Chow rings of Complex Algebraic Groups

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Outline

- Introduction
  - Our problem (algebraic geometory)
- Cohomology of flag variety
  - Borel presentation (algebraic topology)
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  - Divided difference operator (combinatorics)
- Computations and Main Theorems (man & computer power)
- Future Work

Notations

- G: simply connected simple complex Lie group
- B: Borel subgroup of G
- l: rank of G
- G/B: a projective variety called the flag variety associated to G
- $H^*(G/B;\mathbb{Z})$: ordinary integral cohomology of G/B
- $A^*(G)$: Chow ring of G

Main goal

General Goal

Determine $A^*(G)$ for all simply connected simple complex Lie groups

- Classification Theorem tells that G is one of the following types: $SL_n, Spin(n), Sp(n), G_2, F_4, E_6, E_7, E_8$
- Grothendieck considered the problem in the 1950's.
  - He gave a formula to compute it from $H^*(G/B;\mathbb{Z})$.
  - Consequently, $A^*(G)$ was determined to be trivial for $G = SL_n, Sp(n)$.
- $A^*(G) \otimes \mathbb{Z}/p$ was determined by Kac(1985) for all G.
- $A^*(G)$ for $G = Spin(n), G_2, F_4$ were determined by R.Marlin(1974).
  - His method seems to be hopeless for other exceptional types. (Note: Nakagawa also checked the result of Marlin by the same method we use here).

Our Goal Today

Determine $A^*(G)$ for $G = E_6, E_7, E_8$.

What is Chow ring

$A^*(X)$: the Chow ring of a non-singular variety X

- $A^*(X) = \bigoplus_{i \geq 0} A^i(X)$
- $A^i(X)$ is a group of the rational equivalence classes of algebraic cycles of codimension i.
  (an algebraic cycle is a linear sum of possibly singular subvarieties)
- *intersection product* $A^i(X) \otimes A^j(X) \to A^{i+j}(X)$

Basic Facts

Theorem (Grothendieck(1958))

- the cycle map $cl : A^*(G/B) \to H^{2*}(G/B;\mathbb{Z})$ is an isomorphism of rings:
  $A^*(G/B) \xrightarrow{\simeq} H^{2*}(G/B;\mathbb{Z}) = H^*(G/B;\mathbb{Z})$.
- the pullback of the projection $p : G \to G/B$ induces a surjection
  $p^* : A^*(G/B) \to A^*(G)$,
  where the kernel is an ideal generated by $A^1(G/B)$.

Corollary

$A^*(G) \cong H^*(G/B;\mathbb{Z})/(H^2(G/B;\mathbb{Z}))$

Note: Since $H^*(G/B;\mathbb{Q})$ is generated by degree 2 elements,
$A^*(G) \otimes \mathbb{Q} = \mathbb{Q}$ for all G.
For $G = SL_n, Sp_n$, $H^*(G/B;\mathbb{Z})$ is also generated by degree 2 elements,
and so $A^*(G) = \mathbb{Z}$.

Strategy

$A^*(G) \cong H^*(G/B;\mathbb{Z})/(H^2(G/B;\mathbb{Z}))$

- A presentation for $H^*(G/B;\mathbb{Z})$ was given by Borel.
  - It is called *Borel presentation*, which is a quotient of a polynomial ring divided by some ideal.
  - Ring structure is clear, but generators have little geometric meaning.
- $H^*(G/B;\mathbb{Z})$ has another module basis consisting of by *Schubert classes*.
  - Schubert classes come from subvarieties called Schubert varieties.
  - Ring structure is complicated, so it is difficult to use Grothendieck's Theorem.

Hence, what we will do are:

easy: Compute $A^*(G)$ purely algebraically from Borel presentation.

difficult: Find Schubert varieties representing the generators.

Main tool

We use the *divided difference operator* given by Demazure and Berstein-Gelfand-Gelfand.

Borel presentation

- $K(= G_{\mathbb{R}})$: maximal compact subgroup of G
- $T(= T_{\mathbb{R}})$: maximal compact torus of K $(=K \cap B = (S^1)^l)$
- BT: classifying space of T $(=(CP^{\infty})^l)$
- W : Weyl group of K $(=N(T)/T)$
- $\{\omega_i\}_{1 \leq i \leq l}$: fundamental weights and $H^*(BT;\mathbb{Z}) = \mathrm{Sym}\mathfrak{h}_{\mathbb{Z}}^* = \mathbb{Z}[\omega_1, \dots, \omega_l]$
- Inclusion $K \hookrightarrow G$ induces a diffeomorphism $K/T \cong G/B$.
- the classifying map $K/T \xrightarrow{\iota} BT$ of the T-bundle $T \to K \to K/T$ induces the *characteristic map* $\iota^* : H^*(BT;\mathbb{Z}) \longrightarrow H^*(K/T;\mathbb{Z})$

Theorem (Borel(1953))

ι^* induces $H^*(BT;\mathbb{Q})/I_W \simeq H^*(K/T;\mathbb{Q})$, where $I_W = (H^+(BT;\mathbb{Q})^W)$ an ideal generated by the W-invariants of positive degrees.

Borel presentation

Toda(1975) extended Borel's work to give $H^*(K/T;\mathbb{Z})$ by a quotient ring of a polynomial ring.
Based on Toda's method, $H^*(G/B;\mathbb{Z}) = H^*(K/T;\mathbb{Z})$ were explictly determined for all G.

Theorem

$$\iota^* : \frac{H^*(BT;\mathbb{Z}) \otimes \mathbb{Z}[\gamma_{d_i}]}{(\text{ideal})} \simeq H^*(K/T;\mathbb{Z}),\ |\gamma_{d_i}| = 2d_i.$$

Schubert presentation

The Bruhat decomposition of G

$$G = \coprod_{w \in W} BwB$$

gives a cell decomposition

$$G/B = \coprod_{w \in W} BwB/B.$$

- $l(w)$: length of $w \in W$, $w_0 \in W$: the longest element
- X_w = closure of $Bw_0wB/B (\cong \mathbb{C}^{l(w_0w)})$: Schubert variety
- $\sigma_w = \{$the cohomology class corresponding to $X_w\} \in H^{2l(w)}(G/B;\mathbb{Z})$: Schubert class corresponding to w
- $\{\sigma_w\}_{w \in W}$ forms an additive basis for $H^*(G/B;\mathbb{Z})$.

Comparison of the two presentations

Hence we have two descriptions for
$H^*(K/T;\mathbb{Z}) = H^*(G/B;\mathbb{Z}) = A^*(G/B)$

	Borel presentation	Schubert presentation
elements	polynomials	Schubert classes
geometry	no	algebraic cycles
ring structure	easy	hard

Demazure and BGG's divided difference operator bridges those two presentations.

Divided difference operator

We can switch between Borel presentation and Schubert presentation.

Theorem (B-G-G(1973), Demazure(1973))

- For $w \in W$, they defined $\Delta_w : H^*(BT;\mathbb{Z}) \to H^{*-2l(w)}(BT;\mathbb{Z})$.
- (characteristic map) $c : H^{2k}(BT;\mathbb{Z}) \to H^{2k}(K/T;\mathbb{Z})$ defined by

$$c(f) = \sum_{l(w)=k} \Delta_w(f)\sigma_w \quad (\text{Note:}\Delta_w(f) \in \mathbb{Z})$$

- (Giambelli formula)
$$\sigma_w = c\left(\Delta_{w^{-1}w_0}\left(\frac{\prod_{\alpha\in\Delta^+}\alpha}{|W|}\right)\right), \quad \Delta^+\text{: the set of positive roots.}$$

How to calculate ?

- $\Delta_\alpha(\omega_\beta) = \delta_{\alpha\beta}$
- $\Delta_\alpha(fg) = \Delta_\alpha(f)g + s_\alpha(f)\Delta_\alpha(g)$

Borel presentation for $H^*(E_6/T;\mathbb{Z})$

Theorem (Toda-Watanabe(1974))

$$H^*(E_6/T;\mathbb{Z}) = \frac{\mathbb{Z}[t_1,t_2,\dots,t_6,t_0,\gamma_3,\gamma_4]}{(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6,\rho_8,\rho_9,\rho_{12})}\ (|t_i|=2, |\gamma_i|=2i)$$

$$\rho_1 = c_1 - 3t_0 \qquad \rho_2 = c_2 - 4t_0^2$$

$$\rho_3 = c_3 - 2\gamma_3 \qquad \rho_4 = c_4 + 2t_0^4 - 3\gamma_4$$

$$\rho_5 = c_5 - 3t_0\gamma_4 + 2t_0^2\gamma_3 \qquad \rho_6 = {\gamma_3}^2 + 2c_6 - 3t_0^2\gamma_4 + t_0^6$$

$$\rho_8 = 3{\gamma_4}^2 - 6t\gamma_3\gamma_4 - 9t_0^2c_6 + 15t_0^4\gamma_4 - 6t_0^5\gamma_3 - t_0^8$$

$$\rho_9 = 2c_6\gamma_3 - 3t_0^3c_6$$

$$\rho_{12} = 3c_6^2 - 2{\gamma_4}^3 + 6t_0\gamma_3{\gamma_4}^2 + 3t_0^2c_6\gamma_4 + 5t_0^3c_6\gamma_3 - 15t_0^4{\gamma_4}^2 - 10t_0^6c_6 + 19t_0^8\gamma_4 - 6t_0^9\gamma_3 - 2t_0^{12}$$

Correspondence

Using the characteristic map, we can translate the generators $\{t_1, t_2, \ldots, t_6, t, \gamma_3, \gamma_4\}$ in Borel presentation into Schubert classes.

Borel	*Schubert*	*Borel*	*Schubert*
t_1	$-\sigma_1 + \sigma_2$	t_6	σ_6
t_2	$\sigma_1 + \sigma_2 - \sigma_3$	t	σ_2
t_3	$\sigma_2 + \sigma_3 - \sigma_4$	γ_3	$\sigma_{342} + 2\sigma_{542}$
t_4	$\sigma_4 - \sigma_5$	γ_4	$\sigma_{1342} + 2\sigma_{3542} + \sigma_{6542}$
t_5	$\sigma_5 - \sigma_6$		

Furthermore, we wish to take a single Schubert class for each generator. In this E_6 case, for example, we can take the following classes:

$$\begin{aligned} \sigma_{342} &= -\gamma_3 + 2t^3 \\ \sigma_{1342} &= \gamma_4 - 2t\gamma_3 + 2t^4 \end{aligned}$$

$A(E_6)$

By Grothendieck's Theorem,

$$\begin{aligned} A^*(G) &= A^*(G/B)/(A^1(G/B)) \\ &= H^*(G/B;\mathbb{Z})/(H^2(G/B;\mathbb{Z})) \\ &= H^*(K/T;\mathbb{Z})/(H^2(K/T;\mathbb{Z})), \end{aligned}$$

where

$$\begin{aligned} H^*(K/T;\mathbb{Z}) &= \mathbb{Z}[t_1,\dots,t_l,t_0,\gamma_{d_1},\dots]/(\rho_{j_1},\dots) \\ H^2(K/T;\mathbb{Z}) &= \mathbb{Z}\{t_1,\dots,t_l,t_0\} \end{aligned}$$

Therefore to obtain $A^*(G)$ from $H^*(K/T;\mathbb{Z})$, we simply put $t_i = 0, (0 \le i \le l)$ in Borel presentation.

$$\begin{aligned} H^*(E_6/T;\mathbb{Z})/(t_1,\dots,t_6,t_0) &= \mathbb{Z}[\gamma_3,\gamma_4]/(2\gamma_3, 3\gamma_4, \gamma_3^2, \gamma_4^3) \\ &= \mathbb{Z}[\sigma_{542},\sigma_{6542}]/(2\sigma_{542}, 3\sigma_{6542}, \sigma_{542}^2, \sigma_{6542}^3) \end{aligned}$$

Main Theorems

$p : G \to G/B$: projection

Theorem (K-Nakagawa)

$$A(E_6) = \mathbb{Z}[X_3, X_4]/(2X_3, 3X_4, X_3^2, X_4^3)$$

$X_3 = p^*(X_{w_0 s_5 s_4 s_2}) = \overline{B(w_0 s_5 s_4 s_2)B} \subset G$
$X_4 = p^*(X_{w_0 s_6 s_5 s_4 s_2}) = \overline{B(w_0 s_6 s_5 s_4 s_2)B} \subset G$

$A(E_7)$

Theorem (K-Nakagawa)

$$A(E_7) = \mathbb{Z}[X_3, X_4, X_5, X_9] \\ /(2X_3, 3X_4, 2X_5, X_3^2, 2X_9, X_5^2, X_4^3, X_9^2)$$

$$X_3 = p^*(X_{w_0 s_5 s_4 s_2}) = \overline{B(w_0 s_5 s_4 s_2)B} \subset G$$
$$X_4 = p^*(X_{w_0 s_6 s_5 s_4 s_2}) = \overline{B(w_0 s_6 s_5 s_4 s_2)B} \subset G$$
$$X_5 = p^*(X_{w_0 s_7 s_6 s_5 s_4 s_2}) = \overline{B(w_0 s_7 s_6 s_5 s_4 s_2)B} \subset G$$
$$X_9 = p^*(X_{w_0 s_6 s_5 s_4 s_3 s_7 s_6 s_5 s_4 s_2}) = \overline{B(w_0 s_6 s_5 s_4 s_3 s_7 s_6 s_5 s_4 s_2)B} \subset G$$

Proposition (K-Nakagawa)

$$A(E_8) = \mathbb{Z}[X_3, X_4, X_5, X_6, X_9, X_{10}, X_{15}] \Big/ \begin{pmatrix} 2X_3, 3X_4, 2X_5, 5X_6, 2X_9, X_5^2 - 3X_{10}, \\ X_4^3, 2X_{15}, X_9^2, 3X_{10}^2, X_3^8, \\ X_{15}^2 + X_{10}^3 + 2X_6^5 \end{pmatrix}$$

$X_i = p^*(\gamma_i)$ $(i = 3, 4, 5, 6, 9, 10, 15)$

Note: here X_i may not be the pull-back of a single Schubert variety but a linear combination of them.

Case of $G = Spin(n), G_2, F_4$

Theorem (Marlin, Nakagawa)

$$A(F_4) = \mathbb{Z}[X_3, X_4]/(2X_3, 3X_4, X_3^2, X_4^3),$$

where $X_3 = \overline{B(w_0 s_1 s_2 s_3)B}$, $X_4 = \overline{B(w_0 s_1 s_2 s_3 s_4)B}$.

$$A(G_2) = \mathbb{Z}[X_3]/(2X_3, X_3^2),$$

where $X_3 = \overline{B(w_0 s_1 s_2 s_1)B}$.

$$A(Spin(2n+1)) = \mathbb{Z}[X_3, X_5, \dots, X_{2[\frac{n+1}{2}]-1}]/(2X_i,\ X_i^{p_i}),$$

where $X_i = \overline{B(w_0 s_{n-i+1} \cdots s_{n-1} s_n)B}$ $\quad (1 \le i \le n)$ and $p_i = 2^{[\log_2 \frac{n}{i}]+1}$.

$$A(Spin(2n)) = \mathbb{Z}[X_3, X_5, \dots, X_{2[\frac{n}{2}]-1}]/(2X_i,\ X_i^{p_i}),$$

where $X_1 = \overline{B(w_0 s_n)B}$, $X_i = \overline{B(w_0 s_{n-i} \dots s_{n-2} s_n)B}$ $\quad (2 \le i \le n-1))$ and $p_i = 2^{[\log_2 \frac{n-1}{i}]+1}$.

Future Work

- Determine which Schubert classes belong to the decomposable ideal. (equivalently, find indecomposable Schubert classes)
- Find a presentation of a given Schubert class σ_w as a polynomial in a fixed set of ring generators.
  (Schubert polynomial of type $G_2, F_4, E_l (l = 6, 7, 8)$)
- Replace B with any parabolic subgroup P in the above problems.
  (Note: there is a ring monomorphism $H^*(G/P; \mathbb{Z}) \hookrightarrow H^*(G/B; \mathbb{Z})$ described in terms of Schubert presentation)

Finding a set of ring generators

How to determine which Schubert classes can be chosen as generators ? This question can be formulated as follows.

Definition

- R: graded commutative ring with $R^0 = \mathbb{Z}$
- R°: non-invertible elements of R
- decomposable ideal: $(R^\circ \cdot R^\circ)$
- $x \in R$ is indecomposable when $x \neq 0 \in R/(R^\circ \cdot R^\circ)$

In our setting when $R = H^*(G/B;\mathbb{Z})$:

- There is at most one ring generator in each degree $H^{*>2}(G/B;\mathbb{Z})$.
- If we find an indecomposable $\sigma_w \in H^{2d}(G/B;\mathbb{Z})$, then we take it as a generator γ_d.

Related question

Which Schubert classes are indecomposable ?

Example (finding indecomposables)

- $H^*(F_4/B;\mathbb{Z})$ has generators only in degrees $2, 6$, and 8.
- $H^2(F_4/B;\mathbb{Z})$ is spanned by σ_w, where $W = [1], [2], [3], [4]$, the length one elements in the Weyl group. Of course they are indecomposable.
- Out of $16 (= \dim H^6(F_4/B;\mathbb{Z}))$, the indecomposables are:

$$W = [1,2,3], [1,3,2], [2,1,3], [2,3,2], [3,2,1], [3,2,3]$$

- Out of $25 (= \dim H^8(F_4/B;\mathbb{Z}))$, the indecomposables are:

$$\begin{aligned} W = & [1,2,3,4], [1,2,3,2], [1,2,4,3], [1,3,2,3], [1,3,2,4], [1,4,3,2] \\ & [2,1,3,4], [2,1,4,3], [2,3,2,1], [2,3,2,4], [2,4,3,2] \\ & [3,2,1,3], [3,2,1,4], [3,2,3,4] \\ & [4,3,2,1], [4,3,2,3] \end{aligned}$$

Note: there are more than one way to express an element of Weyl group by the products of the simple reflections.

Thank you for listening