1. Introduction

My research concerns computational aspects of algebraic topology. More specifically, I compute homotopy invariants such as the cohomology of spaces related to mapping spaces, as well as the homotopy nilpotency of Lie groups. A *mapping space* $\mathrm{Map}(X, Y)$ is a space of all continuous maps from a topological space X to another space Y, equipped with the compact-open topology. When X and Y are based spaces, i.e. there is a specified basepoint x_0 and y_0 in X and Y respectively, we can also consider the set of based maps $\mathrm{Map}^*(X, Y) \subseteq \mathrm{Map}(X, Y)$ which respects basepoints.

In some sense, the whole field of algebraic topology may be described as a careful analysis of mapping spaces by associating to them various algebraic invariants. For instance, the n-th homotopy group $\pi_n(Y)$ of homotopy classes of maps from the n-dimensional sphere S^n to Y is nothing other than the set of connected components of $\mathrm{Map}^*(S^n, Y)$, and the n-th complex K-group $K^n(X)$ is exactly $\pi_n\mathrm{Map}^*(X, BU \times Z)$, where here BU is the classifying space of the infinite unitary group. In this sense, mapping spaces encode a wealth of important information well worth study. Mapping spaces sometimes have product structures when, for example, the target spaces have products, or the source spaces have coproducts. In such a case, these product structures can be revealed through the cohomology rings of their classifying spaces or their commutator maps, and yields rich structures on the topology of mapping spaces.

My recent research has focussed on the various product structures associated to the following particular types of mapping spaces. First, I study gauge groups $\mathcal{G}$, which are groups of automorphisms of principal G-bundles for G a compact Lie group. The reason that these are related to mapping spaces is that classifying spaces of gauge groups can be homotopically regarded as a connected component of $\mathrm{Map}(M, BG)$, where M is the closed manifold which is the base of the principal G-bundle, and BG is the classifying space of G. Gauge groups are of importance in many areas of mathematics and, in particular, they offer a powerful method to study the topology of manifolds. For example, I compute in [K2] the cohomology of the classifying space of a gauge group over a 3-manifold; this work is described in detail in Section 2. Second, I study the free loop groups $LG := \mathrm{Map}(S^1, X)$ of compact Lie groups G; these are also spaces which arise as the simplest example of infinite dimensional Lie groups, and are studied from various points of view, including geometry, analysis, and string theory. As an example, in [K1] I determined the mod 2 cohomology of the classifying space BLG of a free loop group LG, and relate it to the mod 2 cohomology of certain finite Chevalley groups; this work is described in Section 3. Finally, I have also studied spaces of based loops $\Omega X := \mathrm{Map}^*(S^1, X)$. This class of examples includes Lie groups, so it is natural that the product structure on these spaces are of great interest. One of my computations [KK1] in this direction, joint with D. Kishimoto, is the homotopy nilpotency of p-compact groups for all but finitely many primes p; this result should be viewed as a relation between the cohomology of BG and the homotopy nilpotency of the based loop space $\Omega BG \cong G$ (see Section 3). In all of the above, my main technical tools are the classical methods of algebraic topology, such as spectral sequences, cohomology operations, and rational homotopy theory.

In a different direction, I have also recently begun to pursue a research program which can be described roughly as an algebraic topologist's approach to Schubert calculus. We first briefly recall the context. Let G be a simple complex Lie group, and B a Borel subgroup. As an example, in the case $G = GL_n(\mathbb{C})$, the subgroup of upper triangular matrices is a Borel subgroup. Then the homogeneous space G/B is a projective variety, often called the *flag variety*. In the case $G = GL_n(\mathbb{C})$, this is the classical flag manifold consisting of a sequence ("flag") of subspaces

$$0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_{n-1} \subseteq V_n = \mathbb{C}^n,$$

where each V_i is complex dimension i in $\mathbb{C}^n$. The explicit computation (and in particular, the derivation of explicit formulae for the structure constants) of the cohomology rings (or Chow rings, K-theory, or quantum cohomology) of these homogeneous spaces for different simple complex Lie groups G is a field known as *Schubert calculus*. These structure constants have a rich variety of interpretations, including as the number of intersection points of subvarieties from the point of view of algebraic geometry, or also as Littlewood-Richardson numbers arising in combinatorics. As such, Schubert calculus is currently actively studied from a number of different directions. From the field of algebraic topology, Borel developed a general point of view from which to compute the ordinary cohomology of these homogeneous spaces, and after the work of Borel, a number of algebraic topologists (Toda, Watanabe, and Nakagawa) have completed this program by giving explicit descriptions of $H^*(G/B;\mathbb{Z})$ for all simply connected, simple Lie groups G. However, this work in algebraic topology seems to be relatively little known outside of its own field. One of the reasons for this may be that the resulting description of the cohomology rings by generators and relations are heavily algebraic and have little evident connection to geometry. My recent work [KN] is a first step toward connecting these results to geometry by describing these algebraic-topological generators explicitly in terms of the more geometrical Schubert classes, which are the generators most commonly used by Schubert calculus practitioners; this is described in detail in Section 4. It is my long-term goal to bring the insights and tools of algebraic topology to answer new questions in Schubert calculus.

2. Homotopy theory of gauge groups

In this section, I explain how I study the homotopical properties of gauge groups. We now briefly recall the context of this work. Let G be a compact Lie group, and let P be a principal G-bundle over a closed manifold M, i.e. G acts freely on P with quotient space $M = P/G$. The *gauge group* $\mathcal{G}$ is then defined to be the group of principal bundle automorphisms of P. A fundamental result of Atiyah and Bott [AB] shows that the classifying space $B\mathcal{G}$ of such a gauge group is homotopy equivalent to $\mathrm{Map}_P(M, BG)$, where here $\mathrm{Map}_P(M, BG)$ denotes the connected component of $\mathrm{Map}(M, BG)$ which contains the classifying map of P. This result enables us to study $B\mathcal{G}$ using the standard machinery of algebraic topology, such as Serre spectral sequences, Eilenberg-Moore spectral sequences, and the commutator maps of Lie groups. Indeed, in both of the works described below, we need to use only the classical tools of algebraic topology. This indicates that in the study of the homotopy types of possibly complicated spaces, it is frequently of use to reformulate the problems in terms of mapping spaces.

I begin by describing some work in which I give computations for the cohomology rings of these $\mathrm{Map}_P(M, BG)$. In [K2], I study the cohomology of $\mathrm{Map}_P(M, BG)$ in the special case where M is a closed orientable 3-manifold and G is a simply-connected compact Lie group. Although P is always trivial in this case, the gauge group is far from being trivial and contains a lot of information about M. For example, it can be seen from this work that the homology of M can be recovered from the cohomology of the gauge group even in degree less or equal to 3. By approximating G using an infinite loop space, I obtained the triviality, in low dimensions, of the evaluation fibration $\mathrm{Map}^*(M, BG) \to \mathrm{Map}(M, BG) \to BG$, which is the most basic and powerful tool for mapping spaces in general. Since the top cell of M stably splits off, the Serre spectral sequence associated to the cofibration of the source space arising from the 2-skeleton of M gives a way to calculate the cohomology of $\mathrm{Map}^*(M, BG)$. Hence, in particular, I was able to determine this cohomology ring in this special case up to degree 3, thus slightly generalizing one of the results in [Ku] by K. Kuribayashi with a much simpler proof. In the future, I hope to compute the cohomology to higher degrees and also for the case when M is 4-dimensional manifold, since the space $\mathrm{Map}^*(M, BG)$ for a particular 4-dimensional manifold M has a deep relation to Bott periodicity.

Now suppose given a fixed manifold M and a compact Lie group G. When we consider all principal G-bundles over M, the number of homotopy types of the classifying spaces of the gauge groups of principal G-bundles over M is usually infinite. However, it is known [CS] that the number of homotopy types of the gauge groups themselves is finite. Hence there are usually infinitely many principal G-bundles P whose gauge groups are not isomorphic, but which nevertheless have the same homotopy type. Thus, even before considering the product structure of gauge groups, it is important to classify their homotopy types. We first recall that the homotopy type of a gauge group $\mathcal{G}$ depends on the non-commutativity of the Lie group G, since the homotopy type is unique if the structure group G is abelian. This follows from an analysis of the following evaluation fibration sequence:

$$\mathcal{G} \longrightarrow G \xrightarrow{\gamma} \mathrm{Map}^*_P(M, BG),$$

where here γ is a map closely related to the commutator $G \wedge G \to G$ of G.

In the special case when $M = S^4$ and $G = \mathrm{SU}(n)$, A. Kono and H. Hamanaka [HK] studied the condition for the gauge groups of two principal $\mathrm{SU}(n)$-bundles over S^4 to be homotopy equivalent. As a first step to consider this kind of problem for more complicated structure groups, A. Kono, H. Hamanaka, and I studied the homotopy types of gauge groups in the special case when $M = S^8$ and $G = \mathrm{Sp}(2)$ in [HKK]. In this case, the map γ above is the adjoint of the following compositions:

$$\mathrm{Sp}(2) \wedge S^8 \xrightarrow{1\wedge k} \mathrm{Sp}(2) \wedge \mathrm{Sp}(2) \xrightarrow{\gamma'} \mathrm{Sp}(2),$$

where here k is the classifying map of P, and γ' is the commutator of $\mathrm{Sp}(2)$. Analyzing the order of γ using the Puppe exact sequence associated to the fibration

$$\mathrm{Sp}(2) \longrightarrow \mathrm{Sp}(\infty) \longrightarrow \mathrm{Sp}(\infty)/\mathrm{Sp}(2)$$

allowed us to give necessary and sufficient conditions for the gauge groups of two principal $\mathrm{Sp}(2)$-bundles over S^8 to be homotopy equivalent. However, our current method only works when M is a sphere, so the next thing to do should be to extend our method to cover the cases when M is any manifold.

3. Product structures on loop spaces

As mentioned in the introduction, a space of free loops $LX = \mathrm{Map}(S^1, X)$ on a space X has rich structures, studied from many different points of view in a variety of mathematical fields. In this section I describe some of my work related to these spaces, and in particular related to their product structures.

In [K1], I compute the mod 2 cohomology rings over the Steenrod algebra of the classifying space BLG for a simply-connected compact Lie group G, under the assumption that the mod 2 cohomology of G is a polynomial ring. The essential point of my computation is that these mod 2 cohomology rings turn out to be related – indeed, isomorphic over the Steenrod algebra– to the mod 2 cohomology $H^*(G(q), \mathbb{Z}/2\mathbb{Z})$ of finite Chevalley groups $G(q)$ for an odd prime power $q = p^s$. My computation uses a technique developed by D. Kishimoto and A. Kono in [KK2], in which they consider the cohomology of the following pullback diagram:

$$\begin{tikzcd} L_\phi X \arrow[r] \arrow[d] & X^{[0,1]} \arrow[d, "e_0\times e_1"] \\ X \arrow[r, "1\times\phi"] & X \times X \end{tikzcd}$$

where here e_i for $i = 0, 1$ is evaluation at i and ϕ is any based self-map. The pullback $L_\phi X$ is called the *homotopy fixed point space* of ϕ and reduces to the free loop space of G when ϕ is the identity. On the other hand, when ϕ is the *unstable Adams operation* of order q, the mod 2 cohomology of $L_\phi BG$ is isomorphic to that of $G(q)$ by a result of Friedlander [F]. Using these techniques, I was able to prove the isomorphism claimed above. This result is suggestive of some deeper relation between these finite groups $G(q)$ and the infinite-dimensional groups LG (although there is no canonical map between them). I hope to explore these ideas further in future work.

A variant of the free loop space is the *based loop space* $\Omega X := \mathrm{Map}^*(S^1, X)$ over a space X. This class of spaces includes Lie groups, so we wish to analyze the product structure on such spaces. As an example, we consider the problem of determining the degree to which a based loop space is non-commutative; this is measured in part by a homotopy invariant called the *homotopy nilpotency*, which is a straightforward homotopy-theoretic analogue of the classical notion of the nilpotency of a group. As an initial step, we may first consider only the *rational homotopy* of the space, i.e. we ignore all the torsion information of the underlying space. In this setting, *rational homotopy theory* gives a powerful way of reducing homotopy-theoretic problems to an associated simple algebraic model, called the *Sullivan model*. Thus, questions such as those related to the homotopy nilpotency of ΩX can then be described and solved in terms of the algebra of the Sullivan model.

In [K3], using these methods, I study the rational homotopy type of the based loop space ΩX. More specifically, I consider an algebraic invariant associated to the Sullivan model for X, and show that it is equal to the homotopy nilpotency of the rationalization of ΩX. I accomplish this by analyzing a Sullivan model for the canonical fiberwise action (given by composition of paths) of ΩX on the path-loop fibration $\Omega X \times PX \to X$, where PX is *the path space* $\mathrm{Map}^*([0,1], X)$ and whose action on the fiber is just the multiplication in ΩX. As a corollary, I also obtained a different proof of the well-known fact (see, for example, [S]) that the homotopy nilpotency of a rational H-space is equal to the ordinary nilpotency of the Lie algebra of its homotopy group.

Although rational homotopy theory and the Sullivan model are powerful techniques, one of course also wants to obtain information about the torsion of the underlying space. In this direction, in [KK1], D. Kishimoto and I determine the homotopy nilpotency of compact Lie groups localized at p (and more generally, of p-compact groups) for all but finitely many primes p. To do this, we use a relation between the cohomology operations on the classifying space and the nontriviality of the commutator map in the underlying Lie group. As mentioned in the introduction, this should be viewed as a relation between the cohomology of BG and the homotopy nilpotency of $\Omega BG \simeq G$. I should mention that in [KK1], we were forced to resort to the classification theorem of compact Lie groups to do a case-by-case analysis. However, in future work, I intend to find a description of this homotopy nilpotency in terms of a general algebraic model, just as in the rational homotopy case mentioned above.

4. An algebraic-topological approach to Schubert calculus

In this section, I describe some of my recent work in the field of Schubert calculus. Since this is a relatively new direction in my research, I also take the opportunity to describe in some detail my intentions for future work.

Schubert calculus is the study of the homogeneous spaces G/B for G a simple complex Lie group and B a Borel subgroup. From an algebraic topological viewpoint, the space G/B may also be regarded as K/T, where K is a maximal compact subgroup of G and T is its maximal torus. In the case $G = GL_n(\mathbb{C})$, we have $K = U(n, \mathbb{C})$ the unitary group and T is the subgroup of diagonal matrices. The series of computations of $H^*(G/B; \mathbb{Z})$ initiated by Borel and recently completed for all simple, simply-connected complex Lie groups G, remain relatively unknown to the mathematical community outside of the field of algebraic topology, mainly due to the fact that the computations are heavily algebraic, and the resulting ring generators have no evident connection to the natural underlying geometry of G/B as exhibited by, for instance, the *Schubert varieties*, which are the closure of BwB/B of G/B (for w elements of the Weyl group). Hence it is of importance to translate these algebraic-topological results in a form which can be interpreted in a more geometric context and can therefore be used by a wider community of mathematicians.

As a first step in this direction, M. Nakagawa and I calculate [KN] the integral cohomology of G/B by giving explicit ring generators represented by *Schubert classes*, which are the cohomology classes corresponding to the Schubert varieties mentioned above. As a consequence, we also determine the Chow ring $A(G)$ for G is a simply connected Lie group of exceptional type E_6, E_7, or E_8. This last result is itself a completion of a series of computations of the Chow rings of the simply-connected simple complex algebraic groups initiated by Grothendieck and Chevalley in the 1950s.

Nakagawa and I are currently working on more general cases of our computations in [KN], and we plan to give similar computations for more general homogeneous spaces G/P, where now P is any parabolic subgroup of G. Moreover, the computations given in [KN] are in some sense ad hoc, so we are working towards giving a more uniform algorithm for our computations and also provide an actual implementation of our methods as a computer program. This should aid in making the algebraic-topological methods more readily available to mathematicians from other fields.

Finally, I mention that in the course of our computations, Nakagawa and I naturally came across some problems which can be phrases purely in terms of Schubert calculus, but to which we may apply algebraic-topological techniques. For instance, we wish to characterize those Schubert classes which are *indecomposable*, i.e. cannot be obtained as polynomials of Schubert classes of lower degrees. Similarly, we would like to give candidates for Schubert polynomials for the exceptional Lie types. We believe that these are questions which are of interest from a general Schubert calculus point of view, and that we can contribute to the field by bringing to them the powerful tools of algebraic topology.

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