Schubert calculus for G-manifolds

Shizuo KAJI
Yamaguchi Univ. Japan

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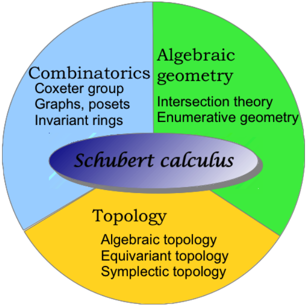
Motivation



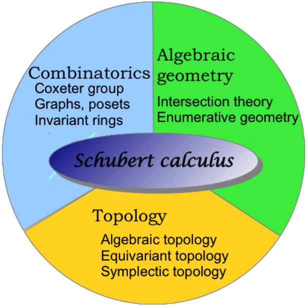
Cohomology of G/T has rich structure, which bridges between combinatorics, topology, and geometry.

The study has attracted mathematicians from different fields and is now called *Schubert calculus*.

Question

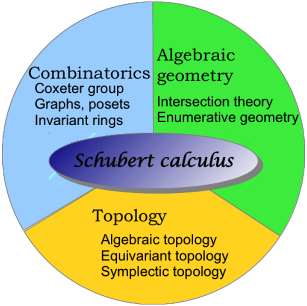
How can we generalize Schubert calculus ?
What is the key to Schubert calculus ?

Different directions of generalizations



- From algebraic geometry: Intersection theory on moduli spaces
- From symplectic topology: Schubert calculus on manifolds with Hamiltonian T-action
- From combinatorics: Schubert calculus for reflection groups
- We turn our attention to the following properties of flag variety:
  - having G-action and
  - being GKM-manifold

Different directions of generalizations



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  - being GKM-manifold

Notation

notation	typical example
G: simple, 1-connected, compact Lie group	$SU(n+1)$
T: maximal torus	diagonal matirices
G/T: flag variety	$Fl_{n+1} = \{0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_{n+1} = \mathbb{C}^{n+1}\}$
E: universal G-bundle	$G * G * \cdots$
(simultaneously, the universal T-bundle)	
$W = N(T)/T$: Weyl group of G	Σ_{n+1}: symmetric group
$s_\beta \in W$: reflection by a root β	
$W \curvearrowright \mathfrak{t}^*$: dual Lie algebra of T	$\mathbb{R}^{n+1} = \langle t_1, \ldots, t_{n+1} \rangle$

$H^*(X) = H^*(X; \mathbb{Q})$: cohomology with the rational coefficients

What is Schubert calculus ?

Schubert calculus

- G/T has a cell decomposition $G/T = \bigcup_{w \in W} \sigma_w$, indexed by the Weyl group W
- It is consistent with the combinatorics on W:
  - $w \leq v$ by the Bruhat order on $W \Leftrightarrow \overline{\sigma_w} \supset \sigma_v$
  - $\dim \sigma_w = 2l(w)$, two times the *length* of $w \in W$
- $\{\sigma_w \mid w \in W\}$ form a free basis for $H^*(G/T)$ and is called the *Schubert classes*

$$H^*(G/T) = \bigoplus_{w \in W} \mathbb{Q}\langle \sigma_w \rangle$$

- The main problem is the computation of *structure constants*

$$\sigma_u \cdot \sigma_v = \sum_{w \in W} c_{uv}^w \sigma_w, \quad c_{uv}^w \in \mathbb{Z}$$

Structure constants

The structure constants are interesting because they are

1. related to enumerative geometry:
   "How many lines are there in the three space which meet all the four given lines ? "
2. related to representation theory:
   tensor multiplicity of irreducible representations
3. related to combinatorics:
   product of Schur and Schubert polynomials
4. related to algebraic topology:
   cup product of Morse cells

Equivariant Schubert calculus

A flag variety G/T admits the left T-action by multiplication.
The action is good because

- G/T is a *GKM-manifold* with respect to this action
- The T-fixed points set corresponds to the Weyl group W
- Schubert classes $\{\sigma_w \mid w \in W\}$ form a $H^*(BT)$-basis for $H^*_T(G/T)$
- σ_w localized at fixed points is *upper-triangular*, which gives a way to compute $c_{uv}^w \in H^*(BT)$
- the action extends to a G-action
- W acts on $H^*_T(G/T)$ and it defines a family of operators on $H^*_T(G/T)$ called the *Divided difference operators*
- Divided difference operators pose a hierarchy on the Schubert classes

Hence, we consider $H^*_T(G/T)$ rather than $H^*(G/T)$,
and try to generalize above properties to *GKM G-manifolds*

Equivariant cohomology and GKM-theory

Definition of the equivariant cohomology

When G acts on a manifold M,
we can define the *G-equivariant cohomology* of M:

- $M \hookrightarrow E \times_G M \to BG$: Borel construction
- $H_G^*(M) := H^*(E \times_G M)$: the G-equivariant cohomology

It captures the G-equivariant topology of M:

- Ex. if G acts freely on M, then $H_G^*(M) = H^*(M/G)$
- Ex. if G acts trivially on M, then $H_G^*(M) = H^*(BG) \otimes H^*(M)$
- In particular, $H_G^*(pt) = H^*(BG) \cong \mathbb{R}[t_1, \ldots, t_n]^W$
- Hence $H_G^*(M)$ is an algebra over $H_G^*(pt)$

If $H_G^*(M)$ is free over $H_G^*(pt)$, the ordinary cohomology can be recovered

$$H^*(M) = \mathbb{Q} \otimes_{H_G^*(pt)} H_G^*(M)$$

GKM (Goresky-Kottwitz-MacPherson) manifold

Let M be a compact manifold with an effective T-action

Definition (Goresky-Kottwitz-MacPherson)

M is GKM $\Leftrightarrow$

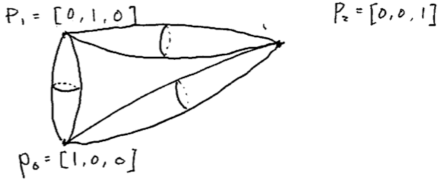
- M^T and 1-dim orbits form a graph
- Serre-SS for $M \hookrightarrow E \times_T M \to BT$ degenerates at E_2 (said to be *equivariantly formal*)
- The weights of the isotropy representation on $T_p(M)$ for all $p \in M^T$ are pairwise linearly independent

Example

- Flag varieties
- Toric manifolds
- Torus manifolds with $H^{odd} = 0$

Figure of GKM manifold

$T^2 = \{((t_1 t_2)^{-1}, t_1, t_2)\} \curvearrowright \mathbb{C}P^2$ by multiplication
$M^T = \{p_0 = [1,0,0], p_1 = [0,1,0], p_2 = [0,0,1]\}$
1-skeleton of this GKM-manifold looks like:



- $H^*(BT) = \dfrac{\mathbb{Q}[t_0, t_1, t_2]}{(t_0 + t_1 + t_2)}$
- $T_{p_0}(M) = \mathbb{C}_{t_0 - t_1} \oplus \mathbb{C}_{t_0 - t_2}$
- $T_{p_1}(M) = \mathbb{C}_{t_1 - t_0} \oplus \mathbb{C}_{t_1 - t_2}$
- $T_{p_2}(M) = \mathbb{C}_{t_2 - t_1} \oplus \mathbb{C}_{t_2 - t_0}$

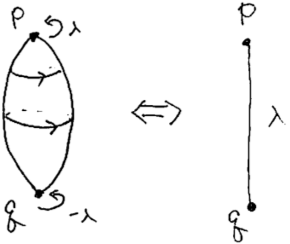
GKM graph

The T-equivariant cohomology of a GKM-manifold has a nice combinatorial description in terms of the *GKM graph*.

The GKM graph $\mathcal{G}$ consists of

- T-fixed points M^T as *vertices*
- 1-dim orbits as *edges*
- weight $\lambda_{p,q} \in H^2(BT)$ of the 1-dim rep on T_pM as *labels*

The GKM graph of $\mathbb{C}P^1 = S^2$ with λ-times the standard T^1-action



$H_T^*(M)$: GKM description

The inclusion of fixed points

$$M^T \xrightarrow{i} M$$

induces an injection on cohomology called *the localization map*

$$H_T^*(M) \xrightarrow{i^*} H_T^*\left(M^T\right) \cong \bigoplus_{p \in M^T} H_T^*(pt)$$

The image of i^* is determined by:

Theorem (Goresky-Kottwitz-MacPherson)

$$H_T^*(M) \cong \mathrm{Im}(i^*) \cong H^*(\mathcal{G}) \subset \bigoplus_{p \in M^T} H^*(BT)$$

$$H^*(\mathcal{G}) \cong \{ \bigoplus_{p \in M^T} h_p(t) \mid h_p(t) \in H^*(BT), \quad \lambda_{p,q} | h_p(t) - h_q(t) \text{ if } p \xrightarrow{\lambda_{p,q}} q \}$$

In the cases when T-action extends to G-action

GKM G-manifold

A flag variety G/T, considered as the right quotient, has the left G-action by the multiplication. We generalize this situation.

Definition

If a Lie group G acts on a manifold M in such a way that M satisfies the GKM condition with respect to the action restricted to T, we say that M is a GKM G-manifold.

Example

- Homogeneous spaces G/H with the multiplication action of K $(H \subset K \subset G)$
- Kuroki determined when a quasitoric manifold is a GKM G-manifold

Schubert calculus for GKM G-manifold

Goal

For a GKM G-manifold M:

1. Find a good description for $H_T^*(M)$ and $H_G^*(M)$
2. Define a nice $H^*(BT)$-basis for $H_T^*(M)$

Note: There is a extreme case when $G = T$.
So we need some data other than the action of G.

Our strategy is to divided $H_T^*(M)$ into two parts:

G-orbits and *G-equivariant* parts.

$H_T^*(M)$: Borel description

In addition to GKM description for $H_T^*(M)$,
we have another description thanks to the G-action:

Proposition

$H_T^*(M) = H^*(BT) \otimes_{H^*(BG)} H_G^*(M)$

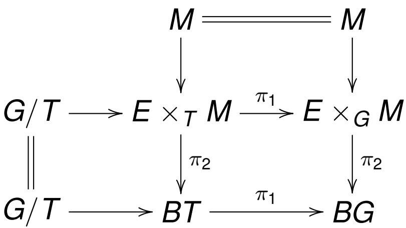
Note: as a special case of flag variety G/T,

$$H_T^*(G/T) = H^*(BT) \otimes_{H^*(BG)} H^*(BT), \quad \left(H^*(BG) = H^*(BT)^W\right)$$
$$= \frac{\mathbb{Q}[t_1, \dots, t_n, z_1, \dots, z_n]}{(e_i(t) - e_i(z))}$$

(*double coinvariant ring of W*)

W-action on $H_T^*(M)$

The Weyl group W acts on the following diagram



$\pi_1(e, m) = (e, m)$
$\pi_2(e, m) = e$
on BT: $w(e) = ew^{-1}$
on $E \times_T M$:
$w(e, p) = (ew^{-1}, wp)$
(Note: the action becomes trivial on $E \times_G M$)

which induces W-actions on cohomology:

- on $H_G^*(M)$: trivial
- on $H^*(BT) = \mathbb{Q}[t_1, \ldots, t_{n+1}]$: $w(f(t)) = f(w^{-1}(t))$
- on $H_T^*(M) = H^*(BT) \otimes_{H^*(BG)} H_G^*(M)$: $w(f(t) \otimes z) = f(w^{-1}(t)) \otimes z$

and $H_G^*(M) = H_T^*(M)^W$

Localization and W-action

Localization at W-orbit points are determined by that at a single point:

Lemma

$$i_{wp}^* = w^{-1} \circ i_p^* \circ w : H_T^*(M) \to H^*(BT)$$

Proof.

$$\begin{array}{ccc} BT & \xrightarrow{i_p} & E \times M \\ \downarrow w & & \downarrow w \\ BT & \xrightarrow{i_{wp}} & E \times M, \end{array}$$

$i_p(e) = (e, p), w(e, p) = (ew^{-1}, wp)$ □

Localization and W-action

The previous Lemma gives the relation between Borel and GKM descriptions:

Corollary

For $f(t) \otimes z \in H_T^*(M) = H^*(BT) \otimes_{H^*(BG)} H_G^*(M)$,

$$i_{wp}^*(f(t) \otimes z) = f(t) \cdot w^{-1}(i_p^*(z)) \in H^*(BT)$$

and for $h \in H^*(\mathcal{G})$

$$w(h)_p(t) = h_{wp}(w^{-1}t)$$

This recovers the well-known "specialization formula" for the double Schubert polynomial.

GKM-theory with W-symmetry

Weyl group action on GKM graph

Consider the GKM graph $\mathcal{G}$ of a GKM G-manifold M.

It has the following W-symmetry:

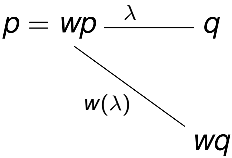
1. There must be an edge $p \overset{\beta}{\text{———}} s_\beta p$
2. If there is an edge

$$p \overset{\lambda}{\text{—————}} q \,,$$

   then so is

$$wp \overset{w(\lambda)}{\text{—————}} wq$$

Note: $W \curvearrowright \mathcal{G}$ is not always free:



Quotient GKM graph

The quotient graph $\mathcal{G}/W$ is defined to be the directed graph with

- M^T/W as vertices: we pick a set of representatives $\{p_1, \ldots, p_N\}$
- $p_i \xrightarrow{(w,\lambda)} p_j \overset{\text{def}}{\iff} p_i \xrightarrow{\lambda} wp_j$ for some $w \in W/W_i$, where W_i is the Weyl group of the isotropy group

$$P_i = \{g \in G \mid gp_i = p_i\}$$

Theorem

$H_G^*(M) \cong H^*(\mathcal{G}/W) \subset \bigoplus_{p_i \in M^T/W} H^*(BT)$

$$H^*(\mathcal{G}/W) \cong \{ \bigoplus_{p_i \in M^T/W} h_{p_i}(t) \mid h_{p_i}(t) \in H^*(BT)^{W_i},$$

$$\lambda | h_{p_i}(t) - w^{-1}(h_{p_j}(t)) \text{ if } p_i \xrightarrow{(w,\lambda)} p_j \}$$

Constructing a Schubert-like basis for $H_T^*(M)$

Schubert class

Now, we consider to define a $H^*(BT)$-basis for $H_T^*(M)$, resembling the Schubert classes for flag varieties.

Facts on Schubert classes

- indexed by T-fixed points
- Upper-triangular: $i_v^*(\sigma_w) = 0$ unless $v \geq w$
- Fits in a hierarchy: $\sigma_w = \partial_{wv^{-1}}\sigma_v$ if $v \geq w$

Upper-triangular class

In our case, $\sum_{i\geq 0} \operatorname{rank}_{H^i(BT)} H_T^*(M) = \chi(M) = \#M^T$.
Hence we may index a basis by $\#M^T$.

Definition

A family of classes $\{\rho_p \mid p \in M^T\}$ of $H^*(\mathcal{G})$ is called *upper-triangular* if

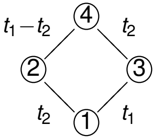
1. ρ_p is not a member of the ideal $(H^+(BT))$ and
2. there exists a total ordering on M^T such that

$$i_q^*(\rho_p) \begin{cases} = 0 & (q < p) \\ \neq 0 & (p = q) \end{cases}$$

- Upper-triangular classes form a $H^*(BT)$-basis of $H_T^*(M)$
- It is *not* known when they exist

Example: computation with upper-triangular classes

$T^2 \curvearrowright M = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, whose GKM-graph is



$$H_T^*(M) = \mathbb{Q}\left\langle \begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 1 & \end{array},\ \begin{array}{ccc} & t_1+t_2 & \\ 2t_2 & & t_1 \\ & 0 & \end{array},\ \begin{array}{ccc} & t_1-t_2 & \\ 0 & & t_1 \\ & 0 & \end{array},\ \begin{array}{ccc} & t_2(t_1-t_2) & \\ 0 & & 0 \\ & 0 & \end{array} \right\rangle$$

$$\left(\begin{array}{ccc} & t_1-t_2 & \\ 0 & & t_1 \\ & 0 & \end{array}\right)^2 = \begin{array}{ccc} & (t_1-t_2)^2 & \\ 0 & & t_1^2 \\ & 0 & \end{array} = t_1\left(\begin{array}{ccc} & t_1-t_2 & \\ 0 & & t_1 \\ & 0 & \end{array}\right) - \begin{array}{ccc} & t_2(t_1-t_2) & \\ 0 & & 0 \\ & 0 & \end{array}$$

Strategy for finding an upper-triangular basis

- Our basis $\{\rho_j\}$ should be indexed by M^T
- M^T is divided into W-orbits $M^T = \bigcup_{i=1}^N Wp_i$
- Each orbit Wp_i corresponds to $H_T^*(G/P_i)$
- The "center points" p_i's are vertices of $\mathcal{G}/W$

1. An upper-triangular basis for $H_T^*(G/P_i)$ can be obtained by using the divided difference operators
2. Classes for p_i's might correspond to classes in $H^*(\mathcal{G}/W) = H_G^*(M)$ $\Rightarrow$ we just assume there is an upper-triangular basis for $H^*(\mathcal{G}/W)$
3. Taking the product of those two, we obtain an upper-triangular basis for $H_T^*(M)$

Basis for $H_T^*(G/P_i)$: Orthogonal class

The G-orbit of $p_i \in M^T$ is G/P_i, (whose T-fixed points are $W/W(P_i)$).

Definition

$o_i \in H_T^{\dim(G/P_i)}(M)$ is *orthogonal* to G/P_i at p_i

$$\Leftrightarrow o_i \cup [G/P_i]^* = [p_i]^*$$

I don't know when such classes exist.
However, recall that

Fact

Schubert classes $\sigma_w, \sigma_{w_0 w} \in H_T^*(G/T)$ are orthogonal, i.e.,

$$\sigma_w \cup \sigma_{w_0 w} = [w]^*$$

In fact, they are the stable and unstable manifolds of the standard Morse function.

Basis for $H_T^*(G/P_i)$: Divided difference for G/T

For a root β, the divided difference operators were defined as a geometric counter part of the partial order on W.

left divided difference: $\partial_\beta : H_T^*(G/T) \to H_T^{*-2}(G/T)$
right divided difference: $\Delta_\beta : H^*(G/T) \to H^{*-2}(G/T)$

They build bridges between combinatorics of W and topology of G/T and have been a crucial tool for Schubert calculus.

For general G-manifold, we construct a generalization of the left one

$$\partial_\beta : H_T^*(M) \to H_T^{*-2}(M)$$

Basis for $H_T^*(G/P_i)$: Divided difference for G-manifold

Let $P_\beta \subset G$ be the minimal parabolic corresponding to a root β, (so that $P_\beta/T \cong \mathbb{C}P^1$)
Consider the fiber bundle

$$P_\beta/T \hookrightarrow E \times_T P_\beta \times_T M \xrightarrow{\pi} E \times_T M,$$

where $\pi(e, g, m) = (eg, m)$.

Definition

$$\partial_{s_\beta}(c) = \pi_1^*(c) \backslash [P_\beta/T]$$

Note: it is defined topologically

Basis for $H_T^*(G/P_i)$: Divided difference for G-manifold

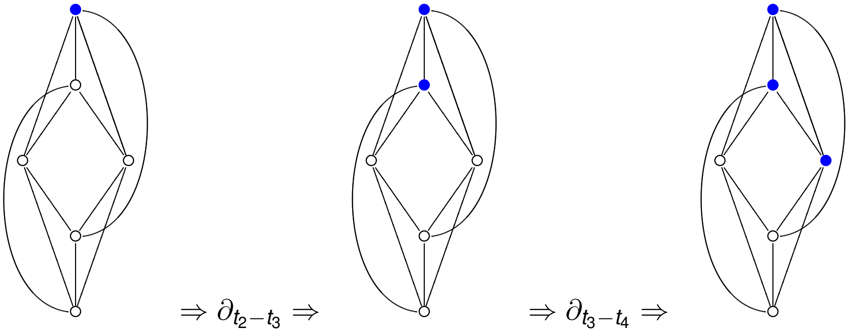
Proposition

- on GKM-description: $\partial_\beta(h)_p = \dfrac{h_p - s_\beta(h_{s_\beta p})}{\beta}$
- on Borel-description: $\partial_\beta(f(t) \otimes z) = \dfrac{f(t) - f(s_\beta(t))}{\beta} \otimes z$
- ∂_β is derivation: $\partial_\beta(fg) = \partial_\beta(f)g + s_\beta(f)\partial_\beta(g)$

- For a flag variety, it coincides with the left divided difference operator
- They are computable both Borel and GKM descriptions.
  In particular, the support of the class is controlled.
- Applied to the orthogonal class, they produce an upper-triangular basis of $H_T^*(G/P_i)$.

Example: GKM-description of ∂_β

$T^4 \curvearrowright Gr_2(\mathbb{C}^4) = U(4)/U(2) \times U(2)$
∂_β distributes the support of a class



black circle: the class vanish at the point
blue circles: the class doesn't vanish at the point

A module basis for $H_T^*(M)$

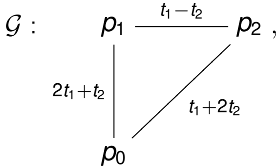
Theorem

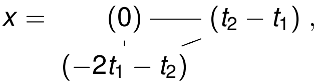
Let $\{u_i \mid i = 1 \dots N\}$ be an upper-triangular basis for $H^*(\mathcal{G}/W)$ and $\{o_i \mid i = 1 \dots N\}$ be a set of orthogonal classes at M^T.
Then $\{\rho_{i,w} = u_i \cdot \partial_w(o_i) \mid i = 1 \dots N, w \in W/W_i\}$ form an upper-triangular basis for $H_T^*(M)$.

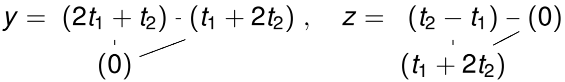
- We pose the lex order on $M^T = V(\mathcal{G}/W) \times \{(G/P_i)^T\}$
- $\mathrm{supp}(u_i \cdot \partial_w(o_i)) = \mathrm{supp}(u_i) \cap \mathrm{supp}(\partial_w(o_i))$
- For $M = G/T$, this is the ordinary Schubert basis
- $\partial_\beta(u_i \cdot o_i) = u_i \cdot \partial_\beta(o_i)$

Example

$T^2 = \{((t_1 t_2)^{-1}, t_1, t_2)\} \subset G = S(U(2) \times U(1)) \curvearrowright \mathbb{C}P^2$ by multiplication
$M^T = \{p_0 = [1,0,0], p_1 = [0,1,0], p_2 = [0,0,1]\}$







$$\Rightarrow H_T^*(M) = H^*(\mathcal{G}) = \frac{\mathbb{Q}[x,y,z]}{(xyz)}$$

Example

W-action on $H_T^*(M)$ is summarized as:
$W = \langle s = s_{t_0 - t_1} \rangle, \quad s(t_1) = -t_1 - t_2, \quad s(t_2) = t_2, \quad s(x) = y, s(z) = z$
The quotient graph is

$$\mathcal{G}/W: \; p_1 \xrightarrow{\;\;t_1 - t_2\;\;} p_2$$

Take classes for $H^*(\mathcal{G}/W)$ as
$\overline{x+y} = \;(2t_1 + t_2) - (3t_2)$,

$\overline{xy} = \;(0) - (t_2 - t_1)(t_1 + 2t_2)$, $\overline{z} = \;(t_2 - t_1) - (0)$,

$$\Rightarrow H_G^*(M) = H^*(\mathcal{G}/W) = \frac{\mathbb{Q}[\overline{x+y}, \overline{xy}, \overline{z}]}{(xyz)}$$

Example

$$\mathcal{G}/W: \; p_1 \xrightarrow{\;t_1 - t_2\;} p_2$$

The orthogonal class at p_1 is $y = \begin{array}{ccc} (2t_1+t_2) & - & (t_1+2t_2) \\ | & \diagup & \\ (0) & & \end{array}$

The orthogonal class at p_2 is $1 = \begin{array}{ccc} (1) & - & (1) \\ | & \diagup & \\ (1) & & \end{array}$

u.t. basis for $H^*(\mathcal{G}/W)$: $\bar{z} = (t_2 - t_1) \text{———} (0)$, $1 = (1) \text{———} (1)$

$$s(y) = \begin{array}{ccc} (2t_1+t_2) & - & (2t_1+t_2) \\ | & \diagup & \\ (2t_1+t_2) & & \end{array}, \partial_s y = 1$$

$$\Rightarrow H_T^*(M) = H^*(BT)\langle 1, z\partial_s y, zy \rangle$$

Future work

- Find interesting examples
- Consider with the integral coefficients (or give geometric treatment)
- Determine when an upper triangular basis exists
- Determine when the orthogonal classes exist
- Fix their indeterminancy (hopefully, by geometric meaning)
- Classify GKM G-manifolds
- Construct an "inverse" to GKM-graph: $\mathcal{G} \to M_{\mathcal{G}}$
  There are some hints:
  1. GKM for a quasi-toric manifold is the 1-skeleton of quotient polytope $P = M/T$,
     from which we can recover P by the fundamental theorem for simple polytopes
  2. GKM graph can be considered as an enriched category whose morphisms are $\mathbb{C}P^1$'s,
     hence we might apply a variant of nerve construction

Thank you very much

Большое спасибо