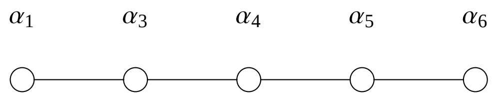
AN ALGEBRAIC TOPOLOGICAL APPROACH TOWARD CONCRETE SCHUBERT CALCULUS OF PROJECTIVE HOMOGENEOUS VARIETIES

SHIZUO KAJI AND MASAKI NAKAGAWA

1. Example: $H^*(E_6/P_2;\mathbb{Z})$

The Dynkin diagram of P_2 is



The semisimple part of P_2 is $\mathrm{SL}_6(\mathbb{C})$, therefore,

$$E_6/P_2 \simeq \mathrm{E}_6/\left(\mathrm{SU}(6)\cdot\mathrm{SU}(2)\right).$$

Theorem (Ishitoya(**Ish**), **KN2**).

$$H^*(E_6/P_2;\mathbb{Z}) = \mathbb{Z}[t,u,v,w]/(\rho_6,\rho_8,\rho_9,\rho_{12}),$$

where

$$t=\sigma_2, u=\sigma_{542}, v=\sigma_{6542}+\sigma_{3542}+\sigma_{1342}, w=\sigma_{136542}.$$

$$\rho_6 = 2t^6 - t^3u - 3t^2v + u^2 + 2w, \rho_8 = t^8 + 3t^2w - 3v^2, \rho_9 = -t^3w + 2uw, \rho_{12} = -t^6w + 15t^4v^2 + 15t^2vw - 26v^3 + 3w^2.$$

2. References

General references:

Ful: W.Fulton, *Intersection theory*, volume 2 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, second edition, **1998**.
Bri: M.Brion, *Equivariant cohomology and equivariant intersection theory*, arXiv:math.AG/9802063v1.
Bou: N. Bourbaki, *Groupes et Algèbre de Lie* IV – VI, Masson, Paris, 1968.
KL: S. Kleiman and D. Laksov, *Schubert calculus*, Amer. Math. Monthly **79** (1972), 1061–1082.
MT: 戸田宏, 三村護, リー群の位相 (上)(下), 紀伊國屋数学叢書. 紀伊國屋書店, 1978.
Oka: 岡田聡一, 古典群の表現論と組合せ論 (上)(下), 数理物理シリーズ３, 培風館, 2006.

Schubert polynomial of various types:

Mac: I. G. Macdonald, Schubert polynomials Surveys in combinatorics, 1991 (Guildford, 1991), 73–99, London Math. Soc. Lecture Note Ser. **166**, Cambridge Univ. Press, Cambridge, 1991.
BGG: I.N. Bernstein, I.M. Gelfand and S.I. Gelfand, *Schubert cells and the cohomology of the spaces G/P*, L.M.S. Lecture Notes vol.69 Cambridge Univ. Press, 1982, 115-140.
Dem: M. Demazure, *Invariants symmétriques des groupes de Weyl et torsion*, Invent. Math. **21** (1973), 287-301.
LS: A. Lascoux and M. Schützenberger, *Polynômes de Schubert*, C. R. Acad. Sci. Paris Sér. I Math. **294** (1982), no. 13, 447–450.

Date: 2008 年 6 月 20 日 九州大学 トポロジー金曜セミナー.

BH: S. Billey and M. Haiman, *Schubert polynomials for the classical groups*, J. Amer. Math. Soc. **8** (1995), no. 2, 443–482.

FM: S. Fomin and A.N. Kirillov, *Combinatorial B_n-analogues of Schubert polynomials*, Trans. Amer. Math. Soc. **348**(1996), no. 9, 3591–3620.

Algebraic topology(Borel presentation):

Bor: A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math. **57** (1953), 115-207.

BS: R. Bott and H. Samelson, *Application of the theory of Morse to the symmetric spaces*, Amer. J. Math. **80** (1958), 964-1029.

Nak1: M. Nakagawa, *The integral cohomology ring of E_7/T*, J. Math. Kyoto Univ. **41** (2001), 303-321.

Nak2: M. Nakagawa, *The integral cohomology ring of E_8/T*, preprint.

Tod: H. Toda, *On the cohomology ring of some homogeneous spaces*, J. Math. Kyoto Univ. **15** (1975), 185-199.

TW: H. Toda and T. Watanabe, *The integral cohomology ring of F_4/T and E_6/T*, J. Math. Kyoto Univ. **14** (1974), 257-286.

Ish: K. Ishitoya, *Integral cohomology ring of the symmetric space EII*, J. Math. Kyoto Univ. **17** (1977), 375–397.

IT: K. Ishitoya and H. Toda, *On the cohomology of irreducible symmetric spaces of exceptional type*, J. Math. Kyoto Univ. **17** (1977), 225–243.

Wat1: T. Watanabe, *Cohomology of a homogeneous space $E_6/T^1 \cdot SU(6)$*, Group representations: cohomology, group actions and topology, 511–518, Proc. Sympos. Pure Math., **63**, Amer. Math. Soc., Providence, RI, 1998.

Wat2: T. Watanabe, *The integral cohomology ring of the symmetric space EV II*, J. Math. Kyoto Univ. **15** (1975), 363–385.

Nak3: M. Nakagawa, *The integral cohomology ring of $E_8/T^1 \cdot E_7$*, preprint.

Chow rings of projective homogeneous varieties:

DZ1: H. Duan and X. Zhao, *The Chow rings of generalized Grassmannians*, arXiv:math.AG/0511332v5

Kac: V. Kac, *Torsion in cohomology of compact Lie groups and Chow rings of reductive algebraic groups*, Invent. Math. **80** (1985), 69-79.

KN1: S. Kaji and M. Nakagawa, *The Chow rings of the algebraic groups E_6 and E_7*, arXiv:math.AT/07093702

KN2: S. Kaji and M. Nakagawa, *Multiplicative Structure of the Chow rings of projective homogeneous varieties*, preprint

Mar: R. Marlin, *Anneaux de Chow des groupes algebriques* $\mathrm{SO}(n), \mathrm{Spin}(n), \mathrm{G}_2$ *et* F_4, Publications Math. d'Orsay, 95-7419 (1974).

Nak4: M. Nakagawa, *A description based on Schubert classes of cohomology of flag manifolds*, arXiv:math.AT/07090785v1.

IM: A. Iliev and L. Manivel, *The Chow ring of the Cayley planes*, Compositio Math. **141** (2005), 146–160.

CMP: P.E. Chaput, L. Manivel, and N. Perrin, *Quantum cohomology of minuscule homogeneous spaces*, arXiv:math.AG/0607492.

NS: S. Nikolenko and N. Semenov, *Chow ring structure made simple*, arXiv:math.AG/0606335.

Department of Applied Mathematics
Faculty of Sciences
Fukuoka University
Fukuoka 814-0180, Japan
E-mail address: kaji@math.sci.fukuoka-u.ac.jp

Department of General Education
Takamatsu National College of Technology
Takamatsu 761-8058, Japan
E-mail address: nakagawa@takamatsu-nct.ac.jp