Computers work exceptionally on Lie groups

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Outline

- Introduction
- Computation of topological invariants
- An example: Chow rings of Lie groups
  - Schubert calculus (geometry)
  - Divided difference operator (combinatorics)
  - Borel presentation (algebraic topology)
  - Computer assisted part
- Open problems

Latex-Beamer

This presentation slide is made with LaTeX-Beamer.

- It is an easy-to-use LaTeXpackage, free of charge
- It produces a PDF file, which is almost environment independent
- There are a lot of people who use it; you can ask, consult web pages, even request new features
- It has the capability of

this kind of gimmicks

- and hyperlinks ▸ Main Theorem

Latex-Beamer

Theorem

A mathematician is an optimist

Proof.

I have discovered a truly remarkable proof which this margin is too small to contain. □

Advantages and Disadvantages

Advantages of using computers are:

- Theorem can be proven while you are sleeping
- Everyone can confirm the computation
- The development of computers and softwares might produce new theorems

Disadvantages of using computers are:

- It requires non-essential, non-mathematical work to make a practical program (sometimes we may have to struggle with bugs in compilers, OS, CPU...)
- It looks less elegant than human proof
- consequently, results are often underestimated by those who don't use computers
- $\Rightarrow$ Where and in what form can we submit results ?

Common process

Problem

Compute some invariant $F(X)$ for a space X,
where F is a functor from spaces to some algebras

- Look for a theory which enables a concrete calculation
- Compute easy cases by hand to get insight
- Translate the mathematical theory into computer algorism
  - Search for "parts" (libraries) made by others
  - Implement of the data structure that corresponds to mathematical objects (polynomial, DGA, Lie algebra, Hopf algebra,. . .)
  - Choose programming language, software, etc.
  - In algebraic topology, we usually resort to symbolic computation rather than numerical one
- Run the program and pray !
- Optimize it from both mathematical and computer's point of view
- Confirm the result by different algorism, softwares, platforms, . . .
- Look at the result to find general theory behind it

Lie group basic

- G: simple, simply-connected, compact Lie group
- $G \supset T$: maximal torus ($\dim T = l$ is the rank of G)
- $\mathfrak{t} \subset \mathfrak{g}$: their Lie algebras with an invariant inner product (,)
- $\mathfrak{t}^* \supset \Pi = \{\alpha_i\}_{1 \le i \le l}$: simple roots
- $\{\omega_i\}_{1 \le i \le l}$: fundamental weights ($(\frac{2\alpha_j}{(\alpha_j, \alpha_j)}, \omega_i) = \delta_{ij}$)
- $H^*(BT; \mathbb{Z}) = \mathbb{Z}[w_1, \dots, w_l]$, where $|w_i| = 2$
- $s_i \in GL(\mathfrak{t}^*)$: simple reflection corresponding to α_i, $s_i \in \mathrm{Aut}(\mathbb{Z}[w_1, \dots, w_l])$ ($s_i(e) = e - (\frac{2\alpha_i}{(\alpha_i, \alpha_i)}, e)\alpha_i$)
- $W = N(T)/T$: Weyl group of G (= a finite group generated by $\{s_i\}_{1 \le i \le l}$)
- Classification:

$$A_n, B_n, C_n, D_n, G_2, F_4, E_6, E_7, E_8$$

These objects are all concrete (symbolic). However, for example, w_i's are possibly irrational vectors so we need to handle with care.
(Stembridge's coxeter/weyl package in Maple is so convenient that this often encourage me to choose Maple)

Computable invariants

- rational cohomology is the invariant ring $H^*(BG;\mathbb{Q}) \cong H^*(BT;\mathbb{Q})^W$
- mod p cohomology $H^*(BG;\mathsf{F}_p)$
  $\leftrightarrow$ the invariant rings $H^*(BT;\mathsf{F}_p)^W$ and $H^*(B(\mathbb{Z}/p)^n;\mathbb{Z}/p)^{W_p}$
- rational cohomology of flag variety is the coinvariant ring

$$H^*(G/T;\mathbb{Q}) \cong H^*(BT;\mathbb{Q})/(H^+(BT;\mathbb{Q})^W)$$

- stable homotopy group $\pi_*^S(G) \leftrightarrow$ free resolution of $H^*(G;\mathsf{F}_p)$ as $\mathcal{A}_p$-algebra
- Grothendieck's torsion index of $G \leftrightarrow$ Gröbner basis of

$$(H^+(BT;\mathbb{Z})^W) \subset H^*(BT;\mathbb{Z})$$

- $H^*(\Omega G), H^*(G/P), K^*(\Omega G), cat(G), \ldots$
- self-equivalence of generalized flag variety Aut$(G/P) \leftrightarrow$ Aut$(H^*(G/P;\mathbb{Q}))$, where P is a parabolic subgroup

An example of invariants: The Chow ring $A^*(G)$

Notation

- G: simply connected simple complex Lie group (SL$(n, \mathbb{C})$)
- B: Borel subgroup of G (the subgroup of upper triangular matrices)
- G$/B$: a projective variety called the flag variety
  (the space of "flags", $0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_{n-1} \subseteq V_n = \mathbb{C}^n$, $\dim_{\mathbb{C}}(V_i) = i$).
- $H^*(\mathrm{G}/B; \mathbb{Z})$: ordinary integral cohomology of G$/B$
- $A^*(\mathrm{G})$: Chow ring of G
  - $A^*(\mathrm{G}) = \bigoplus_{i \geq 0} A^i(\mathrm{G})$
  - $A^i(\mathrm{G})$ is a group of the rational equivalence classes of algebraic cycles of codimension i.
    (an algebraic cycle is a linear sum of possibly singular subvarieties)
  - *intersection product* $A^i(\mathrm{G}) \otimes A^j(\mathrm{G}) \to A^{i+j}(\mathrm{G})$

Chow ring of G

Goal

Determine $A^*(\mathsf{G})$ for all simply connected simple complex Lie groups

- Classification Theorem tells that G is one of the following: $\mathsf{SL}_n, \mathsf{Spin}_n, \mathsf{Sp}_n, \mathsf{G}_2, \mathsf{F}_4, \mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$
- Grothendieck considered the problem in the 1950's
  - (Grothendieck) $A^*(\mathsf{G}) \cong H^*(\mathsf{G}/B;\mathbb{Z})/(H^2(\mathsf{G}/B;\mathbb{Z}))$
  - Consequently, $A^*(\mathsf{G}) \otimes \mathbb{Q} = \mathbb{Q}$ for all G and $A^*(\mathsf{G}) = \mathbb{Z}$ for $G = \mathsf{SL}_n, \mathsf{Sp}_n$
- $A^*(\mathsf{G})$ for G $= \mathsf{Spin}_n, \mathsf{G}_2, \mathsf{F}_4$ were determined by R.Marlin(1974)
- Remaining cases are when G $= \mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$.

Grothendieck's Theorem

Theorem (Grothendieck(1958))

- the cycle map $cl : A^*(G/B) \to H^{2*}(G/B;\mathbb{Z})$ is an isomorphism of rings:

$$A^*(G/B) \xrightarrow{\simeq} H^{2*}(G/B;\mathbb{Z})$$

- the pullback of the projection $p : G \to G/B$ induces a surjection

$$p^* : A^*(G/B) \to A^*(G),$$

where the kernel is an ideal generated by $A^1(G/B)$.

Corollary

$A^*(G) \cong H^*(G/B;\mathbb{Z})/(H^2(G/B;\mathbb{Z}))$
$A^*(G) = \mathbb{Z}, \quad G = SL_n, Sp_n$

Schubert class

The Bruhat decomposition gives a cell decomposition

$$G/B = \coprod_{w \in W} BwB/B$$

- X_w = closure of $BwB/B (\cong \mathbb{C}^{l(w)})$: Schubert variety
- $Z_w = \{$the cohomology class corresponding to $[X_{w_0 w}]\} \in H^{2l(w)}(G/B;\mathbb{Z})$: Schubert class
- $\{Z_w\}_{w \in W}$ forms an additive basis for $H^*(G/B;\mathbb{Z})$ (indexed by W)
  In particular, $H^*(G/B;\mathbb{Z})$ is torsion free

Structure constant

The intersection product of two Schubert classes Z_w, Z'_w can be written in the linear sum of Schubert classes:

$$Z_w \cdot Z'_w = \sum_{l(v)=l(w)+l(w')} c^v_{ww'} Z_v$$

The coefficients $c^v_{ww'} \in \mathbb{Z}$ are called the structure constants

A goal in Schubert calculus

Give a combinatorial formula for $c^v_{ww'}$

- Littlewood-Richardson rule for Grassmaniann
- Chevalley formula for $Z_w \cdot Z_{w'}$ when $l(w) = 1$
- Schubert polynomial

Divided difference operator

Definition (B-G-G(1973), Demazure(1973))

1. For $\alpha_i \in \Pi$, $\Delta_i : H^*(BT;\mathbb{Z}) \to H^{*-2}(BT;\mathbb{Z})$

$$\Delta_i(f) = \frac{f - s_i(f)}{\alpha_i}, f \in H^*(BT;\mathbb{Z}) = \mathbb{Z}[\omega_1, \dots, \omega_l]$$

2. For $w \in W$, $w = s_{i_1} s_{i_2} \cdots s_{i_k}$: a reduced decomposition,

$$\Delta_w = \Delta_{i_1} \circ \Delta_{i_2} \circ \cdots \circ \Delta_{i_k} : H^*(BT;\mathbb{Z}) \to H^{*-2k}(BT;\mathbb{Z})$$

Theorem (B-G-G(1973), Demazure(1973))

- the characteristic map $c : H^{2k}(BT;\mathbb{Z}) \to H^{2k}(G/B;\mathbb{Z})$:

$$c(f) = \sum_{l(w)=k} \Delta_w(f) Z_w \quad (\text{Note:}\Delta_w(f) \in \mathbb{Z})$$

- the following composition is induced by the inclusion of $\mathbb{Z} \hookrightarrow \mathbb{Q}$:

$$H^*(G/B;\mathbb{Z}) \hookrightarrow H^*(G/B;\mathbb{Q}) \cong H^*(BT;\mathbb{Q})/H^+(BT;\mathbb{Q})^W \xrightarrow{c} H^*(G/B;\mathbb{Q})$$

- (Giambelli formula)

$$Z_w = c\left(\Delta_{w^{-1}w_0}\left(\frac{\prod_{\alpha\in\Delta^+}\alpha}{|W|}\right)\right)$$

- Inductive formula:

$$\begin{aligned} \Delta_\alpha(\omega_\beta) &= \delta_{\alpha\beta} \\ \Delta_\alpha(fg) &= \Delta_\alpha(f)g + s_\alpha(f)\Delta_\alpha(g) \\ \Delta_w &= \Delta_{i_1}\circ\Delta_{i_2}\circ\cdots\circ\Delta_{i_k} \end{aligned}$$

Ring structure of $H^*(G/B;\mathbb{Z})$

$$\text{polynomial} \underset{\text{Giambelli formula}}{\overset{\text{characteristic map}}{\Leftrightarrow}} \text{Schubert classes (Weyl group)}$$

1. Given elements $Z_w, Z'_w \in H^*(G/B;\mathbb{Z})$
2. by Giambelli formula, we have polynomials $f, f' \in H^*(BT;\mathbb{Q})$ which correspond to Z_w, Z'_w
3. by characteristic map, we have $c(f \cdot f') \in H^*(G/B;\mathbb{Z})$ which correspond to the intersection product $Z_w \cdot Z'_w$
4. we obtain the ring structure of $H^*(G/B;\mathbb{Z})$
5. hence we obtain $A(G) \cong H^*(G/B;\mathbb{Z})/H^2(G/B;\mathbb{Z})$

Computational complexity

characteristic map:

$$c(f) = \sum_{l(w)=k} \Delta_w(f) Z_w$$

Giambelli formula:

$$Z_w = c\left(\Delta_{w^{-1}w_0}\left(\frac{\prod_{\alpha\in\Delta^+}\alpha}{|W|}\right)\right)$$

For $G = E_8$,

- $|W| = 696729600 = 8064 * 24 * 3600$
- $l(w_0) = |\Delta^+| = \frac{1}{2}\dim G/B = 120$

Today's computer cannot handle polynomials of degree 120 !
$\Rightarrow$ The above strategy is not practical as it is

Borel presentation

There are two descriptions for $H^*(G/B;\mathbb{Z}) = A^*(G/B)$

	Borel presentation	Schubert presentation
elements	quotient of a polynomial ring polynomials	$\mathbb{Z}$-basis indexed by Weyl group Schubert classes
geometry	no	algebraic cycles
ring structure	easy	hard (main theme of Schubert calculus)

$$\text{Borel presentation} \underset{\text{Giambelli formula}}{\overset{\text{characteristic map}}{\Leftrightarrow}} \text{Schubert presentation}$$

Results from algebraic topology

By spectral sequence argument, Borel presentation for each $H^*(G/B;\mathbb{Z})$ was computed by Borel, Toda-Watanabe, Bott-Samelson, and Nakagawa.

They have the following form in general.

$$H^*(G/B;\mathbb{Z}) \cong \mathbb{Z}[t_i, \gamma_j]/(\text{ideal}), \quad (|t_i| = 2, |\gamma_j| > 2$$

Our strategy is:

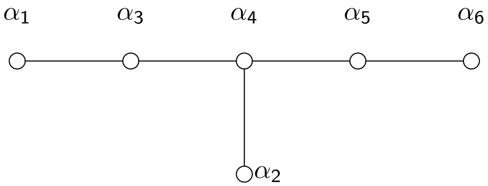
1. Compute $A^*(G)$ purely algebraically from Borel presentation

$$A^*(G) \cong H^*(G/B;\mathbb{Z})/(H^2(G/B;\mathbb{Z})) \cong \mathbb{Z}[\gamma_j]/(\text{ideal})$$

2. Find Schubert varieties representing the generators γ_j

Convenient presentation of $H^*(BT;\mathbb{Z})$

Let G be either E_6, E_7, or E_8.



If we take away α_2, then the Dynkin diagram becomes type A.
By this observation, We take another set of generators for $H^*(BT;\mathbb{Z}) = \mathbb{Z}[\omega_1, \omega_2, \ldots, \omega_l]$:

$$
\begin{aligned}
t_l &= \omega_l \\
t_i &= s_{i+1}(t_{i+1}) = \begin{cases} \omega_i - \omega_{i+1} & (4 \le i \le l-1) \\ \omega_{i-1} + \omega_i - \omega_{i+1} & (i = 2, 3) \end{cases} \\
t_1 &= s_1(t_2) = -\omega_1 + \omega_2 \\
t &= \omega_2
\end{aligned}
$$

Toda-Watanabe's magical basis

Let $c_i = i$-th elementary symmetric function in $t_1, \dots, t_l$ $(1 \leq i \leq l)$

$$\begin{aligned} H^*(BT;\mathbb{Z}) &= \mathbb{Z}[\omega_1, \omega_2, \dots, \omega_l] \\ &= \mathbb{Z}[t_1, t_2, \dots, t_l, t]/(c_1 - 3t). \end{aligned}$$

- s_i $(i \neq 2)$ act on $\{t_i\}_{1 \leq i \leq l}$ as permutations and trivially on t.
- $\Rightarrow$ For example, for $f \in \mathbb{Z}[t, c_2, \dots, c_l]$, $\Delta_i f = 0$ if $i \neq 2$
- $\Rightarrow$ this reduces the computation

Theorem (Nakagawa(2001))

$$H^*(E_7/B;\mathbb{Z}) \cong \mathbb{Z}[t_1, t_2, \ldots, t_7, t, \gamma_3, \gamma_4, \gamma_5, \gamma_9] \\ /(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \rho_8, \rho_9, \rho_{10}, \rho_{12}, \rho_{14}, \rho_{18}),$$

$$\begin{aligned}
\rho_1 &= c_1 - 3t, \\
\rho_2 &= c_2 - 4t^2, \\
\rho_3 &= c_3 - 2\gamma_3, \\
\rho_4 &= c_4 + 2t^4 - 3\gamma_4, \\
\rho_5 &= c_5 - 3t\gamma_4 + 2t^2\gamma_3 - 2\gamma_5, \\
\rho_6 &= {\gamma_3}^2 + 2c_6 - 2t\gamma_5 - 3t^2\gamma_4 + t^6, \\
\rho_8 &= 3{\gamma_4}^2 - 2\gamma_3\gamma_5 + t(2c_7 - 6\gamma_3\gamma_4) - 9t^2c_6 \\
&\quad + 12t^3\gamma_5 + 15t^4\gamma_4 - 6t^5\gamma_3 - t^8, \\
&\ldots
\end{aligned}$$

From the result in previous slide, an easy calculation by hand shows

$$\begin{aligned} A^*(E_7) &\cong A^*(E_7/B)/(A^1(E_7/B)) \\ &\cong H^*(E_7/B;\mathbb{Z})/(H^2(E_7/B;\mathbb{Z})) \\ &\cong \mathbb{Z}[\gamma_3,\gamma_4,\gamma_5,\gamma_9]/(2\gamma_3,3\gamma_4,2\gamma_5,\gamma_3^2,2\gamma_9,\gamma_5^2,\gamma_4^3,\gamma_9^2) \end{aligned}$$

By using a Maple script, we obtain

$$\begin{aligned} \gamma_3 &= Z_{342} + 2Z_{542} \\ \gamma_4 &= Z_{1342} + 2Z_{3542} + Z_{6542} \\ \gamma_5 &= Z_{76542} \\ \gamma_9 &= 2Z_{154376542} + Z_{654376542} \end{aligned}$$

Note that we abbreviate $s_{i_1}s_{i_2}\cdot s_{i_k} \in W$ as $i_1\cdots i_k$

$$A^*(E_7) = \mathbb{Z}[Z_{542}, Z_{6542}, Z_{76542}, Z_{654376542}] \Big/ \begin{pmatrix} 2Z_{542}, 3Z_{6542}, 2Z_{76542}, Z_{542}^2, 2Z_{654376542}, \\ Z_{76542}^2, Z_{6542}^3, Z_{654376542}^2 \end{pmatrix}$$

Theorem (K-Nakagawa)

$$A(E_6) = \mathbb{Z}[Z_{542}, Z_{6542}]/(2Z_{542}, 3Z_{6542}, Z_{542}^2, Z_{6542}^3),$$
$$(Z_{542} = \overline{B(w_0 s_6 s_5 s_4 s_2)B} \subset G \text{ etc.})$$

$$\begin{aligned} A(E_7) = \mathbb{Z}[X_3, X_4, X_5, X_9] \\ /(2X_3, 3X_4, 2X_5, X_3^2, 2X_9, X_5^2, X_4^3, X_9^2) \end{aligned}$$

$$(X_3 = Z_{542}, X_4 = Z_{6542}, X_5 = Z_{76542}, X_9 = Z_{654376542})$$

$$A(E_8) = \mathbb{Z}[X_3, X_4, X_5, X_6, X_9, X_{10}, X_{15}] \Big/ \begin{pmatrix} 2X_3, 3X_4, 2X_5, 5X_6, 2X_9, X_5^2 - 3X_{10}, \\ X_4^3, 2X_{15}, X_9^2, 3X_{10}^2, X_3^8, \\ X_{15}^2 + X_{10}^3 + 2X_6^5 \end{pmatrix}$$

We encounter spin-off problems during the process.

Definition

Z_w is indecomposable $\Leftrightarrow Z_w \notin \langle Z_v \rangle_{l(v)<l(w)}$

- torsion index and decomposability

$$t(G) = \mathrm{min}\{t | t \cdot Z_w \in \mathrm{Im}(c), \forall w \in W\}$$

- combinatorics of Weyl group and decomposability
- Schubert polynomial for exceptional types
  (Schubert polynomials live in $H^*(BT;\mathbb{Z}) \otimes \mathbb{Z}[\gamma_i]$, where γ_i's correspond to indecomposable Schubert classes)
- Cohomology (Chow rings) of generalized flag varieties G/P

I hope further experimentation and visualization will lead to the solution.

Open problems

- Cohomology ring
  - Invariant ring of Weyl group $\mathbb{Z}[w_1, \ldots, w_l]^W$
  - Invariant ring of mod p Weyl group $\mathsf{F}_p[v_1, \ldots, v_m]^{W_p}$
  - Action of Steenrod operations on $H^*(BG; \mathsf{F}_p)$
- Homotopy groups
  - Handy free resolutions for algebras which arose as cohomology rings
  - Homology of the λ-algebra (E_2-term of Adams spectral sequence)
- How to handle, for example, algebras over Steenrod algebras ?
- or generally, algebras over operads ?
- How to deal with spectral sequences ?