group
  $G_p^\wedge$ is homotopy nilpotent $\Leftrightarrow$ $\mathbf{H}_*(G;\mathbb{Z})$ has no p-torsion.
  $\Rightarrow$ We consider only on larger primes.

Our goal today

- For which primes ?
  $\Rightarrow$ The homotopy nilpotency seems to increase rapidly as the prime gets smaller, and gets harder to calculate. As a first step, we only consider all but finite many primes, namely the so called *regular primes*.

Goal

Determine $\mathrm{nil}(G)$ for all the pairs (G, p), where G is an irreducible p-compact, p-regular group.

Our strategy is divided into two steps

- First, we give an upper bound in terms of rational cohomology of the H-groups.
- Second, we specialize to p-compact, p-regular groups and give lower bounds by case by case analysis.
- these bounds therefore enable us to explicitly determine the homotopy nilpotency for all the p-compact, p-regular groups.

The type of H-group

For a H-group X,

Definition

- X has *type* $(n_1, n_2, \ldots, n_l)$ with $n_1 \leq \cdots \leq n_l$ $\overset{\text{def}}{\Longleftrightarrow} X \simeq_0 S^{2n_1-1} \times \cdots \times S^{2n_l-1}$.
- X of type $(n_1, n_2, \ldots, n_l)$ is *p-regular* $\overset{\text{def}}{\Longleftrightarrow} X \simeq_p S^{2n_1-1} \times \cdots \times S^{2n_l-1}$.

- (Kumpel(1972), Wilkerson(1973)) If X is a p-compact group, $p \geq n_l \Leftrightarrow X$ is p-regular.
- Types of compact simple Lie groups are completely known:

$A_l = SU(l+1)$	$(2, 3, \ldots, l+1)$	G_2	$(2, 6)$
$B_l = Spin(2l+1)$	$(2, 4, \ldots, 2l)$	F_4	$(2, 6, 8, 12)$
$C_l = Sp(l)$	$(2, 4, \ldots, 2l)$	E_6	$(2, 5, 6, 8, 9, 12)$
$D_l = Spin(2l)$	$(2, 4, \ldots, 2l-2, l)$	E_7	$(2, 6, 8, 10, 12, 14, 18)$
		E_8	$(2, 8, 12, 14, 18, 20, 24, 30)$

(Note: there is a similar classification for all p-compact groups)

Previous results in our direction

Theorem (James and Thomas(1962))

For a loop space G of type $(n_1, \ldots, n_l)$,

- $n_l < p < 2n_l \Rightarrow G_p^\wedge$ is not homotopy commutative.

Theorem (McGibbon(1984))

For a compact simple Lie group G of type $(n_1, \ldots, n_l)$,

- If $p > 2n_l$, then $G_p^\wedge$ is homotopy commutative.
- If $p < 2n_l$, then $G_p^\wedge$ is not homotopy commutative except for the cases that $(G, p) = (Sp(2), 3)$, $(G_2, 5)$.

Note: L.Saumell generalized this work in two directions:

- *Homotopy commutativity of finite loop spaces (1991).*
- *Higher homotopy commutativity in localized groups (1995).*

Main Theorems

Theorem (K-Kishimoto "Upper bound")

X: H-group of type $(n_1, \ldots, n_l)$, $p > n_l$

- $\mathrm{nil}(X_p^\wedge) \leq 3$
- For $2n_l < p$, $\mathrm{nil}(X_p^\wedge) = 1$
- For $p < 2n_l$, $\mathrm{nil}(X_p^\wedge) \leq 2$ if $n_1 \neq 2$ or we cannot choose n_i, n_j, n_k, n_s satisfying $n_i + n_j = n_s + p - 1,\ n_k + n_s = p + 1$
  In particular, $\mathrm{nil}(X_p^\wedge) \leq 2$ if $\frac{3}{2}n_l < p < 2n_l$

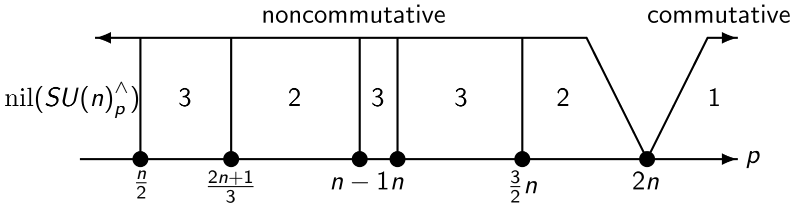
Theorem (K-Kishimoto "Lower bound")

G: p-compact group, p: regular prime ($\Leftrightarrow p \geq n_l$)

- $\mathrm{nil}(G) = 3$ iff $n_1 = 2$ and $n_i + n_j = n_s + p - 1,\ n_k + n_s = p + 1$ for some n_i, n_j, n_k, n_s and $(G, p) \neq (SU(2), 2)$

Remarks on Main Theorems

- For a regular prime p, $\mathrm{nil}(X_p^\wedge) = \mathrm{nil}(X_{(p)})$, where $X_{(p)}$ is the p-localization of X.
- First one slightly generalizes McGibbon's proof, since we don't require X to be a loop space: $2n_l < p \Rightarrow \mathrm{nil}(X_p^\wedge) = 1$,
- When we consider a p-compact group G, we don't need simply connectedness, since G splits into the product of a p-completed Lie group and a simply connected p-compact group.
- Kishimoto has obtained some results for quasi regular primes:



Outline of Proof

- For the upper bounds of homotopy nilpotency:
  - Decompose the commutator by elementary group theory
  - Use the computation of the p-primary part of $\pi_*(S^{2i-1})$ due to Toda
- For the lower bounds of homotopy nilpotency:
  - Use the classification for a case by case analysis
  - Bott's calculation of the Samelson product in $SU(n)$
  - Some facts on classical Lie groups
  - Find non-trivial Samelson products in BG using Steenrod operations

Basic Tools - Samelson Product

We want to see the triviality of the commutator through maps from well known spaces, i.e. spheres.

Definition

A, B: space, X: H-group
Samelson product of maps $\alpha : A \to X$ and $\beta : B \to X$, denoted by $\langle \alpha, \beta \rangle$, is the composition

$$A \wedge B \xrightarrow{\alpha \wedge \beta} X \wedge X \xrightarrow{\gamma} X$$

Note: $A \wedge B = A \times B / (A \times \{*\} \cup \{*\} \times B)$, in particular $S^n \wedge S^m = S^{n+m}$.

By Definition, $\mathrm{nil}(X) < k \Leftrightarrow \underbrace{\langle 1_X, \langle 1_X, \cdots \langle 1_X, 1_X \rangle \rangle \cdots \rangle}_{k} \simeq *$

Note: usually both A and B are taken to be spheres.

Elementary group theory

We want to decompose the commutator into atomic pieces, namely Samelson products of maps from spheres.

For H : any discrete group

Let $[a, b]$ denote the usual commutator of a and b:
$[a, b] = aba^{-1}b^{-1}$ $(a, b \in H)$

Then, for example, we have:

$$[xy, z] = [x, [y, z]][y, z][x, z]$$

Repeatedly applying this, we can decompose commutators into atomic pieces.
$\Rightarrow$ We only have to care about the commutators for generators.

Samelson Product

X : p-regular H-group and $X \simeq S_1 \times \cdots \times S_l$, where $S_i = (S^{2n_i-1})^\wedge_p$.
$\epsilon_i : S_i \to X$: inclusion
Define $\epsilon'_i : X \to X$ as $S_1 \times \cdots \times S_l \xrightarrow{\pi_i} S_i \xrightarrow{\epsilon_i} S_1 \times \cdots \times S_l$

$$\Rightarrow 1_X = \mu(\epsilon'_1 \cdots \epsilon'_l)$$

Thus we can take ϵ_i's as atom.
Then the argument in the previous slide gives:

Lemma

$$\begin{aligned} \mathrm{nil}X < k &\Leftrightarrow \langle 1_X, \langle 1_X, \cdots \langle 1_X, 1_X\rangle\rangle \cdots \rangle \simeq * \\ &\Leftrightarrow \langle \epsilon_{i_1}, \langle \epsilon_{i_2}, \cdots \langle \epsilon_{i_k}, \epsilon_{i_{k+1}}\rangle \cdots \rangle\rangle \simeq *, \qquad 1 \le \forall i_j \le l \\ &\Leftrightarrow \pi_h \circ \langle \epsilon_{i_1}, \langle \epsilon_{i_2}, \cdots \langle \epsilon_{i_k}, \epsilon_{i_{k+1}}\rangle \cdots \rangle\rangle \simeq *, \qquad 1 \le \forall i_j, h \le l \end{aligned}$$

$\pi_{2n_i-1}((S^{2n_j-1})^\wedge_p) \ni \pi_j \circ \epsilon_i : (S^{2n_i-1})^\wedge_p \to (S^{2n_j-1})^\wedge_p$

Upper bound

- In our setting, triviality of $\gamma : X \wedge X \to X$ is reduced to that of the following composition: (here we omit completion sign)

$$S^{2n_i-1} \wedge S^{2n_j-1} \hookrightarrow X \wedge X \xrightarrow{\gamma} X \xrightarrow{\pi_k} S^{2n_k-1}.$$

- Then the problem of the triviality of the iterated commutator is reduced to that of iterated Samelson products from a sphere to another sphere

$$\gamma \circ \Sigma^i \gamma : S^i \wedge S^j \wedge S^k \xrightarrow{1\wedge\gamma} S^i \wedge S^{j+k} \xrightarrow{\gamma} S^{i+j+k}.$$

- A homotopy set of maps from a sphere to a p-completed sphere is called the p-primary part of homotopy groups, and is known to high enough degrees for our purpose.
- Fortunately there are few non-trivial elements. Moreover we can list all possible non-trivial Samelson products for degree reason.
- This argument gives an upper bound for the homotopy nilpotency.

p-primary components of homotopy groups of spheres

We recall some facts on p-primary components of homotopy groups of spheres due to Toda.
Note: ${}^p\pi_*(S^{2n-1}) = \pi_*((S^{2n-1})^{\wedge}_p)$

- $${}^p\pi_{2n-1+k}(S^{2n-1}) = \begin{cases} \mathbb{Z}/p & k = 2p-3 \\ 0 & 0 < k < 4p-6, k \neq 2p-3 \end{cases}$$
- $\alpha_1(3) \in {}^p\pi_{2p}(S^3) = \mathbb{Z}/p$: generator
- $\alpha_1(n) \overset{\text{def}}{=} \Sigma^{2n-4}\alpha_1(3)$
  $\Rightarrow {}^p\pi_{2n+2p-4}(S^{2n-1}) = \mathbb{Z}/p$ is generated by $\alpha_1(2n-1)$.
- $\alpha_1(3) \circ \alpha_1(2p) \neq 0$.
- $\alpha_1(2n-1) \circ \alpha_1(2n+2p-4) = 0$, $(n > 2)$

Samelson products of p-regular H-groups

X : H-group of type $(n_1, \dots, n_l)$ with $n_1 \leq \cdots \leq n_l$.

$\Rightarrow X_p^\wedge \simeq (S^{2n_1-1})_p^\wedge \times \cdots \times (S^{2n_l-1})_p^\wedge, (p \geq n_l)$,

we define ϵ_{n_i} and π_{n_j} as

$$(S^{2n_i-1})_p^\wedge \xrightarrow{\epsilon_{n_i}} (S^{2n_1-1})_p^\wedge \times \cdots \times (S^{2n_l-1})_p^\wedge \xrightarrow{\pi_{n_j}} (S^{2n_j-1})_p^\wedge$$

- Since $X_{(0)}$ is homotopy commutative,
  - $\langle \epsilon_{n_i}, \epsilon_{n_j} \rangle \in \pi_{2(n_i+n_j-1)}(X_p^\wedge)$: torsion
  - $\pi_{n_s} \circ \langle \epsilon_{n_i}, \epsilon_{n_j} \rangle = \begin{cases} N\alpha_1(2n_s - 1) & n_i + n_j = n_s + p - 1 \\ 0 & n_i + n_j \neq n_s + p - 1 \end{cases}$.
  - $p > 2n_l \Rightarrow \mathrm{nil} X_{(p)} < 2$.
- Since $\langle \epsilon_{n_i}, \epsilon_{n_j} \circ \alpha_1(2n_j - 1) \rangle = \langle \epsilon_{n_i}, \epsilon_{n_j} \rangle \circ \Sigma^{2n_i - 1} \alpha_1(2n_j - 1)$,
  - If a 3-fold commutator $\langle \epsilon_{n_k}, \langle \epsilon_{n_i}, \epsilon_{n_j} \rangle \rangle$ is non-trivial,
  - $\Rightarrow \; n_t = 2, \; n_i + n_j = n_s + p - 1, \; n_i + n_j + n_k = 2p$.
  - Especially $p > \frac{3}{2} n_l \Rightarrow \mathrm{nil} G_p^\wedge < 3$.
- $p > n_l \Rightarrow \mathrm{nil} G_p^\wedge \leq 3$

Theorem for upper bound

Theorem (K-Kishimoto "Upper bound")

X: H-group of type $(n_1, \ldots, n_l)$, $p > n_l$

- $\mathrm{nil}(X_p^\wedge) \leq 3$
- For $2n_l < p$, $\mathrm{nil}(X_p^\wedge) = 1$
- For $p < 2n_l$, $\mathrm{nil}(X_p^\wedge) \leq 2$ if $n_1 \neq 2$ or we cannot choose n_i, n_j, n_k, n_s satisfying $n_i + n_j = n_s + p - 1,\ n_k + n_s = p + 1$
  In particular, $\mathrm{nil}(X_p^\wedge) \leq 2$ if $\frac{3}{2}n_l < p < 2n_l$

Now we proceed to give a lower bound.

Samelson products in classical Lie groups

We give a lower bound for the homotopy nilpotency by explictly giving non-trivial Samelson products.

- Bott(1960) calculated the Samelson products for $SU(n)$ in the integral case.
- The result can be used also to find non-trivial Samelson products in p-completed cases.
- For other classical groups,

Theorem (well known facts)

Inclusions below induces monomorphism on homotopy groups when localized at an odd prime p.

$$\begin{aligned} Sp(n) &\hookrightarrow SU(2n) \\ Spin(2n+1) &\hookrightarrow Spin(2n+2). \end{aligned}$$

and by Friedlander(1975) there are isomorphisms of H-groups:

$$Spin(2n+1) \simeq_p Sp(n)$$

Samelson products in $SU(n)$

We give a lower bound for $\mathrm{nil}(SU(n)^\wedge_p)$.

- $\hat{\epsilon}_i \in \pi_{2i-1}(SU(n)) = \mathbb{Z},\ (i = 2, \dots, n)$: generator
- (Bott) The order of $\langle \hat{\epsilon}_i, \hat{\epsilon}_j \rangle$ is divisible by $\frac{(i+j-1)!}{(i-1)!(j-1)!}$, $(i + j > n)$

This leads to the following:

- $$\begin{aligned}(n \le p < 2n) &\Rightarrow \langle \epsilon_n, \epsilon_{p-n} \rangle \neq 0 \\ &\Rightarrow \mathrm{nil}(SU(n)^\wedge_p) \geq 2\end{aligned}$$
- $$\begin{aligned}(n \le p < \frac{3}{2}n) &\Rightarrow \langle \epsilon_n, \epsilon_{2p-2n} \rangle \neq 0 \\ &\text{and } \langle \epsilon_n, \langle \epsilon_n, \epsilon_{2p-2n} \rangle \rangle \neq 0 \\ &\Rightarrow \mathrm{nil}(SU(n)^\wedge_p) \geq 3\end{aligned}$$

$\mathcal{P}^1$ and Samelson Product

Here we give a cohomological criterion for non-triviality of Samelson products.

G: loop space, BG: the classifying space of G, p: regular prime.

- $\bar{\epsilon}_{n_j} : S^{2n_j} \to BG_p^\wedge$: suspension of $\epsilon_{n_j} : S^{2n_j - 1} \hookrightarrow G_p^\wedge$.
- x_j: the dual of the hurwicz image of $\bar{\epsilon}_{n_j}$
- $\Rightarrow \mathbf{H}^*(BG; \mathbb{Z}/p) = \mathbb{Z}/p[x_1, \dots, x_l]$
- $\mathcal{P}^1 : \mathbf{H}^*(BG; \mathbb{Z}/p) \to \mathbf{H}^{*+2(p-1)}(BG; \mathbb{Z}/p)$: Steenrod's first reduced power operation.

$$\langle \epsilon_{n_i}, \epsilon_{n_j} \rangle = 0 \Leftrightarrow \begin{array}{ccc} S^{2n_i} \vee S^{2n_j} & \xrightarrow{\bar{\epsilon}_{n_i} \vee \bar{\epsilon}_{n_j}} & BG_p^\wedge \vee BG_p^\wedge \\ \downarrow & & \downarrow \\ S^{2n_i} \times S^{2n_j} & \xrightarrow{\exists \theta} & BG_p^\wedge \end{array}$$

$\Rightarrow \forall x_k, \theta \mathcal{P}^1 x_k = \mathcal{P}^1 \theta x_k = 0$

$\Rightarrow \forall x_k$, the component $\alpha x_i \cdot x_j$ in $\mathcal{P}^1 x_k$ is 0

Calculation of $\mathcal{P}^1$ for $\mathbf{H}^*(BG;\mathbb{Z}/p)$

Non-triviality of the action of $\mathcal{P}^1$ detects non-triviality of certain Samelson product.

Thus, we can find non-trivial Samelson products if we know the action of $\mathcal{P}^1$ on the cohomology of the classifying space $\mathbf{H}^*(BG;\mathbb{Z}/p)$.

We can calculate the cohomology operation $\mathcal{P}^1$ by the facts below:

- The cohomology of BG for a p-compact group is known to be the ring of invariants of the corresponding Weyl group.
- $\mathbf{H}^*(BG;\mathbb{Z}/p) = \mathbb{Z}/p[y_1, \ldots, y_l]^W, |y_i| = 2.$
- Mehta(1988) calculated the invariant ring for all Weyl groups.
- $\mathcal{P}^1 y_i = y_i^p$ enables us to calculate $\mathcal{P}^1$ on $\mathbf{H}^*(BG;\mathbb{Z}/p)$.

Calculation of $\mathcal{P}^1$ for groups of large rank

Theoretically the method in the previous slide gives the way to calculate $\mathcal{P}^1$ action, however the calculation is very hard in practice.

One possible strategies to reduce the computations are listed below:

- If there is an inclusion of Weyl groups $W_1 \to W_2$, we have a ring homomorphism $\mathbf{H}^*(BG_2; \mathbb{Z}/p) \to \mathbf{H}^*(BG_1; \mathbb{Z}/p)$.
- For example,
  - $Spin(10) \to E_6 \to E_7$
  - $Spin(16) \to E_8$
  - $I_5 \to H_3 \to H_4$

Future Work

- Find a systematic description for the homotopy nilpotency.
- Find a relation to homotopical properties of gauge groups.

Let $\mathcal{G}$ be a gauge group of a principal G-bundle over a sphere S^n. There is a fibration:

$$\mathcal{G} \to G \xrightarrow{\delta} B\mathcal{G}_0,$$

where $\delta : G \to B\mathcal{G}_0 \simeq \Omega^{n-1}G$ can be regarded as a (adjoint of) Samelson product

$$G \wedge S^{n-1} \hookrightarrow G \wedge G \to G$$