Chow rings of Complex Algebraic Groups

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Outline

- Introduction
  - Our problem (algebraic geometory)
- Cohomology of flag variety
  - Borel presentation (algebraic topology)
  - Schubert presentation (geometry)
  - Divided difference operator (combinatorics)
- Computations and Main Theorems (man & computer power)
- Future Work

Notations

Notice for simplicity, above examples are NOT simply connected.

- G: simply connected simple complex algebraic group ($GL(n, \mathbb{C})$)
- B: Borel subgroup of G (the subgroup of upper triangular matrices)
- G/B: a projective variety called the flag variety (the space of "flags", $0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_{n-1} \subseteq V_n = \mathbb{C}^n$, $\dim_{\mathbb{C}}(V_i) = i$).
- $H^*(G/B; \mathbb{Z})$: ordinary integral cohomology of G/B
- $A^*(G)$: Chow ring of G

Main goal

General Goal

Determine $A^*(G)$ for all simply connected simple complex algebraic groups

- Classification Theorem tells that G is one of the following types: $SL_n, Spin_n, Sp_n, G_2, F_4, E_6, E_7, E_8$
- Chevalley and Grothendieck considered the problem in the 1950's.
  - They gave a formula to compute it from $H^*(G/B; \mathbb{Z})$.
  - Consequently, $A^*(G)$ was determined to be trivial for $G = SL_n, Sp_n$.
- $A^*(G) \otimes \mathbb{Z}/p$ was determined by Kac(1985) for all G.
- $A^*(G)$ for $G = Spin_n, G_2, F_4$ were determined by R.Marlin(1974).

Our Goal Today

Determine $A^*(G)$ for $G = E_6, E_7, E_8$.

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Chow ring

$A^*(X)$: the Chow ring of a non-singular variety X

- $A^*(X) = \bigoplus_{i \geq 0} A^i(X)$
- $A^i(X)$ is a group of the rational equivalence classes of algebraic cycles of codimension i.
  (an algebraic cycle is a linear sum of possibly singular subvarieties)
- *intersection product* $A^i(X) \otimes A^j(X) \to A^{i+j}(X)$

Basic Facts

Theorem (Grothendieck(1958))

- the cycle map $cl : A^*(G/B) \to H^{2*}(G/B;\mathbb{Z})$ is an isomorphism of rings:
  $A^*(G/B) \xrightarrow{\simeq} H^{2*}(G/B;\mathbb{Z})$.
- the pullback of the projection $p : G \to G/B$ induces a surjection $p^* : A^*(G/B) \to A^*(G)$,
  where the kernel is an ideal generated by $A^1(G/B)$.

Corollary

$A^*(G) \cong H^*(G/B;\mathbb{Z})/(H^2(G/B;\mathbb{Z}))$

Note: Since $H^*(G/B;\mathbb{Q})$ is generated by degree 2 elements,
$A^*(G) \otimes \mathbb{Q} = \mathbb{Q}$ for all G.
For $G = SL_n, Sp_n$, $H^*(G/B;\mathbb{Z})$ is also generated by degree 2 elements,
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Comparison of the two presentations

We will see two descriptions for
$H^*(G/B;\mathbb{Z}) = A^*(G/B)$

	Borel presentation	Schubert presentation
elements	quotient of a polynomial ring polynomials	$\mathbb{Z}$-basis indexed by Weyl group Schubert classes
geometry	no	algebraic cycles
ring structure	easy	hard (main(?) theme of Schubert calc

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easy Compute $A^*(G)$ purely algebraically from Borel presentation.

difficult Find Schubert varieties representing the generators.

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Borel presentation

- K: maximal compact subgroup of G (=real compact form of G)
- T: maximal compact torus of K ($=K \cap B = (S^1)^l$)
- BT: classifying space of T ($=(CP^\infty)^l$)
- W : Weyl group of K ($=N(T)/T$)
- $\{\omega_i\}_{1 \leq i \leq l}$: fundamental weights and $H^*(BT; \mathbb{Z}) = \mathbb{Z}[\omega_1, \ldots, \omega_l]$
- Inclusion $K \hookrightarrow G$ induces a diffeomorphism $K/T \cong G/B$.
- the classifying map $K/T \xrightarrow{\iota} BT$ of the T-bundle $T \to K \to K/T$ induces the *characteristic map* $\iota^* : H^*(BT; \mathbb{Z}) \longrightarrow H^*(K/T; \mathbb{Z})$

Theorem (Borel(1953))

1. $\iota^* : H^*(BT; \mathbb{Q}) \longrightarrow H^*(K/T; \mathbb{Q})$ is surjective.
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Borel presentation

Toda(1975) extended Borel's work to give $H^*(K/T;\mathbb{Z})$ by a quotient ring of a polynomial ring.
Based on Toda's method, $H^*(G/B;\mathbb{Z}) = H^*(K/T;\mathbb{Z})$ were explictly determined:

- $H^*(SU(n)/T;\mathbb{Z})$ $\cdots$Borel (1953)
- $H^*(Spin(n)/T;\mathbb{Z})$ $\cdots$Toda-Watanabe (1974)
- $H^*(Sp(n)/T;\mathbb{Z})$ $\cdots$Borel (1953)
- $H^*(G_2/T;\mathbb{Z})$ $\cdots$Bott-Samelson (1955)
- $H^*(F_4/T;\mathbb{Z})$ $\cdots$Toda-Watanabe (1974)
- $H^*(E_6/T;\mathbb{Z})$ $\cdots$Toda-Watanabe (1974)
- $H^*(E_7/T;\mathbb{Z})$ $\cdots$Nakagawa (2001)
- $H^*(E_8/T;\mathbb{Z})$ $\cdots$Nakagawa (2007, preprint)

Schubert presentation

The Bruhat decomposition of G

$$G = \coprod_{w \in W} BwB$$

gives a cell decomposition

$$G/B = \coprod_{w \in W} BwB/B.$$

- X_w = closure of $BwB/B (\cong \mathbb{C}^{l(w)})$: Schubert variety
- Z_w = {the cohomology class corresponding to $[X_{w_0 w}]$} $\in H^{2l(w)}(G/B;\mathbb{Z})$: Schubert class
- $\{Z_w\}_{w \in W}$ forms an additive basis for $H^*(G/B;\mathbb{Z})$.

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Divided difference operator

We can switch between Borel presentation and Schubert presentation.

Theorem (B-G-G(1973), Demazure(1973))

- For $w \in W$, they defined $\Delta_w : H^*(BT;\mathbb{Z}) \to H^{*-2l(w)}(BT;\mathbb{Z})$.
- (characteristic map) $c : H^{2k}(BT;\mathbb{Z}) \to H^{2k}(K/T;\mathbb{Z})$ defined by

$$c(f) = \sum_{l(w)=k} \Delta_w(f) Z_w \quad (\text{Note:}\Delta_w(f) \in \mathbb{Z})$$

- (Giambelli formula)
$$Z_w = c\left(\Delta_{w^{-1}w_0}\left(\frac{\prod_{\alpha\in\Delta^+}\alpha}{|W|}\right)\right)$$

Borel presentation for $H^*(E_6/T;\mathbb{Z})$

Theorem (Toda-Watanabe(1974))

$$H^*(E_6/T;\mathbb{Z}) = \frac{\mathbb{Z}[t_1,t_2,\dots,t_6,t_0,\gamma_3,\gamma_4]}{(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6,\rho_8,\rho_9,\rho_{12})} \quad (|t_i| = 2, |\gamma_i| = 2i)$$

$$\rho_1 = c_1 - 3t_0 \qquad \rho_2 = c_2 - 4t_0^2$$

$$\rho_3 = c_3 - 2\gamma_3 \qquad \rho_4 = c_4 + 2t_0^4 - 3\gamma_4$$

$$\rho_5 = c_5 - 3t_0\gamma_4 + 2t_0^2\gamma_3 \qquad \rho_6 = {\gamma_3}^2 + 2c_6 - 3t_0^2\gamma_4 + t_0^6$$

$$\rho_8 = 3{\gamma_4}^2 - 6t\gamma_3\gamma_4 - 9t_0^2c_6 + 15t_0^4\gamma_4 - 6t_0^5\gamma_3 - t_0^8$$

$$\rho_9 = 2c_6\gamma_3 - 3t_0^3c_6$$

$$\begin{aligned}\rho_{12} =& 3c_6^2 - 2{\gamma_4}^3 + 6t_0\gamma_3{\gamma_4}^2 + 3t_0^2c_6\gamma_4 + 5t_0^3c_6\gamma_3 - 15t_0^4{\gamma_4}^2 - 10t_0^6c_6 \\ &+ 19t_0^8\gamma_4 - 6t_0^9\gamma_3 - 2t_0^{12}\end{aligned}$$

$A(E_6)$

By Grothendieck's Theorem,

$$\begin{aligned} A^*(G) &= A^*(G/B)/(A^1(G/B)) \\ &= H^*(G/B;\mathbb{Z})/(H^2(G/B;\mathbb{Z})) \\ &= H^*(K/T;\mathbb{Z})/(H^2(K/T;\mathbb{Z})), \end{aligned}$$

where

$$H^*(K/T;\mathbb{Z}) = \mathbb{Z}[t_1,\ldots,t_l,t_0,\gamma_{i_1},\ldots]/(\rho_{j_1},\ldots)$$
$$H^2(K/T;\mathbb{Z}) = \mathbb{Z}\{t_1,\ldots,t_l,t_0\}$$

Therefore to obtain $A^*(G)$ from $H^*(K/T;\mathbb{Z})$, we simply put $t_i = 0, (0 \le i \le l)$ in Borel presentation.

$$H^*(E_6/T;\mathbb{Z})/(t_1,\ldots,t_6,t_0) = \mathbb{Z}[\gamma_3,\gamma_4]/(2\gamma_3, 3\gamma_4, \gamma_3^2, \gamma_4^3)$$
$$\text{(using the div. diff. op.)} = \mathbb{Z}[Z_{542}, Z_{6542}]/(2Z_{542}, 3Z_{6542}, Z_{542}^2, Z_{6542}^3)$$

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Main Theorems

$p : G \to G/B$: projection

Theorem (K-Nakagawa)

$$A(E_6) = \mathbb{Z}[X_3, X_4]/(2X_3, 3X_4, X_3^2, X_4^3)$$

$$X_3 = p^*(X_{w_0 s_5 s_4 s_2}) = \overline{B(w_0 s_5 s_4 s_2)B} \subset G$$
$$X_4 = p^*(X_{w_0 s_6 s_5 s_4 s_2}) = \overline{B(w_0 s_6 s_5 s_4 s_2)B} \subset G$$

$A(E_7)$

Theorem (K-Nakagawa)

$$A(E_7) = \mathbb{Z}[X_3, X_4, X_5, X_9] \\ /(2X_3, 3X_4, 2X_5, X_3^2, 2X_9, X_5^2, X_4^3, X_9^2)$$

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$$X_9 = p^*(X_{w_0 s_6 s_5 s_4 s_3 s_7 s_6 s_5 s_4 s_2}) = \overline{B(w_0 s_6 s_5 s_4 s_3 s_7 s_6 s_5 s_4 s_2)B} \subset G$$

$A(E_8)$

Proposition (K-Nakagawa)

$$A(E_8) = \mathbb{Z}[X_3, X_4, X_5, X_6, X_9, X_{10}, X_{15}] \Big/ \begin{pmatrix} 2X_3, 3X_4, 2X_5, 5X_6, 2X_9, X_5^2 - 3X_{10}, \\ X_4^3, 2X_{15}, X_9^2, 3X_{10}^2, X_3^8, \\ X_{15}^2 + X_{10}^3 + 2X_6^5 \end{pmatrix}$$

$X_i = p^*(\gamma_i)$ $(i = 3, 4, 5, 6, 9, 10, 15)$

Note: here X_i may not be the pull-back of a single Schubert variety but a linear combination of them.

Future Work

- Determine which Schubert classes belong to the decomposable ideal. (equivalently, find indecomposable Schubert classes)
- Find a presentation of a given Schubert class Z_w as a polynomial in a fixed set of ring generators.
  (Schubert polynomial of type $G_2, G_4, E_l(l = 6, 7, 8)$)
- Replace B with any parabolic subgroup P in the above problems.
  (Note: there is a ring monomorphism $H^*(G/P; \mathbb{Z}) \hookrightarrow H^*(G/B; \mathbb{Z})$ described in terms of Schubert presentation)

Thank you for listening