An algebraic topological approach toward concrete Schubert calculus

Shizuo Kaji
joint with
Masaki Nakagawa

The 2nd East Asia Conference on Algebraic Topology
at National University of Singapore
Dec. 19, 2008

Outline

- Introduction
- Cohomology of flag variety
- Our results
- Future Work

Notations

- G: connected compact Lie group
- T: maximal torus of G
- $l = \dim T$: rank of G
- $\mathfrak{g}, \mathfrak{t}$: Lie algebras of G and T
- for $X \in \mathfrak{t}$, $G/P = \{Ad(g)X \mid g \in G\} \subset \mathfrak{g}$: (generalized) flag variety of type G
  - $P = \{g \in G \mid Ad(g)X = X\}$
  - G/P is a projective variety
- W, W_P: Weyl groups of G and P
  - $\alpha_1, \ldots, \alpha_l$: simple roots of G
  - $s_1, \ldots, s_l$: simple reflections corresponding to simple roots
  - W is the finite group generated by $s_1, \ldots, s_l$
  - $l(w)$ is the length of $w \in W$
- $\omega_1, \ldots, \omega_l$: fundamental weights of G
  - $H^*(BT; \mathbb{Z}) \cong \mathbb{Z}[\omega_1, \ldots, \omega_l]$

Examples

We can assume that G is simple, 1-connected without losing any generality.

- $G = SU(n)$, $T = \mathrm{diag}(e^{it_1}, \ldots, e^{it_n})$, $\sum t_i = 0$
- $W = S_n$: n-th symmetric group
- $s_i = (i, i+1)$: simple transposition
- take $X \in \mathfrak{t}$ as regular point ($\{i \mid s_i X = X\} = \emptyset$)
  - $P = T$
  - $W_P = *$
  - G/P is the ordinary flag manifold $SU(n)/T$: the space of *flags*, $0 \subseteq V^1 \subseteq V^2 \subseteq \cdots \subseteq V^{n-1} \subseteq V^n = \mathbb{C}^n$
- take $X \in \mathfrak{t}$ with $\{i \mid s_i X \neq X\} = \{m\}$
  - $P_m = SU(m) \times SU(n-m)$
  - $W_P = S_m \times S_{n-m}$
  - G/P is a Grassmann manifold $SU(n)/SU(m) \times SU(n-m)$: the space of m-dim linear subspace $V^m \subset \mathbb{C}^n$

Goal

General Goal

Determine the cohomology ring $H^*(G/P;\mathbb{Z})$

Borel answered this question for the rational coefficients as:

Theorem (Borel)

$$H^*(G/P;\mathbb{Q}) \cong (\mathbb{Q}[\omega_1,\dots,\omega_l])^{W_P}/((\mathbb{Q}[\omega_1,\dots,\omega_l])^{W})$$

Theorem by Bott-Samelson says:

Theorem (Bott-Samelson)

$H^*(G/P;\mathbb{Z})$ is concentrated in even degrees and torsion free

So the problem reduces to the understanding of the inclusion:

$$H^*(G/P;\mathbb{Z}) \hookrightarrow H^*(G/P;\mathbb{Q}) \cong (\mathbb{Q}[w_1,\dots,w_l])^{W_P}/((\mathbb{Q}[w_1,\dots,w_l])^{W})$$

Schubert classes

Let $W^P = W/W_P$ the left coset.
(There is a canonical set of minimal length left coset representatives:
$W^P = \{w \in W \mid \forall w' \in wW_P, l(w') = l(w) + 1\})$

- $W^P = W$ when $P = T$
- W^P is $(m, n-m)$-partition when $G/P = SU(n)/SU(m) \times SU(n-m)$

$H^*(G/P; \mathbb{Z})$ has a good basis which consists of *Schubert classes*.

Theorem (Basis theorem)

$H^*(G/P; \mathbb{Z})$ has a free $\mathbb{Z}$-basis $\{Z_w | w \in W^P\}$, where $|Z_w| = 2l(w)$.

Definition

A product of two classes $Z_w Z_v$ is a linear sum of Schubert classes:

$$Z_w Z_v = \sum_{u \in W^P} c^u_{w,v} Z_u$$

$c^u_{w,v} \in \mathbb{Z}$ is called the *structure constants*.

Motivation

Why do we consider $H^*(G/P;\mathbb{Z})$?

- Chow ring $A^*(G/P)$ is isomorphic to $H^*(G/P;\mathbb{Z})$ and closely related to $A^*(G^{\mathbb{C}})$
- $H^*(G/P;\mathbb{Z})$ is related to $H^*(G;\mathbb{Z})$ and $H^*(BG;\mathbb{Z})$
- structure constants have various interpretations in enumerative geometry, representation theory, etc ...

Goal in Schubert calculus

Determine the structure constants $c_{w,v}^{u}$

- More generally, the structure constants for H_T^*, K_T^*, Q_T^*, etc...

Previous results

- Algorithmic formula for structure constant
  - Littlewood-Richardson rule
  - Chevalley formula
  - GKM type descriptions
  - Duan's formula
- *Schubert polynomials* (Polynomial representatives for Z_w)
  - Schur function for $SU(n)/SU(m) \times SU(n-m)$
  - Several definitions for Schubert polynomials (only for classical type)
- *Borel presentations* for $H^*(G/P;\mathbb{Z})$ using Toda's method.

We are especially interested in the case of *exceptional Lie types*

example in Schubert calculus

- $G/P = SU(4)/SU(2) \times SU(2)$.
- $H^*(G/P;\mathbb{Z}) = \langle Z_{[]}, Z_{[1]}, Z_{[2]}, Z_{[1,1]}, Z_{[2,1]}, Z_{[2,2]} \rangle$
- $H^*(G/P;\mathbb{Z}) = \mathbb{Z}[x_1, x_2, y_1, y_2]/(1 + c_1 + c_2)(1 + c_1' + c_2')$, where $c_1 = x_1 + x_2, c_2 = x_1 x_2, c_1' = y_1 + y_2, c_2' = y_1 y_2$
- Schur polynomials $X_{[]} = 1, X_{[1]} = c_1, X_{[2]} = c_1^2 - c_2, \ldots, X_{[2,2]} = c_2^2 - c_3 c_1 = c_2^2$
- We can compute $X_{[1]}^4 = 2x_{[2,2]}$
- From this one can tell that
    1. The number of lines which intersects all given 4 lines in $\mathbb{C}P^3$ is 2
    2. $(\pi_{[1]}^{\times 4})_{\text{ind}}$ has $\pi_{[2,2]}$ with multiplicity 2 in irreducible decmposition (Note there is an 1-1 correspondance between irr-rep of sym. gp. and partitions)

Borel presentation

Classification Theorem tells that G is one of the following types:

$$SU(n), Spin(n), Sp(n), G_2, F_4, E_6, E_7, E_8$$

An algebraic argument using the fibration sequence

$$G \to G/P \to BP,$$

$H^*(G/P; \mathbb{Z})$ can be calculated as a quotient of polynomial algebra. And the following list of calculations has been obtained:

- (Bott-Samelson1958) G_2/T
- (Toda-Watanabe1974) $Spin(n)/T, F_4/T, E_6/P_1 \cong E_6/P_6, E_6/T$
- (Ishitoya-Toda1977) F_4/P_4
- (Ishitoya1977, Watanabe1998) E_6/P_2
- (Watanabe1975) E_7/P_7
- (Nakagawa2001) $E_7/P_1, E_7/T$
- (Nakagawa(preprint)) $E_8/P_8, E_8/T$

Comparison of the two presentations

We have two descriptions for $H^*(G/P;\mathbb{Z})$

	Borel presentation	Schubert presentation
elements	polynomials	Schubert classes
geometry	no	algebraic cycles
ring structure	easy	hard

Using *divided difference operator*, we can bridge the two.

Divided difference operator

Theorem (B-G-G(1973), Demazure(1973))

- There are well defined operators called the divided difference operators:

$$\Delta_w : H^*(BT;\mathbb{Z}) \to H^{*-2l(w)}(BT;\mathbb{Z}), (w \in W)$$

- A map $c : H^{2k}(BT;\mathbb{Z})^{W_p} \to H^{2k}(G/P;\mathbb{Z})$ defined by

$$c(f) = \sum_{l(w)=k} \Delta_w(f) Z_w \quad (\text{Note:} \Delta_w(f) \in \mathbb{Z})$$

  "converts" Borel presentation to Schubert presentation

- (Giambelli formula)

  $Z_w = c\left(\Delta_{w^{-1}w_0}\left(\frac{\prod_{\alpha\in\Delta^+}\alpha}{|W|}\right)\right)$

  "converts" Schubert presentation to Borel presentation

Translation

$$
\begin{array}{ccccc}
\text{Borel} & & H^*(G/T;\mathbb{Q}) & & \text{Schubert} \\
 & & \| & & \\
H^*(G/T;\mathbb{Z}) & \subset & H^*(BT;\mathbb{Q})/(H^+(BT;\mathbb{Q})^W) & \supset & H^*(G/T;\mathbb{Z}) \\
\| & & \| & & \| \\
\mathbb{Z}[\omega_1,\ldots]/(\rho_1,\ldots) & \subset & \mathbb{Q}[\omega_1,\ldots,\omega_l]/(\phi_1,\ldots) & \supset & \bigoplus_{w\in W}\mathbb{Z}\{Z_w\} \\
 & \searrow & \uparrow & \nearrow & \\
 & & \mathbb{Q}[\omega_1,\ldots,\omega_l] = H^*(BT;\mathbb{Q}) & &
\end{array}
$$

(Note that for $P \neq T$, the canonical map $H^*(G/P;\mathbb{Z}) \hookrightarrow H^*(G/T;\mathbb{Z})$ maps Schubert classes to themselves)

Our result

Take X_i such that $\{i \mid s_i X \neq X\} = \{i\}$ and let P_i be the corresponding stabilizer subgroup.

We give a description of $H^*(G/P; \mathbb{Z})$ for the following cases:

$$\begin{array}{llllllll} G & = F_4 & F_4 & E_6 & E_6 & E_7 & E_7 & E_8 \\ P & = P_1 & P_4 & P_1 & P_2 & P_1 & P_7 & P_8, \end{array}$$

as a quotient of a polynomial algebra whose generators correspond to Schubert classes.
(the above list includes all (co)minuscules of exceptional type)

This can be considered as a intermediate step to finding a candidate for Schubert polynomial

Borel presentation for $H^*(E_6/P_2;\mathbb{Z})$

$$(\mathbb{Q}[\omega_1,\dots,\omega_l])^W = \mathbb{Q}[I_2, I_5, I_6, I_8, I_9, I_{12}], \quad |I_k| = 2k$$

$$\mathbb{Z}[\omega_1,\dots,\omega_l]^{W_2} = \mathbb{Z}[\omega_2, c_2, c_3, c_4, c_5, c_6], \quad |c_k| = 2k$$

$$\text{Let } u = \frac{1}{2}c_3 - \omega_2^3,\ v = \frac{1}{3}(c_4 + 2\omega_2^4) - \omega_2 u,$$

Theorem (Ishitoya(1977))

$$H^*(E_6/P_2;\mathbb{Z}) = \mathbb{Z}[\omega_2, u, v, c_6]/(\rho_6, \rho_8, \rho_9, \rho_{12}),$$

$$\begin{aligned}
r_6 &= 2\omega_2^6 - \omega_2^3 u - 3\omega_2^2 v + u^2 + 2c_6,\\
r_8 &= \omega_2^8 + 3\omega_2^2 c_6 - 3v^2,\\
r_9 &= -\omega_2^3 c_6 + 2uc_6,\\
r_{12} &= -\omega_2^6 c_6 + 15\omega_2^4 v^2 + 15\omega_2^2 v c_6 - 26v^3 + 3c_6^2.
\end{aligned}$$

Schubert presentation for $H^*(E_6/P_2;\mathbb{Z})$

Denote $Z_w = Z_{i_1 i_2 \cdots}$ when $w = s_{i_1} s_{i_2} \cdots$.

Using divided difference operator, we have

$$\begin{aligned}
\omega_2 &= Z_2 \\
u &= Z_{542} \\
v &= Z_{6542} + Z_{3452} + Z_{1342} \\
c_6 &= Z_{136542} \\
-u + \omega_2 &= Z_{342} \\
v - \omega_2 u &= Z_{1342}
\end{aligned}$$

Theorem (c.f. Duan-Zhao)

$$H^*(E_6/P_2;\mathbb{Z}) = \mathbb{Z}[Z_2, Z_{342}, Z_{1342}, Z_{136542}]/(r_6, r_8, r_9, r_{12})$$

Similarly, we obtained $H^*(G/P;\mathbb{Z})$ for the cases listed above.

What is Schubert polynomial

"Theorem"

$H^*(G/T;\mathbb{Z}) \cong \mathbb{Z}[\omega_1,\ldots,\omega_l] \otimes \mathbb{Z}[u_1,\ldots]/(\text{ideal})$,
where $\omega_i = Z_{s_i}$, $|\omega_i| = 2$ and $|u_i| > 2$.

Schubert polynomial $\{X_w | w \in W\}$ can be considered as a family of representatives of Z_w in $\mathbb{Z}[\omega_1,\ldots,\omega_l] \otimes \mathbb{Z}[u_1,\ldots]$
Thus,

$$X_w X_v = \sum_{u \in W} c^u_{w,v} X_u \text{ mod (ideal)}$$

Desirable properties:

- coefficients of X_w are positive
- $\Delta_i X_{ws_i} = X_w$ if $l(ws_i) = l(w) + 1$
- stable under the inclusion $G_n \hookrightarrow G_{n+1}$ for classical types

Bernstein-Gelfand-Gelfand (1982), Lascoux and Schützenberger (1982), Billey-Haiman (1995), Fomin and Kirillov (1996), etc...

Future works

1. Give a reasonable characterization of Schubert polynomial
2. Determine a polynomial ring in which Schubert polynomial resides
3. Characterize indecomposable Schubert classes (which makes a set of ring generators)
4. Find a presentation of a given Schubert class Z_w as a polynomial in a fixed set of ring generators.

Fin

Thank you for listening

a variety of flags

(from left to right)Singapore, China, Korea, Vietnam, Taiwan, India, Mexico, Spain, UK, Japan