Chow rings of Complex Algebraic Groups

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Outline

- 10 days ago, Duan and Zhao posted a preprint (arXiv:math.AT/0711.2541v1) which announces a series of results similar to those I will discuss today.
- Introduction
- Cohomology of flag variety
  - Borel presentation
  - Schubert presentation
- Connection between the two presentations
  - Divided difference operator
- Computations and Main Theorems
- Future Work

Notations

- G: simply connected simple complex algebraic group
  (a complexification of a simple 1-connected compact Lie group)
- B: Borel subgroup of G
- G/B is a projective variety called the flag variety
- $H^*(G/B; \mathbf{Z})$: ordinary integral cohomology of G/B
- $A^*(G)$: Chow ring of G

Main goal

General Goal

Determine $A^*(G)$ for all simply connected simple complex algebraic groups

- Classification Theorem tells that G is one of the following types: $SL_n, Spin_n, Sp_n, G_2, F_4, E_6, E_7, E_8$
- Chevalley and Grothendieck considered the problem in the 1950's.
  - They gave a formula to compute it from $H^*(G/B; \mathbf{Z})$.
  - Consequently, $A^*(G)$ was determined to be trivial for $G = SL_n, Sp_n$.
- $A^*(G) \otimes \mathbf{Z}/p$ was determined by Kac(1985) for all G.
- $A^*(G)$ for $G = Spin_n, G_2, F_4$ were determined by R.Marlin(1974).
  - He resorted to Schubert calculus to determine $H^*(G/B; \mathbf{Z})$.
  - His method seems to be hopeless for other exceptional types. (Note: Nakagawa also checked the result of Marlin by the same method we use here).

Our Goal Today

Determine $A^*(G)$ for $G = E_6, E_7, E_8$.

Chow ring

$A^*(X)$: the Chow ring of a non-singular variety X

- $A^*(X) = \bigoplus_{i \geq 0} A^i(X)$
- $A^i(X)\,(= A_{\dim X - i}(X))$ is a group of the rational equivalence classes of algebraic cycles of codimension i.
  (an algebraic cycle is a linear sum of possibly singular subvarieties)
- *intersection product* $A^i(X) \otimes A^j(X) \to A^{i+j}(X)$
- *cycle map* $cl : A^*(X) \to H^{2*}(X; \mathbf{Z})$: ring homomorphism
  (Note: taking the Poincare dual of the fundamental class of a cycle)

Basic Facts

Theorem (Grothendieck(1958))

- the cycle map is an isomorphism of rings: $A^*(G/B) \xrightarrow{\simeq} H^{2*}(G/B; \mathbf{Z})$.
- the pullback of the projection $p : G \to G/B$ induces a surjection $p^* : A^*(G/B) \to A^*(G)$, where the kernel is an ideal generated by $A^1(G/B)$.

Corollary

$A^*(G) \cong H^*(G/B; \mathbf{Z})/(H^2(G/B; \mathbf{Z}))$

Note: Since $H^*(G/B; \mathbf{Q})$ is generated by degree 2 elements, $A^*(G) \otimes \mathbf{Q} = \mathbf{Q}$ for all G.
For $G = SL_n, Sp_n$, $H^*(G/B; \mathbf{Z})$ is also generated by degree 2 elements, and so $A^*(G) = \mathbf{Z}$.

Strategy

- A presentation for $H^*(G/B; \mathbf{Z})$ was given by Borel.
  - It is called *Borel presentation*, which is a quotient of a polynomial ring divided by some ideal.
  - Ring structure is clear, but generators have little geometric meaning.
- $H^*(G/B; \mathbf{Z})$ has another module basis consisting of by *Schubert classes*.
  - Schubert classes come from subvarieties.
  - But we don't know structure constants, so it is difficult to use Grothendieck's Theorem.

Hence, what we will do are:

1. Compute $A^*(G)$ purely algebraically from Borel presentation.
2. Find Schubert varieties representing the generators.

Main tool

We use the *divided difference operator* given by Demazure and Berstein-Gelfand-Gelfand.

Borel presentation

- K: maximal compact subgroup of G (=real compact form of G)
- T: maximal compact torus of K $(=(S^1)^l)$
- BT: classifying space of T $(=(CP^\infty)^l)$
- W : Weyl group of K $(=N(T)/T)$
- $\{\omega_i\}_{1\leq i\leq l}$: fundamental weights and $H^*(BT;\mathbf{Z}) = \mathbf{Z}[\omega_1,\ldots,\omega_l]$
- Inclusion $K \hookrightarrow G$ induces a diffeomorphism $K/T \cong G/B$.
- the classifying map $K/T \xrightarrow{\iota} BT$ of the T-bundle $T \to K \to K/T$ induces the *characteristic map* $\iota^* : H^*(BT;\mathbf{Z}) \longrightarrow H^*(K/T;\mathbf{Z})$

Theorem (Borel(1953))

1. $\iota^* : H^*(BT;\mathbf{Q}) \longrightarrow H^*(K/T;\mathbf{Q})$ is surjective.
2. The kernel is $(H^+(BT;\mathbf{Q})^W)$ an ideal generated by the W-invariants of positive degrees.

Borel presentation

Toda(1975) extended Borel's work to give $H^*(K/T;\mathbf{Z})$ by a quotient ring of a polynomial ring.
Based on Toda's method, $H^*(K/T;\mathbf{Z})$ were explictly determined:

· $H^*(SU(n)/T;\mathbf{Z})$	$\cdots$Borel (1953)
· $H^*(Spin(n)/T;\mathbf{Z})$	$\cdots$Toda-Watanabe (1974)
· $H^*(Sp(n)/T;\mathbf{Z})$	$\cdots$Borel (1953)
· $H^*(G_2/T;\mathbf{Z})$	$\cdots$Bott-Samelson (1955)
· $H^*(F_4/T;\mathbf{Z})$	$\cdots$Toda-Watanabe (1974)
· $H^*(E_6/T;\mathbf{Z})$	$\cdots$Toda-Watanabe (1974)
· $H^*(E_7/T;\mathbf{Z})$	$\cdots$Nakagawa (2001)
· $H^*(E_8/T;\mathbf{Z})$	$\cdots$Nakagawa (2007, preprint)

Schubert presentation

The Bruhat decomposition of G

$$G = \coprod_{w \in W} BwB$$

gives a cell decomposition

$$G/B = \coprod_{w \in W} BwB/B.$$

- $l(w)$: length of $w \in W$, $w_0 \in W$: the longest element
- $X_w^\circ = BwB/B \cong \mathbb{C}^{l(w)}$: Schubert cell
- X_w = closure of X_w°: Schubert variety
- $Z_w = \{$Poincare dual of the fundamental class $[X_{w_0 w}]\} \in H^{2l(w)}(G/B; \mathbf{Z})$: Schubert class
- $\{Z_w\}_{w \in W}$ forms an additive basis for $H^*(G/B; \mathbf{Z})$.

Comparison of the two presentations

Hence we have two descriptions for
$H^*(K/T; \mathbf{Z}) = H^*(G/B; \mathbf{Z}) = A^*(G/B)$

	Borel presentation	Schubert presentation
elements	polynomials	Schubert classes
geometry	no	algebraic cycles
ring structure	easy	hard

Demazure and BGG's divided difference operator bridges those two presentations.

Divided difference operator

- K: maximal compact subgroup of G (=real compact form of G)
- T: maximal compact torus of K $(=(S^1)^l)$
- $\Pi = \{\alpha_i\}_{1\leq i\leq l}$: simple roots
- $\{\omega_i\}_{1\leq i\leq l}$: fundamental weights ($(\frac{2\alpha_j}{(\alpha_j,\alpha_j)},\omega_i) = \delta_{ij}$)
- s_i : simple reflection corresponding to α_i
  ($s_i(e) = e - (\frac{2\alpha_i}{(\alpha_i,\alpha_i)}, e)\alpha_i$)
- W: Weyl group of G (= a finite group generated by $\{s_i\}_{1\leq i\leq l}$)

Definition (B-G-G(1973), Demazure(1973))

1. For $\alpha_i \in \Pi$, $\Delta_i : H^*(BT;\mathbf{Z}) \to H^{*-2}(BT;\mathbf{Z})$

$$\Delta_i(f) = \frac{f - s_i(f)}{\alpha_i}, f \in H^*(BT;\mathbf{Z}) = \mathbf{Z}[\omega_1,\ldots,\omega_l]$$

2. For $w \in W$, $w = s_{i_1}s_{i_2}\cdots s_{i_k}$: a reduced decomposition
   $\Delta_w = \Delta_{i_1}\circ\Delta_{i_2}\circ\cdots\circ\Delta_{i_k}$

Divided difference operator

Theorem (B-G-G(1973), Demazure(1973))

- $\Delta_w : H^*(BT;\mathbf{Z}) \to H^{*-2l(w)}(BT;\mathbf{Z})$ is well-defined.
- A map $c : H^{2k}(BT;\mathbf{Z}) \to H^{2k}(K/T;\mathbf{Z})$ defined by

$$c(f) = \sum_{l(w)=k} \Delta_w(f) Z_w \quad (\text{Note:} \Delta_w(f) \in \mathbf{Z})$$

is identical to the characteristic map ι^*. ▸ cf

How to calculate ?

- $\Delta_\alpha(\omega_\beta) = \delta_{\alpha\beta}$
- $\Delta_\alpha(fg) = \Delta_\alpha(f)g + s_\alpha(f)\Delta_\alpha(g)$

Translation

Note: $H^*(K/T;\mathbf{Z}) = H^*(G/B;\mathbf{Z})$ has no torsion (Bott(1958)).

$$
\begin{array}{ccccc}
\text{Borel} & & H^*(K/T;\mathbf{Q}) & & \text{Schubert} \\
 & & \| & & \\
H^*(K/T;\mathbf{Z}) & \subset & H^*(BT;\mathbf{Q})/(H^+(BT;\mathbf{Q})^W) & \supset & H^*(K/T;\mathbf{Z}) \\
\| & & \| & & \| \\
\mathbf{Z}[\omega_1,\ldots]/(\rho_1,\ldots) & \subset & \mathbf{Q}[\omega_1,\ldots,\omega_l]/(\phi_1,\ldots) & \supset & \bigoplus_{w\in W}\mathbf{Z}\{Z_w\} \\
\text{(representative)} & \searrow & \uparrow & \nearrow & \text{(characteristic)} \\
 & & \mathbf{Q}[\omega_1,\ldots,\omega_l] = H^*(BT;\mathbf{Q}) & &
\end{array}
$$

Convenient presentation of $H^*(BT;\mathbf{Z})$

- $K = E_l$ $(l = 6, 7, 8)$
- $\{\omega_i\}_{1\leq i\leq l}$: fundamental weights
- $H^*(BT;\mathbf{Z}) = \mathbf{Z}[\omega_1, \omega_2, \ldots, \omega_l]$

We take another set of generators for $H^*(BT;\mathbf{Z})$:

$$
\begin{aligned}
t_l &= \omega_l \\
t_i &= s_{i+1}(t_{i+1}) = \begin{cases} \omega_i - \omega_{i+1} & (4 \leq i \leq l-1) \\ \omega_{i-1} + \omega_i - \omega_{i+1} & (i = 2, 3) \end{cases} \\
t_1 &= s_1(t_2) = -\omega_1 + \omega_2 \\
t &= \omega_2
\end{aligned}
$$

Convenient presentation of $H^*(BT;\mathbf{Z})$

Let c_i = i-th elementary symmetric function in $t_1, \ldots, t_l$ $(1 \leq i \leq l)$

$$\begin{aligned} H^*(BT;\mathbf{Z}) &= \mathbf{Z}[\omega_1, \omega_2, \ldots, \omega_l] \\ &= \mathbf{Z}[t_1, t_2, \ldots, t_l, t]/(c_1 - 3t). \end{aligned}$$

- s_i $(i \neq 2)$ act on $\{t_i\}_{1\leq i\leq l}$ as permutations and trivially on t.
- The action of s_2 on $\{t_i\}_{1\leq i\leq l}$, and t is given by

$$\begin{aligned} s_2(t_i) &= \begin{cases} t - t_1 - t_2 - t_3 + t_i & (1 \leq i \leq 3) \\ t_i & (4 \leq i \leq l) \end{cases} \\ s_2(t) &= 2t - t_1 - t_2 - t_3. \end{aligned}$$

- Thus, for $f \in \mathbf{Z}[t, c_2, \ldots, c_l]$, $\Delta_i f = 0$ if $i \neq 2$.

We sometimes call $\{t_1, t_2, \ldots, t_l, t\}$ *Toda-Watanabe's magical basis*.

Borel presentation for $H^*(E_6/T;\mathbf{Z})$

Theorem (Toda-Watanabe(1974))

$$H^*(E_6/T;\mathbf{Z}) = \frac{\mathbf{Z}[t_1,t_2,\ldots,t_6,t,\gamma_3,\gamma_4]}{(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6,\rho_8,\rho_9,\rho_{12})}\ (|\gamma_i| = 2i)$$

$$\rho_1 = c_1 - 3t \qquad \rho_2 = c_2 - 4t^2$$

$$\rho_3 = c_3 - 2\gamma_3 \qquad \rho_4 = c_4 + 2t^4 - 3\gamma_4$$

$$\rho_5 = c_5 - 3t\gamma_4 + 2t^2\gamma_3 \qquad \rho_6 = {\gamma_3}^2 + 2c_6 - 3t^2\gamma_4 + t^6$$

$$\rho_8 = 3{\gamma_4}^2 - 6t\gamma_3\gamma_4 - 9t^2c_6 + 15t^4\gamma_4 - 6t^5\gamma_3 - t^8$$

$$\rho_9 = 2c_6\gamma_3 - 3t^3c_6$$

$$\begin{aligned}\rho_{12} = &3c_6^2 - 2{\gamma_4}^3 + 6t\gamma_3{\gamma_4}^2 + 3t^2c_6\gamma_4 + 5t^3c_6\gamma_3 - 15t^4{\gamma_4}^2 - 10t^6c_6 \\ &+ 19t^8\gamma_4 - 6t^9\gamma_3 - 2t^{12}\end{aligned}$$

Correspondence

Using the characteristic map, we can translate the generators $\{t_1, t_2, \ldots, t_6, t, \gamma_3, \gamma_4\}$ in Borel presentation into Schubert classes.

Borel	*Schubert*	*Borel*	*Schubert*
t_1	$-Z_1 + Z_2$	t_6	Z_6
t_2	$Z_1 + Z_2 - Z_3$	t	Z_2
t_3	$Z_2 + Z_3 - Z_4$	γ_3	$Z_{342} + 2Z_{542}$
t_4	$Z_4 - Z_5$	γ_4	$Z_{1342} + 2Z_{3542} + Z_{6542}$
t_5	$Z_5 - Z_6$		

Furthermore, we wish to take a single Schubert class for each generator. In this E_6 case, for example, we can take the following classes:

$$\begin{aligned} Z_{342} &= -\gamma_3 + 2t^3 \\ Z_{1342} &= \gamma_4 - 2t\gamma_3 + 2t^4 \end{aligned}$$

Finding a set of ring generators

**How to determine which Schubert classes can be chosen as generators ?

This question can be formulated as follows.

Definition

- R: ring
- R°: non-invertible elements of R
- decomposable ideal: $R^\circ \cdot R^\circ$
- $x \in R$ is indecomposable when $x \neq 0 \in R/R^\circ \cdot R^\circ$

In our setting when $R = H^*(K/T; \mathbf{Z})$:

- There is at most one ring generator in each degree $H^{*>2}(K/T; \mathbf{Z})$.
- If we find an indecomposable $Z_w \in H^*(K/T; \mathbf{Z})$, then we take it as a generator.

Related question

Which Schubert classes are indecomposable ?

$A(E_6)$

By Grothendieck's Theorem,

$$\begin{aligned} A^*(G) &= A^*(G/B)/(A^1(G/B)) \\ &= H^{2*}(G/B;\mathbf{Z})/(H^2(G/B;\mathbf{Z})) \\ &= H^{2*}(K/T;\mathbf{Z})/(H^2(K/T;\mathbf{Z})), \end{aligned}$$

(Note: $H^{2*-1}(K/T;\mathbf{Z}) = 0$)

where

$$\begin{aligned} H^*(K/T;\mathbf{Z}) &= \mathbf{Z}[t_1,\ldots,t_l,t,\gamma_{i_1},\ldots]/(\rho_{j_1},\ldots) \\ H^2(K/T;\mathbf{Z}) &= \mathbf{Z}\{t_1,\ldots,t_l,t\} \end{aligned}$$

Therefore to obtain $A^*(G)$ from $H^*(K/T;\mathbf{Z})$, we simply put $t = 0, t_i = 0, (1 \le i \le l)$ in Borel presentation.

$$\begin{aligned} H^*(E_6/T;\mathbf{Z})/(t_1,\ldots,t_6,t) &= \mathbf{Z}[\gamma_3,\gamma_4]/(2\gamma_3, 3\gamma_4, \gamma_3^2, \gamma_4^3) \\ \text{(using the correspondence)} &= \mathbf{Z}[Z_{542}, Z_{6542}]/(2Z_{542}, 3Z_{6542}, Z_{542}^2, Z_{6542}^3) \end{aligned}$$

Main Theorems

$p : G \longrightarrow G/B$: projection
Since a Schubert class Z_w corresponds to Schubert variety $X_{w_0 w}$, we have

Theorem (K-Nakagawa)

$$A(E_6) = \mathbf{Z}[X_3, X_4]/(2X_3, 3X_4, X_3^2, X_4^3)$$

$$X_3 = p^*(X_{w_0 s_5 s_4 s_2}) = \overline{B(w_0 s_5 s_4 s_2)B} \subset G$$
$$X_4 = p^*(X_{w_0 s_6 s_5 s_4 s_2}) = \overline{B(w_0 s_6 s_5 s_4 s_2)B} \subset G$$

$H^*(E_7/T;\mathbf{Z})$

Similarly, from the Borel presentation of $H^*(E_7/T;\mathbf{Z})$, we have

$$
\begin{aligned}
&H^*(E_7/T;\mathbf{Z})/(t_1,\ldots,t_7,t)\\
&\quad = \mathbf{Z}[\gamma_3,\gamma_4,\gamma_5,\gamma_9]/(2\gamma_3,3\gamma_4,2\gamma_5,\gamma_3^2,2\gamma_9,\gamma_5^2,\gamma_4^3,\gamma_9^2)\\
&\quad = \mathbf{Z}[Z_{542},Z_{6542},Z_{76542},Z_{654376542}]\\
&\qquad \Big/ \begin{pmatrix} 2Z_{542},3Z_{6542},2Z_{76542},Z_{542}^2,2Z_{654376542},\\ Z_{76542}^2,Z_{6542}^3,Z_{654376542}^2 \end{pmatrix}
\end{aligned}
$$

$A(E_7)$

Theorem (K-Nakagawa)

$$A(E_7) = \mathbf{Z}[X_3, X_4, X_5, X_9]$$
$$/(2X_3, 3X_4, 2X_5, X_3^2, 2X_9, X_5^2, X_4^3, X_9^2)$$

$$X_3 = p^*(X_{w_0 s_5 s_4 s_2}) = \overline{B(w_0 s_5 s_4 s_2)B} \subset G$$
$$X_4 = p^*(X_{w_0 s_6 s_5 s_4 s_2}) = \overline{B(w_0 s_6 s_5 s_4 s_2)B} \subset G$$
$$X_5 = p^*(X_{w_0 s_7 s_6 s_5 s_4 s_2}) = \overline{B(w_0 s_7 s_6 s_5 s_4 s_2)B} \subset G$$
$$X_9 = p^*(X_{w_0 s_6 s_5 s_4 s_3 s_7 s_6 s_5 s_4 s_2}) = \overline{B(w_0 s_6 s_5 s_4 s_3 s_7 s_6 s_5 s_4 s_2)B} \subset G$$

$A(E_8)$

Proposition (K-Nakagawa)

$$A(E_8) = \mathbf{Z}[X_3, X_4, X_5, X_6, X_9, X_{10}, X_{15}] \Big/ \begin{pmatrix} 2X_3, 3X_4, 2X_5, 5X_6, 2X_9, X_5^2 - 3X_{10}, \\ X_4^3, 2X_{15}, X_9^2, 3X_{10}^2, X_3^8, \\ X_{15}^2 + X_{10}^3 + 2X_6^5 \end{pmatrix}$$

$X_i = p^*(\gamma_i)$ $(i = 3, 4, 5, 6, 9, 10, 15)$

Note: here X_i may not be the pull-back of a single Schubert variety but a linear combination of them.

Future Work

- Determine which Schubert classes belong to the decomposable ideal. (equivalently, find indecomposable Schubert classes)
- Find a presentation of a given Schubert class Z_w as a polynomial in a fixed set of ring generators.
  (Schubert polynomial of type $G_2, G_4, E_l (l = 6, 7, 8)$)
- Replace B with any parabolic subgroup P in the above problems.
  (Note: there is a ring monomorphism $H^*(G/P; \mathbf{Z}) \hookrightarrow H^*(G/B; \mathbf{Z})$ described in terms of Schubert presentation)

Note: Once a set of simple roots is fixed, a parabolic group P corresponds to a subset Π_P of simple roots. G/P has a Schubert basis $\{Z_w\}_{w \in S}$ indexed by the left coset of $W(G)$ by $W(P)$, where $W(P)$ is generated by simple reflections corresponding to the complement of Π_P. S can be injected into $W(G)$ as $\{w \in W(G) | l(ws_i) > l(w), \forall s_i \in \Pi_P^{\complement}\}$, and this induces a ring monomorphism $H^*(G/P; \mathbf{Z}) \hookrightarrow H^*(G/B; \mathbf{Z})$.

Example (finding indecomposables)

- $H^*(F_4/B;\mathbf{Z})$ has generators only in degrees $2, 6$, and 8.
- $H^2(F_4/B;\mathbf{Z})$ is spanned by Z_w, where $W = [1], [2], [3], [4]$, the length one elements in the Weyl group. Of course they are indecomposable.
- Out of $16 (= \dim H^6(F_4/B;\mathbf{Z}))$, the indecomposables are:

$$W = [1,2,3], [1,3,2], [2,1,3], [2,3,2], [3,2,1], [3,2,3]$$

- Out of $25 (= \dim H^8(F_4/B;\mathbf{Z}))$, the indecomposables are:

$$\begin{aligned} W =& [1,2,3,4], [1,2,3,2], [1,2,4,3], [1,3,2,3], [1,3,2,4], [1,4,3,2] \\ & [2,1,3,4], [2,1,4,3], [2,3,2,1], [2,3,2,4], [2,4,3,2] \\ & [3,2,1,3], [3,2,1,4], [3,2,3,4] \\ & [4,3,2,1], [4,3,2,3] \end{aligned}$$

Note: there are more than one way to express an element of Weyl group by the products of the simple reflections.

Thank you for listening

and

for your patience with my "exceptional" English