Mod 2 cohomology of some low rank 2-local finite groups

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Outline

- Introduction
- Main result
- List of computational results

- G: compact simple, simply-connected Lie group
- p: prime
- q: prime power, coprime to p
- $\mathbf{F}_q$: Finite field with order q
- $X_p^{\wedge}$: Bousfield-Kan p-completion of a space X
- $G(q)$: Chevalley group of type G over $\mathbf{F}_q$

Some finite groups occur as matrix groups with finite field coefficients. Chevalley constructed a series of finite groups of Lie type, including exceptional ones, over finite fields.

General Goal

Compute the cohomology of a Chevalley group $G(q)$

- $H^*(GL(q); \mathbf{F}_p)$ was calculated by Quillen (1972)
- Fiedorowicz-Priddy, Mitchell, Kleinerman, Shapiro, ...

Goal Today

Compute the mod 2 cohomology $H^*(G(q); \mathbf{F}_2)$ as a ring over the Steenrod algebra, when $H^*(BG; \mathbf{F}_2)$ is a polynomial algebra.

- $H^*(G(q); \mathbf{F}_2)$ for $G = SU_n, Sp_n, G_2$ was calculated by Kishimoto-Kono
- $H^*(G(q); \mathbf{F}_2)$ for $G = Sol, G_2$ was calculated by Grbić
- Remaining cases are $G = Spin_7, Spin_8, Spin_9, F_4$

Strategy

- The group cohomology $H^*(G(q); \mathbf{F}_p)$ is isomorphic to the ordinary cohomology of the classifying space $H^*(BG(q); \mathbf{F}_p)$
- $BG(q)$ is p-good, i.e. $H^*(BG(q); \mathbf{F}_p) \cong H^*(BG(q)^\wedge_p; \mathbf{F}_p)$
- There is an homotopy equivalence $BG(q)^\wedge_p \simeq \mathbf{L}_{\psi^q} BG^\wedge_p$
- $\mathbf{L}_{\psi^q} BG$ is close to the free loop space LBG so that we can calculate its cohomology

Main Theorem

Theorem

Let q is an odd prime power and G be either $Spin_7, Spin_8, Spin_9$ or F_4.
There are isomorphisms of algebras over the mod 2 Steenrod algebra.

$$H^*(BLG;\mathbf{F}_2) \cong H^*(LBG;\mathbf{F}_2) \cong H^*(BG(q);\mathbf{F}_2)$$

$$H^*(BLDI(4);\mathbf{F}_2) \cong H^*(LBDI(4);\mathbf{F}_2) \cong H^*(BSol(q);\mathbf{F}_2)$$

Remark: $BLG \simeq LBG$

The space of free loops $LG = \mathrm{Map}(S^1, G)$ over G equipped with the pointwise multiplication is called the free loop group on G.
Let B_nG be the n-fold join of G divide by the diagonal action of G, then $BG \simeq \mathrm{colim}_n B_nG$.
Define a map

$$\begin{array}{rcl} h_n : B_nLG & \to & LB_nG \\ (t_1l_1, t_2l_2, \ldots, t_nl_n) & \mapsto & (s \to (t_1l_1(s), t_2l_2(s), \ldots, t_nl_n(s))\,, \end{array}$$

where $l_i \in LG$ and $0 \leq t_i \leq 1, \sum t_i = 1$.
Then it induces a homotopy equivalence $h_\infty : BLG \simeq LBG$.

Remark: $BDI(4)$

Theorem (Dickson)

Let V_n be a n dimensional vector space over $\mathbf{F}_2$, acted on by the general linear group $GL(V_n)$. Then

$$S(V_n)^{GL(V_n)} = \mathbf{F}_2[x_0, \ldots, x_{n-1}], \qquad |x_i| = 2^n - 2^i$$

Can the Dickson invariant $S(V_n)^{GL(V_n)}$ be realized as the cohomology of a space ?

FACT

- $H^*(RP^\infty) = S(V_1)^{GL(V_1)} = \mathbf{F}_2[y_1]$
- $H^*(BSO(3)) = S(V_2)^{GL(V_2)} = \mathbf{F}_2[y_2, y_3]$
- $H^*(BG_2) = S(V_3)^{GL(V_3)} = \mathbf{F}_2[y_4, y_6, y_7]$
- $H^*(BDI(4)) = S(V_4)^{GL(V_4)} = \mathbf{F}_2[y_8, y_{12}, y_{14}, y_{15}]$,
  where $BDI(4)$ is the Dwyer-Wilkerson space
- for $n > 4$ there are no such spaces

Remark: $BSol(q)$

There is a family of 2-local finite groups associated to $BDI(4)$ called *Solomon's group*.

FACT

- There is a saturated fusion system $\mathcal{F}_{Sol(q)}$ over the Sylow 2-group of $Spin_7(q)$ with a unique centric linking system associated to it [Levi-Oliver (2002)]
- $\mathcal{F}_{Sol(q)}$ cannot be realized by any finite group [Solomon (1974)]
- Its classifying space is homotopy equivalent to $BSol(q)(\simeq \mathbf{L}_{\psi^q} BDI(4))$, the space first considered by Benson (1998)

Twisted loop space

In the following, X denotes any simply-connected space, although we have always $X = BG$ in mind.
Let $f : X \to X$ be a self-equivalence of X. *The homotopy fixed points space* or *the twisted loop space* $\mathbf{L}_f X$ of f is defined as the following pullback:

$$\begin{array}{ccc} \mathbf{L}_f X & \longrightarrow & X^{[0,1]} \\ \downarrow & & \downarrow{\scriptstyle e_0 \times e_1} \\ X & \xrightarrow{1 \times f} & X \times X \end{array}$$

where e_i $(i = 0, 1)$ is the evaluation at i.

- $\mathbf{L}_f X = \{l : [0,1] \to X | f(l(0)) = l(1)\}$
- if $f = 1$ the identity map, then $\mathbf{L}_1 X$ is the usual free loop space LX

Unstable Adams operation

Theorem (Wilkerson(1974))

There is an automorphism of $BG^{\wedge}_p$ called *unstable Adams operation of exponent* q

$$\psi^q : BG^{\wedge}_p \to BG^{\wedge}_p,$$

with the following property

$$\begin{array}{rcl} (\psi^q)^* : H^{2n}(BG^{\wedge}_p; \mathbf{F}^{\wedge}_p) & \longrightarrow & H^{2n}(BG^{\wedge}_p; \mathbf{F}^{\wedge}_p) \\ x & \longmapsto & q^n x \end{array}$$

The following Theorem is crucial for the study of p-local structure of finite groups of Lie type.

Theorem (Friedlander(1982))

$$BG(q)^{\wedge}_p \simeq \mathbf{L}_{\psi^q} BG^{\wedge}_p$$

What is difficult ?

From now on, we concentrate on the case of $p = 2$ and

$$H^*(BG; \mathbf{F}_2) \simeq \mathbf{F}_2[x_1, \dots, x_l].$$

- We may use the Eilenberg-Moore spectral sequence for the pullback square of $\mathbf{L}_{\psi^q} BG_p^{\wedge}$
- Then, $H^*(\mathbf{L}_{\psi^q} BG_p^{\wedge}) \simeq \mathbf{F}_2[x_1, \dots, x_l] \otimes \Delta[x'_1, \dots, x'_l], |x'_i| = |x_i| - 1$
- However we cannot determine the ring structure for the simple system part
- Kishimoto and Kono gave an alternative method, which we use today

Cohomology of free loop space

First, we consider the case of a free loop space.
We have the evaluation map:

$$\begin{array}{rcl} ev : S^1 \times LX & \longrightarrow & X \\ (t,l) & \longmapsto & l(t) \end{array}$$

Let $s \in H^1(S^1)$ be a generator. Then we have the following map:

$$\begin{array}{rcl} \sigma : H^*(X) & \rightarrow & H^{*-1}(LX), \\ ev^*(x) & = & s \otimes \sigma(x) + 1 \otimes e_0^*(x) \in H^*(S^1) \otimes H^*(LX). \end{array}$$

Lemma

Assume that $H^*(X) \cong \mathbf{F}_2[x_1, \dots, x_l]$. Then

- $H^*(LX) \cong \mathbf{F}_2[e_0^*(x_1), e_0^*(x_2), \dots, e_0^*(x_l)] \otimes \Delta(\sigma(x_1), \sigma(x_2), \dots, \sigma(x_l))$
- σ commutes with the action of Steenrod operations
- $\sigma(xy) = \sigma(x)e_0^*(y) + e_0^*(x)\sigma(y)$ for $x, y \in H^*(X)$

$$H^*(LBG) \cong \mathbf{F}_2[e_0^*(x_1), e_0^*(x_2), \ldots, e_0^*(x_l)] \otimes \Delta(\sigma(x_1), \sigma(x_2), \ldots, \sigma(x_l))$$
$$e_0^* : H^*(BG) \to H^*(LBG), \qquad \sigma : H^*(BG) \to H^{*-1}(LBG)$$

- the generators in the simple system part are related to those in the polynomial part,
- which in turn are related to those of $H^*(BG)$.
- That is, the algebra structure and the action on the Steenrod algebras of $H^*(LBG)$ is completely determined by those on the generators $x_1, \ldots, x_l \in H^*(BG)$.

$H^*(BLSpin(7))$

Proposition

$$\begin{aligned} H^*(BLSpin(7)) &\cong H^*(LBSpin(7)) \\ &\cong \mathbf{F}_2[v_4, v_6, v_7, v_8, y_3, y_5, y_7]/I \quad (|v_i| = i, |y_i| = i), \\ & \quad v_i = e_0^*(w_i), \quad y_{i-1} = \sigma(w_i) \end{aligned}$$

where I is the ideal generated by

$$\{y_5^2 + y_3^2 v_4 + y_3 v_7, y_3^4 + y_3^2 v_6 + y_5 v_7, y_7^2 + y_3^2 v_8 + y_7 v_7\}.$$

	v_4	v_6	v_7	v_8	y_3	y_5	y_7
Sq^1	0	v_7	0	0	0	y_3^2	0
Sq^2	v_6	0	0	0	y_5	0	0
Sq^4	v_4^2	$v_4 v_6$	$v_4 v_7$	$v_4 v_8$	0	$y_3 v_6 + y_5 v_4$	$y_3 v_8 + y_7 v_4$

$H^*(BSpin(7)) \cong \mathbf{F}_2[w_4, w_6, w_7, w_8]$

$$\begin{array}{ccccc} & w_4 & w_6 & w_7 & w_8 \\ Sq^1 & 0 & w_7 & 0 & 0 \\ Sq^2 & w_6 & 0 & 0 & 0 \\ Sq^4 & w_4^2 & w_4w_6 & w_4w_7 & w_4w_8. \end{array}$$

- $H^*(LBSpin(7)) \cong$ $\mathbf{F}_2[e_0^*(w_4), e_0^*(w_6), e_0^*(w_7), e_0^*(w_8)] \otimes \Delta(\sigma(w_4), \sigma(w_6), \sigma(w_7), \sigma(w_8))$
- σ commutes with the action of Steenrod operations;

$$Sq^j\sigma(w_i) = \sigma(Sq^j w_i)$$

- For ring structure, we only have to determine $\sigma(w_i)^2$, which is equal to $Sq^{i-1}\sigma(w_i) = \sigma(Sq^{i-1}w_i)$
- $\sigma(xy) = \sigma(x)e_0^*(y) + e_0^*(x)\sigma(y)$

Steenrod squares

$$Sq^n : H^k(X; \mathbf{F}_2) \to H^{k+n}(X; \mathbf{F}_2)$$

satisfy the following equations:

1. $Sq^0 = 1$
2. $Sq^n(x) = x^2$ if $|x| = n$
3. $Sq^n(x) = 0$ if $|x| < n$
4. $Sq^n(xy) = \sum_{i+j=n} Sq^i(x)Sq^j(y)$ (Cartan formula)
5. $Sq^iSq^j = \sum_{0 \le l \le \lfloor i/2 \rfloor} \binom{j-l-1}{i-2l} Sq^{i+j-l}Sq^l$ (Adem relation)

Easy exercise using above relations completes the calculation.
For example, we calculate y_5^2 in

$$H^*(LBSpin(7)) \cong \mathbf{F}_2[v_4, v_6, v_7, v_8] \otimes \Delta(y_3, y_5, y_6, y_7).$$

$$y_5^2 = Sq^5 y_5 = Sq^1 Sq^4 y_5 = Sq^1 Sq^4 \sigma(v_6)$$

$$= Sq^1 \sigma(Sq^4 v_6) = Sq^1 \sigma(v_4 v_6)$$

$$= Sq^1(v_4 y_5 + y_3 v_6) = v_4 Sq^1(y_5) + y_3 Sq^1(v_6) = v_4 y_3^2 + y_3 v_7$$

Cohomology of $\mathbf{L}_f X$

Kishimoto and Kono developed a method to compute the cohomology of $\mathbf{L}_f X$, which is parallel to the above case of LX.
First we need to introduce another space $\mathbf{T}_f X$ on X.
For a self-equivalence $f : X \to X$,

- *the homotopy quotient space* or *the twisted tube* $\mathbf{T}_f X$ of X is defined by

$$\mathbf{T}_f X = \frac{[0,1] \times X}{(0,x) \simeq (1, f(x))}$$

- a canonical inclusion $\iota : X \hookrightarrow \mathbf{T}_f X$, where $\iota(x) = (0, x)$.
- When $f = 1$ the identity map, then $\mathbf{T}_1 X = S^1 \times X$.
- When $f^* = 1$, we have a short exact sequence:

$$0 \to H^{n-1}(X) \xrightarrow{\delta} H^n(\mathbf{T}_f X) \xrightarrow{\iota^*} H^n(X) \to 0$$

Let $i : \mathbf{L}_f X \to L\mathbf{T}_f X$ be the following map

$$\begin{array}{rcl} \mathbf{L}_f X & \to & L\mathbf{T}_f X \\ l & \mapsto & t \mapsto (t, l(t)). \end{array}$$

Then *the twisted cohomology suspension* is defined by

$$\sigma_f : H^*(\mathbf{T}_f X) \xrightarrow{\sigma} H^{*-1}(L\mathbf{T}_f X)) \xrightarrow{i^*} H^{*-1}(\mathbf{L}_f X).$$

If we have a section $r : H^*(X) \to H^*(\mathbf{T}_f X)$ of ι^*, which commutes with the action of the Steenrod algebras, we can define another map

$$\sigma'_f = \sigma_f \circ r : H^*(X) \to H^{*-1}(\mathbf{L}_f X)$$

which satisfies the same properties as $\sigma : H^*(X) \to H^{*-1}(LX)$.

Proposition (Kishimoto-Kono)

Assume that

- $H^*(X) \cong \mathbf{F}_2[x_1, \dots, x_l]$
- There is a section $r : H^*(X) \to H^*(\mathbf{T}_f X)$ of ι^* which commutes with the action of the Steenrod algebras

then we have

- $H^*(\mathbf{L}_f X) \cong \mathbf{F}_2[e_0^*(x_1), e_0^*(x_2), \dots, e_0^*(x_l)] \otimes \Delta(\sigma'_f(x_1), \sigma'_f(x_2), \dots, \sigma'_f(x_l))$
- σ'_f commutes with the action of Steenrod operations
- $\sigma'_f(xy) = \sigma'_f(x)e_0^*(y) + e_0^*(x)\sigma'_f(y)$ for $x, y \in H^*(X)$

Corollary

Under the same hypothesis, $H^*(\mathbf{L}_f X) \cong H^*(LX)$ as algebras over the Steenrod algebra.

We apply this Proposition to the case of $X = BG_2^\wedge$ for

$$G = Spin(7), Spin(8), Spin(9), F_4, DI(4),$$

by constructing a section r in ad-hoc ways.

$H^*(BSpin_7(q); \mathbf{F}_2)$

Proposition

$$\begin{aligned} H^*(BSpin_7(q)) &\cong H^*(BLSpin(7)) \\ &\cong \mathbf{F}_2[v_4, v_6, v_7, v_8, y_3, y_5, y_7]/I \end{aligned}$$

$$\begin{aligned} I = \{ & y_5^2 + y_3^2 v_4 + y_3 v_7, y_3^4 + y_3^2 v_6 + y_5 v_7, \\ & y_7^2 + y_3^2 v_8 + y_7 v_7 \} \end{aligned}$$

	v_4	v_6	v_7	v_8	y_3	y_5	y_7
Sq^1	0	v_7	0	0	0	y_3^2	0
Sq^2	v_6	0	0	0	y_5	0	0
Sq^4	v_4^2	$v_4 v_6$	$v_4 v_7$	$v_4 v_8$	0	$y_3 v_6 + y_5 v_4$	$y_3 v_8 + y_7 v_4$

$H^*(BSpin_8(q); \mathbf{F}_2)$

Proposition

$$\begin{aligned} H^*(BSpin_8(q); \mathbf{F}_2) &\cong H^*(LBSpin(8); \mathbf{F}_2) \\ &\cong \mathbf{F}_2[v_4, v_6, v_7, v_8, f_8, y_3, y_5, y_7, z_7]/I \end{aligned}$$

$$\begin{aligned} I = \{ & y_5^2 + y_3^2 v_4 + y_3 v_7, y_3^4 + y_3^2 v_6 + y_5 v_7, \\ & y_7^2 + y_3^2 v_8 + y_7 v_7, z_7^2 + y_3^2 f_8 + z_7 v_7 \} \end{aligned}$$

	v_4	v_6	v_7	v_8	f_8	y_3	y_5	y_7	z_7
Sq^1	0	v_7	0	0	0	0	y_3^2	0	0
Sq^2	v_6	0	0	0	0	y_5	0	0	0
Sq^4	v_4^2	$v_4 v_6$	$v_4 v_7$	$v_4 v_8$	$v_4 f_8$	0	$y_3 v_6 + y_5 v_4$	$y_3 v_8 + y_7 v_4$	$y_3 f_8 + z_7 v_4$

$H^*(BSpin_9(q); \mathbf{F}_2)$

Proposition

$$\begin{aligned} H^*(BSpin_9(q); \mathbf{F}_2) &\cong H^*(LBSpin(9); \mathbf{F}_2) \\ &\cong \mathbf{F}_2[v_4, v_6, v_7, v_8, f_{16}, y_3, y_5, y_7, z_{15}]/I \end{aligned}$$

$$\begin{aligned} I = \{ & y_5^2 + y_3 v_7 + v_4 y_3^2, y_3^4 + y_3^2 v_6 + y_5 v_7, \\ & y_7^2 + y_3^2 v_8 + y_7 v_7, z_{15}^2 + v_7 v_8 z_{15} + v_7 y_7 f_{16} + y_3^2 v_8 f_{16} \} \end{aligned}$$

	v_4	v_6	v_7	v_8	f_{16}	y_3	y_5	y_7	z_{15}
Sq^1	0	v_7	0	0	0	0	y_3^2	0	0
Sq^2	v_6	0	0	0	0	y_5	0	0	0
Sq^4	v_4^2	$v_4 v_6$	$v_4 v_7$	$v_4 v_8$	0	0	$y_3 v_6 + y_5 v_4$	$y_3 v_8 + y_7 v_4$	0
Sq^8	0	0	0	v_8^2	$v_8 f_{16} + v_4^2 f_{16}$	0	0	0	J_1

where $J_1 = y_7 f_{16} + v_8 z_{15} + v_4^2 z_{15}$.

$H^*(BF_4(q);\mathbf{F}_2)$

Proposition

$$H^*(BF_4(q);\mathbf{F}_2) \cong H^*(LBF_4;\mathbf{F}_2) \cong \mathbf{F}_2[v_4, v_6, v_7, v_{16}, v_{24}, y_3, y_5, y_{15}, y_{23}]/I$$

$$I = \{y_5^2 + y_3v_7 + v_4y_3^2, y_3^4 + v_6y_3^2 + y_5v_7, y_{15}^2 + v_7y_{23} + v_{24}y_3^2, y_{23}^2 + y_3^2v_{16}v_{24} + v_7v_{24}y_{15} + v_7v_{16}y_{23}\}$$

	v_4	v_6	v_7	v_{16}	v_{24}
Sq^1	0	v_7	0	0	0
Sq^2	v_6	0	0	0	0
Sq^4	v_4^2	v_4v_6	v_4v_7	0	v_4v_{24}
Sq^8	0	0	0	$v_{24} + v_4^2v_{16}$	$v_4^2v_{24}$
Sq^{16}	0	0	0	v_{16}^2	$v_{16}v_{24} + v_4v_6^2v_{24}$

	y_3	y_5	y_{15}	y_{23}
Sq^1	0	y_3^2	0	0
Sq^2	y_5	0	0	0
Sq^4	0	$y_3v_6 + v_4y_5$	0	$y_3v_{24} + v_4y_{23}$
Sq^8	0	0	$y_{23} + v_4^2y_{15}$	$v_4^2y_{23}$
Sq^{16}	0	0	0	$v_{24}y_{15} + v_{16}y_{23} + y_3v_6^2v_{24} + v_4v_6^2y_{23}$

$H^*(BSol(q); \mathbf{F}_2)$

Proposition

$$\begin{aligned} H^*(BSol(q); \mathbf{F}_2) &\cong H^*(LBDI(4); \mathbf{F}_2) \\ &\cong \mathbf{F}_2[v_8, v_{12}, v_{14}, v_{15}, y_7, y_{11}, y_{13}]/I \end{aligned}$$

$$\begin{aligned} I = \{ & y_{11}^2 + y_7 v_{15} + v_8 y_7^2, \\ & y_{13}^2 + y_{11} v_{15} + v_{12} y_7^2, \\ & y_7^4 + y_{13} v_{15} + v_{14} y_7^2 \} \end{aligned}$$

	v_8	v_{12}	v_{14}	v_{15}	y_7	y_{11}	y_{13}
Sq^1	0	0	v_{15}	0	0	0	y_7^2
Sq^2	0	v_{14}	0	0	0	y_{13}	0
Sq^4	v_{12}	0	0	0	y_{11}	0	0
Sq^8	v_8^2	$v_8 v_{12}$	$v_8 v_{14}$	$v_8 v_{15}$	0	$v_8 y_{11} + y_7 v_{12}$	$v_8 y_{13} + y_7 v_{14}$

Future Work

Question

When $H^*(BLG; \mathbf{F}_p)$ and $H^*(BG(q); \mathbf{F}_p)$ is isomorphic as an algebra over $\mathcal{A}_p$
$\Rightarrow$ give an example of a concrete computation for $H^*(BG(q); \mathbf{F}_p)$ when $H^*(BG; \mathbf{F}_p)$ is NOT polynomial.

End of the talk

Thank you for listening.