MOD 2 COHOMOLOGY OF $BLP\mathrm{Sp}(2m+1)$

SHIZUO KAJI AND AKIRA KONO

Abstract. We calculate the mod-2 cohomology ring of $BLP\mathrm{Sp}(2m+1)$, the classifying space of the free loop group over $P\mathrm{Sp}(2n-1)$ as an algebra over the mod-2 Steenrod algebra.

Let $\mathrm{Sp}(n)$ be the (compact) symplectic group, whose center consists of the two diagonal matrices $\Delta_n = \{\mathrm{diag}(1,\dots,1), \mathrm{diag}(-1,\dots,-1)\}$. The quotient group $P\mathrm{Sp}(n) = \mathrm{Sp}(n)/\Delta_n$ is called the projective symplectic group. The purpose of this note is to determine the mod-2 cohomology ring of $BP\mathrm{Sp}(n)$ with the action of the mod-2 Steenrod algebra and that of $LBP\mathrm{Sp}(n)$ when n is an odd integer. In [2], the mod-2 cohomology ring of $BP\mathrm{Sp}(2m+1)$ was given, however, the action of the mod-2 Steenrod algebra on it, which is necessary for the calculation for $LBP\mathrm{Sp}(n)$, is yet to be determined. We use an alternative method here.

Throughout this note, we denote by $H^*(X)$ the cohomology of a space X with the coefficient $\mathbb{F}_2$, the finite field of order two. For a Lie group homomorphism $\rho : G \to G'$, we use the same notation ρ for its induced map on the classifying spaces $BG \to BG'$, and ρ^* for the map on the mod-2 cohomology $H^*(BG') \to H^*(BG)$.

Our strategy is as follows: Recall that the inclusion $\mathrm{Sp}(1)^n \hookrightarrow \mathrm{Sp}(n)$ induces an isomorphism $H^*(B\mathrm{Sp}(n)) \cong H^*(B\mathrm{Sp}(1)^n)^{\Sigma_n}$, where the n-th symmetric group Σ_n acts on $\mathrm{Sp}(1)^n$ as the permutation of n-factors. Similarly we regard $H^*(BP\mathrm{Sp}(n))$ as an invariant ring in $H^*(B\mathrm{Sp}(1)^n/\Delta_n)$ and reduce its computation to that of the latter. For the structure of $H^*(B\mathrm{Sp}(1)^n/\Delta_n)$, we make use of the low rank isomorphisms $\mathrm{Sp}(1)/\Delta_1 \cong SO(3)$ and $\mathrm{Sp}(1)^2/\Delta_2 \cong SO(4)$.

1. Cohomology of $B\mathrm{Sp}(1)^n/\Delta_n$

First we observe the low rank cases when $n = 1, 2$. The following digram

$$\begin{array}{ccccc}
\Delta_1 & \longrightarrow & \mathrm{Sp}(1) & \longrightarrow & SO(3) \\
\| & & \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle i} \\
\Delta_2 & \longrightarrow & \mathrm{Sp}(1)\times\mathrm{Sp}(1) & \longrightarrow & SO(4),
\end{array}$$

where Δ_n is the diagonal and i the inclusion, gives rise to the diagram of fibrations:

$$\begin{array}{ccccc}
B\mathrm{Sp}(1) & \xrightarrow[\pi]{} & BSO(3) & \longrightarrow & K(\mathbb{F}_2, 2) \\
\downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle i} & & \| \\
B\mathrm{Sp}(1)\times B\mathrm{Sp}(1) & \xrightarrow[\pi]{} & BSO(4) & \longrightarrow & K(\mathbb{F}_2, 2),
\end{array}$$

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where $K(\mathbb{F}_2, 2)$ is regarded as the classifying space $B\Delta_n$. Looking at the cohomology level, we obtain

$$\begin{array}{ccccc}
\mathbb{F}_2[x] & \xleftarrow{\pi^*} & \mathbb{F}_2[w_2, w_3] & \longleftarrow & \mathbb{F}_2[u_2, \mathrm{Sq}^1 u_2, \mathrm{Sq}^2\mathrm{Sq}^1 u_2, \dots] \\
\uparrow{\scriptstyle \Delta^*} & & \uparrow{\scriptstyle i^*} & & \| \\
\mathbb{F}_2[x_1, x_2] & \xleftarrow{\pi^*} & \mathbb{F}_2[w_2, w_3, w_4] & \longleftarrow & \mathbb{F}_2[u_2, \mathrm{Sq}^1 u_2, \mathrm{Sq}^2\mathrm{Sq}^1 u_2, \dots],
\end{array}$$

where $x \in H^4(B\mathrm{Sp}(1))$, $u_2 \in H^2(K(\mathbb{F}_2, 2))$ are the generators and $w_i \in H^i(B\mathrm{SO}(n))$ is the i-th Stiefel-Whitney class. Since

$$\Delta^*(x_j) = x \ (j = 1, 2), \quad i^*(w_j) = \begin{cases} w_j & (j = 2, 3) \\ 0 & (j = 4) \end{cases}, \pi^*(w_4) \neq 0, \text{ and } \Delta^*\pi^*(w_4) = 0,$$

we have $\pi^*(w_4) = x_1 + x_2$.

Let $p_{1k} : \mathrm{Sp}(1)^n \to \mathrm{Sp}(1)^2$ be the projection to the first and the k-th factor, then we have

$$\begin{array}{ccccc}
B\mathrm{Sp}(1)^n & \xrightarrow{\pi} & B\mathrm{Sp}(1)^n/\Delta_n & \longrightarrow & K(\mathbb{F}_2, 2) \\
\downarrow{\scriptstyle p_{1k}} & & \downarrow{\scriptstyle \bar{p}_{1k}} & & \| \\
B\mathrm{Sp}(1)^2 & \xrightarrow{\pi} & B\mathrm{SO}(4) & \longrightarrow & K(\mathbb{F}_2, 2),
\end{array}$$

where

$$\begin{cases} p_{1k}^*(x_1) = x_1 \\ p_{1k}^*(x_2) = x_k. \end{cases}$$

From the Serre spectral sequence applied to the above diagram, we have

Lemma 1.1.

$$H^*(B\mathrm{Sp}(1)^n/\Delta_n) = \mathbb{F}_2[u_2', u_3', y_2, \dots, y_n],$$

where $u_2' = \bar{p}_{12}^*(w_2), u_3' = \bar{p}_{12}^*(w_3) = \mathrm{Sq}^1 u_2', y_i = \bar{p}_{1i}^*(w_4)\ (2 \le i \le n)$.

Furthermore, the Wu-fomura $\mathrm{Sq}^1 w_4 = 0, \mathrm{Sq}^2 w_4 = w_2 w_4$ *in* $H^*(B\mathrm{SO}(4))$ *yields* $\mathrm{Sq}^1 y_i = 0, \mathrm{Sq}^2 y_i = u_2' y_i$.

Now we concentrate on the case when $n = 2m + 1$ is odd. Then we can rephrase the above Lemma as:

$$H^*(B\mathrm{Sp}(1)^n/\Delta_n) = \mathbb{F}_2[u_2', u_3', d_1, \dots, d_n]/(d_1 + \cdots + d_n),$$
$$\mathrm{Sq}^1 u_2' = u_3', \mathrm{Sq}^1 d_i = 0, \mathrm{Sq}^2 d_i = u_2' d_i,$$

where $d_1 = y_2 + \cdots + y_n, d_i = d_1 + y_i\ (2 \le i \le n)$ so that Σ_n acts on d_i as permutation.

2. Cohomology of $BP\mathrm{Sp}(2m+1)$

In this section, we continue to assume $n = 2m+1$ is an odd integer. Consider the following diagram of fibrations with Σ_n-action:

$$\begin{array}{ccccc}
 & & K(\mathbb{F}_2, 2) & = & K(\mathbb{F}_2, 2) \\
 & & \uparrow & & \uparrow \\
\mathrm{Sp}(n)/\mathrm{Sp}(1)^n & \longrightarrow & B(\mathrm{Sp}(1)^n/\Delta_n) & \xrightarrow{\delta} & BP\mathrm{Sp}(n) \\
\| & & \uparrow{\scriptstyle \pi} & & \uparrow \\
\mathrm{Sp}(n)/\mathrm{Sp}(1)^n & \longrightarrow & B\mathrm{Sp}(1)^n & \longrightarrow & B\mathrm{Sp}(n).
\end{array}$$

Since the Serre spectral sequences for the middle row fibration collapses, δ^* maps injectively into $H^*(B(\mathrm{Sp}(1)^n/\Delta_n))^{\Sigma_n} \cong \mathbb{F}_2[u_2', u_3', e_2', \dots, e_n']$, where e_i' is the i-th elementary symmetric function on $d_1, \dots, d_n$. (Note that $e_1' = 0$.) In fact, δ^* is an isomorphism $H^*(BP\mathrm{Sp}(n)) \cong H^*(B(\mathrm{Sp}(1)^n/\Delta_n))^{\Sigma_n}$ by the Poincaré polynomial argument. To sum up, we have on the cohomology

$$\begin{array}{ccccc}
 & & \mathbb{F}_2[u_2, \mathrm{Sq}^1 u_2, \mathrm{Sq}^2\mathrm{Sq}^1 u_2, \dots] & = & \mathbb{F}_2[u_2, \mathrm{Sq}^1 u_2, \mathrm{Sq}^2\mathrm{Sq}^1 u_2, \dots] \\
 & & \downarrow & & \downarrow \\
\mathbb{F}_2[d_1, \dots, d_n]/(e_1', \dots, e_n') & \longleftarrow & \mathbb{F}_2[u_2', u_3', d_1, \dots, d_n]/(d_1 + \cdots + d_n) & \hookleftarrow & \mathbb{F}_2[u_2', u_3', e_2', \dots, e_n'] \\
\| & & \downarrow{\scriptstyle \pi^*} & & \downarrow \\
\mathbb{F}_2[x_1, \dots, x_n]/(e_1, \dots, e_n) & \longleftarrow & \mathbb{F}_2[x_1, \dots, x_n] & \hookleftarrow & \mathbb{F}_2[e_1, \dots, e_n],
\end{array}$$

where e_i is the i-th elementary symmetric function on $x_1, \dots, x_n$. We already know that $\mathrm{Sq}^1 u_2' = u_3'$ and $\mathrm{Sq}^2 u_3' = 0$ and hence we investigate the action of the Steenrod squares on e_i' $(2 \leq i \leq n)$. Since $\mathrm{Sq}^1 d_i = 0, \mathrm{Sq}^2 d_i = u_2' d_i$, we have $\mathrm{Sq}^1 e_i' = 0, \mathrm{Sq}^2 e_i' = \begin{cases} u_2' e_i' & (i : \text{odd}) \\ 0 & (i : \text{even}) \end{cases}$. By the Adem relation, we have

$$\mathrm{Sq}^{4j+1} = \mathrm{Sq}^1\mathrm{Sq}^{4j}, \mathrm{Sq}^{4j+2} = \mathrm{Sq}^2\mathrm{Sq}^{4j} + \mathrm{Sq}^1\mathrm{Sq}^{4j}\mathrm{Sq}^1, \mathrm{Sq}^{4j+3} = \mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^{4j}.$$

Therefore to complete the computation, we have only to show

Lemma 2.1.

$$\mathrm{Sq}^{4j} e_i' = \sum_{h=0}^{\min(i-j,j)} \binom{i-j+h}{2h} u_2'^{2h} \sum_{k=0}^{j-h} \binom{i-k-1}{j-h-k} e_{i+j-h-k}' e_k' \quad (0 \leq j < i),$$

where $e_1' = e_k' = 0$ $(k > n)$.

Proof. For a monomial $d_{l_1} \cdots d_{l_i}$, the Cartan formula computes

$$\mathrm{Sq}^{4j}(d_{l_1} \cdots d_{l_i}) = \sum_{h=0}^{\min(i-j,j)} \sum_{\epsilon_1 + \cdots + \epsilon_i = i+j-h, \epsilon_1, \dots, \epsilon_i \in \{1,2\}} \binom{i-j+h}{2h} u_2'^{2h} d_{l_1}^{\epsilon_1} \cdots d_{l_i}^{\epsilon_i} \quad (0 \leq j < i).$$

Since $e'_i = \sum_{\{l_1,\dots,l_i\}\subset\{1,\dots,n\}} d_{l_1}\cdots d_{l_i}$, applying the following general property of monomial symmetric function yields the statement:

$$\sum_{\{l_1,\dots,l_i\}\subset\{1,\dots,n\}} \sum_{\epsilon_1+\dots+\epsilon_i=q,\epsilon_1,\dots,\epsilon_i\in\{1,2\}} d_{l_1}^{\epsilon_1}\cdots d_{l_i}^{\epsilon_i} \equiv \sum_{k=0}^{q-i} \binom{i-k-1}{q-i-k} e'_{q-k}e'_k \bmod 2, \qquad (i \le q < 2i).$$

□

So far we obtained

Theorem 2.2. *When n is an odd interger,*

$$H^*(BP\mathrm{Sp}(n)) \cong \mathbb{Z}_2[u'_2, u'_3, e'_2, \dots, e'_n],$$

where $\mathrm{Sq}^1 u'_2 = u'_3$, $\mathrm{Sq}^1 u'_3 = \mathrm{Sq}^2 u'_3 = 0$, $\mathrm{Sq}^1 e'_i = 0$, $\mathrm{Sq}^2 e'_i = \begin{cases} u'_2 e'_i & (i : odd) \\ 0 & (i : even) \end{cases}$, *and*

$$\mathrm{Sq}^{4j} e'_i = \sum_{h=0}^{\min(i-j,j)} \binom{i-j+h}{2h} u'^{2h}_2 \sum_{k=0}^{j-h} \binom{i-k-1}{j-h-k} e'_{i+j-h-k} e'_k \quad (0 \le j < i),$$

with $e'_1 = e'_k = 0$ $(k > n)$.

3. Classifying space of the free loop group

Using the result of the previous section and the method developed in [1], we now proceed to the calculation of $H^*(BLP\mathrm{Sp}(n))$, where $LP\mathrm{Sp}(n)$ is the group of free loops on $P\mathrm{Sp}(n)$. First, note that there is a well-known homotopy equivalence $BLG \simeq LBG$ for a Lie group G, and hence $H^*(BLP\mathrm{Sp}(n)) \cong H^*(LBP\mathrm{Sp}(n))$.

Let ev be the evaluation map

$$\begin{array}{rcl} S^1 \times LBP\mathrm{Sp}(n) & \to & BP\mathrm{Sp}(n) \\ (t, l) & \mapsto & l(t). \end{array}$$

Then a map $\sigma : H^*(BP\mathrm{Sp}(n)) \to H^{*-1}(LBP\mathrm{Sp}(n))$ is defined by the following equation:

$$ev^*(x) = s \otimes \sigma(x) + 1 \otimes x, \quad (x \in H^*(BP\mathrm{Sp}(n))),$$

where $s \in H^1(S^1)$ is the generator. By [1, Prop. 3],

- $H^*(LBP\mathrm{Sp}(n)) \cong \mathbb{F}_2[u'_2, u'_3, e'_2, \dots, e'_n] \otimes \Delta(\sigma(u'_2), \sigma(u'_3), \sigma(e'_2), \dots, \sigma(e'_n))$, where we use the same symbol for the cohomology classes and their pull-back via the evaluation at 0: $LBP\mathrm{Sp}(n) \to BP\mathrm{Sp}(n)$,
- the action of the Steenrod squares commutes with σ,
- and σ is a derivation: $\sigma(xy) = \sigma(x)y + x\sigma(y)$ for $x, y \in H^*(BP\mathrm{Sp}(n))$

The ring structure of the simple system part is determined by looking at the action of Sq on the generators:

$$
\begin{aligned}
\sigma(u_2')^2 &= \mathrm{Sq}^1\sigma(u_2') = \sigma(\mathrm{Sq}^1 u_2') = \sigma(u_3') \\
\sigma(u_3')^2 &= \mathrm{Sq}^2\sigma(u_3') = \sigma(\mathrm{Sq}^2 u_3') = 0 \\
\sigma(e_i')^2 &= \mathrm{Sq}^{4i-1}\sigma(e_i') = \sigma(\mathrm{Sq}^{4i-1} e_i') = \sigma(\mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^{4(i-1)} e_i') \\
&= \sigma\left(\mathrm{Sq}^1\mathrm{Sq}^2\left(\sum_{k=0}^{i-1} e_{2i-1-k}' e_k' + {u_2'}^2 \sum_{k=0}^{i-2}(i-1-k) e_{2i-2-k}' e_k'\right)\right) \\
&= \sigma(u_3' \sum_{k=0}^{i-1} e_{2i-1-k}' e_k') = \sigma(u_3') \sum_{k=0}^{i-1} e_{2i-1-k}' e_k' + u_3' \sum_{k=0}^{i-1} \left(\sigma(e_{2i-1-k}') e_k' + e_{2i-1-k}' \sigma(e_k')\right)
\end{aligned}
$$

Put $v_2 = \sigma(u_2')$ and $f_i = \sigma(e_i')$ ($f_0 = f_1 = 0$) then

Theorem 3.1. *When n is an odd interger,*

$$
H^*(BLP\mathrm{Sp}(n)) \cong H^*(LBP\mathrm{Sp}(n)) \cong \frac{\mathbb{F}_2[u_2', u_3', e_2', \ldots, e_n', v_2, f_2, \ldots, f_n]}{\left(v_2^4 = 0, f_i^2 = v_2^2 \sum_{k=0}^{i-1} e_{2i-1-k}' e_k' + u_3' \sum_{k=0}^{i-1}\left(f_{2i-1-k} e_k' + e_{2i-1-k}' f_k\right)\right)},
$$

where $|u_2'| = 2, |u_3'| = 3, |e_i'| = 4i, |v_2| = 1$, *and* $|f_i| = 4i-1$.

References

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Department of Mathematical Sciences, Faculty of Science
Yamaguchi University
1677-1, Yoshida, Yamaguchi 753-8512, Japan
E-mail address: skaji@yamaguchi-u.ac.jp

Department of Mathematics
Kyoto University
Kyoto 606-8502, Japan
E-mail address: kono@math.kyoto-u.ac.jp