A concise proof of the interval decomposition of persistent homology

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Advert / Motivation

We are finishing an introductory book on persistent homology (in Japanese)

池祐一・エスカラ エマソン ガウ・大林一平・鍛冶静雄 著

位相的データ解析から構造発見へ パーシステントホモロジーを中心に

サイエンス社AI/データサイエンスシリーズ 近刊

The topic today originates from a question we had during the preparation of the book.

What is the quickest way to introduce the fundamental structure theorem of persistent homology?

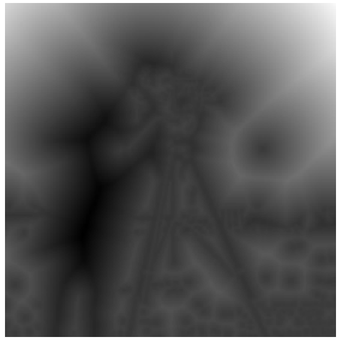
Persistent homology as a feature extractor

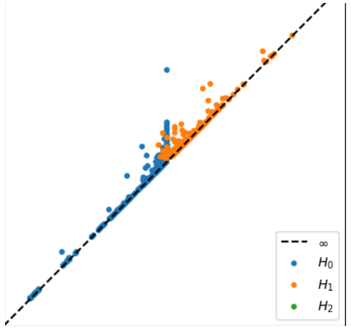
For $K_* = K_1 \subset K_2 \subset \cdots \subset K_M$: finite sequence of finite cell complexes,

its persistent homology $PH(K_*; k)$ with coefficients in a field k is *represented* by a multi-set of points of the form $(b, d) \in \{1,2, \ldots, M, \infty\}^2$

This presentation of $PH(K_*; k)$ as a *persistence diagram* or *barcode* makes persistent homology powerful machinery as a feature extractor of data

Data → Filtered complex K_* → Data feature $PH(K_*; k)$ → Classification Regression etc.





This example is from "**Tutorial on Topological Data Analysis**", which introduces TDA packages for Python.
Google **"shizuo kaji tutorial"**

Persistent homology

$$K_1 \subset K_2 \subset \cdots \subset K_M$$

$H_*(\quad ; k)$

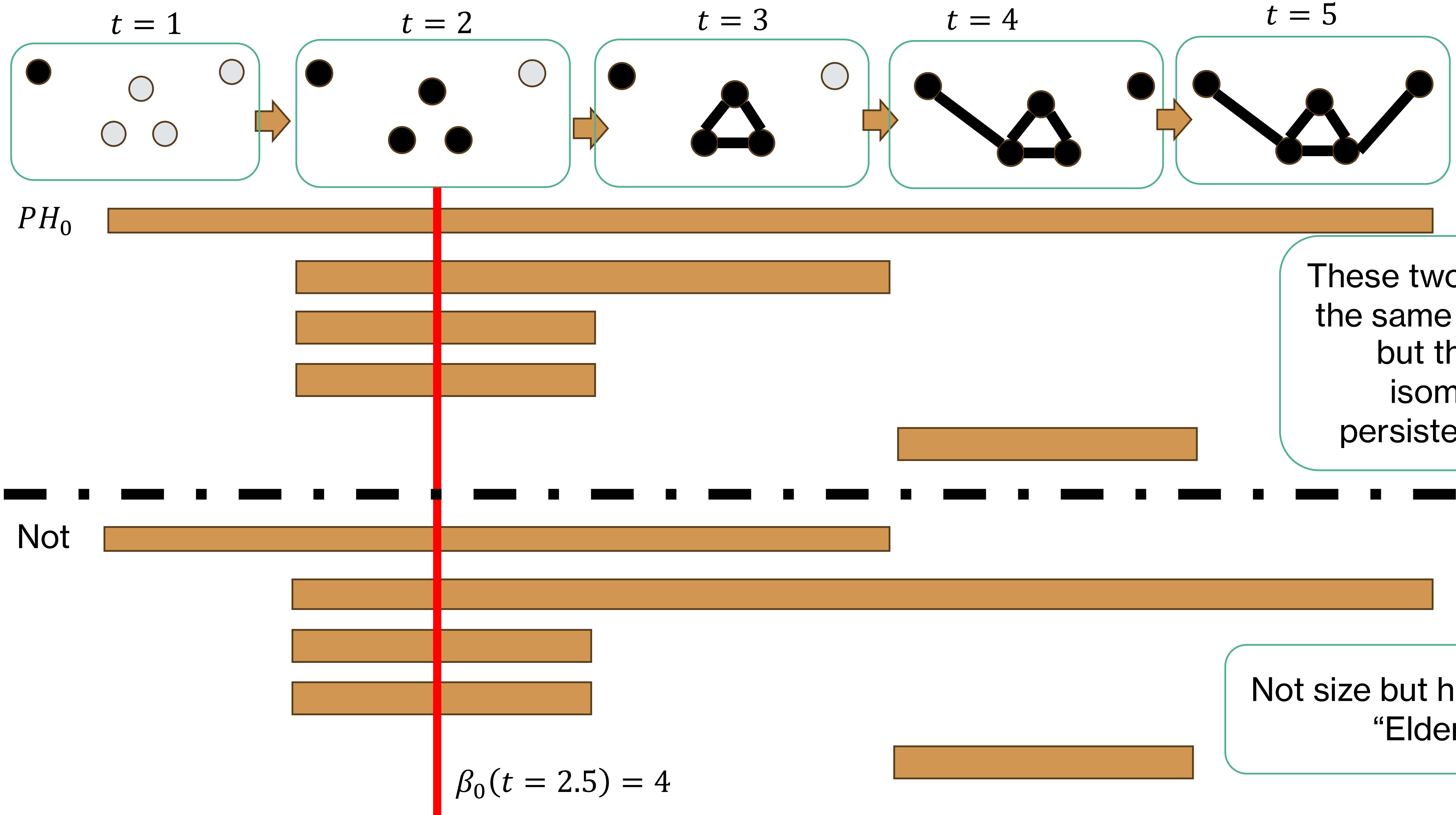
Sequence of "computable" objects

$$H_*(K_1; k) \to H_*(K_2; k) \to \cdots \to H_*(K_M; k)$$

The algebraic structure of the latter is more tractable
than the combinatorial/topological structure of the former.

If we will focus only on the linear structure on the algebraic side,
the persistent diagram provides a complete invariant.

Ex: look at the whole sequence, not slice by slice



These two barcodes give the same betti numbers, but they are not isomorphic as persistence modules

Not size but history matters
"Elder rule"

Interval

R : a totally ordered set
$Vect$: the category of vector spaces

Persistence module: a functor $V: R \to Vect$

Interval: $I \subset R$ s.t. $x, y \in I \Rightarrow z \in I \ (x \leq \forall z \leq y)$

Interval module: $k_I(x) = \begin{cases} k \ (x \in I) \\ 0 \ (x \notin I) \end{cases}$ (maps are defined in an obvious way)

Interval decomposition theorem

Theorem
Any persistent module can be expressed “uniquely” by a direct sum of interval modules under a “mild condition”.

Decomposition: $V \cong \bigoplus_{I \in \Lambda} V_I$

Uniqueness: Λ is unique as a multi-set (factors I’s are unique up to permutation)

the multiset of the endpoints of I provide a (almost complete) invariant

- When R is finite => Gabriel’s theorem 1972
- When every V_t is finite dimensional => Crawley-Boevey 2012
- When V is q-tame (i.e., all maps have finite rank)
  => Chazal-Vin de Silva-Glisse-Oudot 2015
- Uniqueness: Krull–Schmidt–Azumaya’s theorem

Today, we focus the simple case:
R: finite and $\dim(V_t) < \infty$

The case of finite R and $\dim(V_t) < \infty$

Theorem

A sequence of finite dimensional vector spaces

$$0 = V_0 \xrightarrow{h_0} V_1 \xrightarrow{h_1} \cdots \xrightarrow{h_{M-1}} V_M \xrightarrow{h_M} V_{M+1} = 0$$

decomposes into a direct sum of intervals of the following form

$$k_{[a,b)}\colon\ 0 \to \cdots \to 0 \to \underset{\substack{\parallel \\ V_a}}{k} \xrightarrow{Id} k \xrightarrow{Id} \cdots \xrightarrow{Id} \underset{\substack{\parallel \\ V_{b-1}}}{k} \to \underset{\substack{\parallel \\ V_b}}{0} \to 0 \to \cdots \to 0$$

That is, (existence) $V \cong \bigoplus_{i=1}^{m} k_{[a_i,b_i)}$

(uniqueness) the multiset $\{[a_i, b_i) | i = 1 \ldots m\}$ is unique.

A standard proof: existence

$$0 = V_0 \xrightarrow{h_0} V_1 \xrightarrow{h_1} \cdots \xrightarrow{h_{M-1}} V_M \xrightarrow{h_M} V_{M+1} = 0$$

View h_i as an action of an indeterminant t and consider the sequence as a $k[t]$-module.

Then invoke the structure theorem of a finitely-generated module over PID:

Theorem
M: A finitely generated, graded module over $k[t]$

$$M \cong \bigoplus_{i=1}^{m} \Sigma^{\mathrm{n_i}}\, k[t]/(t^{d_i})$$

(we allow (t^{d_i})=0 for some i)

A standard proof is given essentially by the matrix reduction algorithm.

Note: R: non-negatively graded PID => $R = R_0$ or $R \cong k[t]$
A graded variant of the structure theorem does not hold when $R = R_0$!

(a good exposition: Loeh, 2023)

A standard proof: uniqueness

Note that morphisms between intervals are very restricted:
if there exists $k_{[a,b)} \to k_{[a',b')}$ injective => $b = b'$ (surjective => $a = a'$)

Let $V \cong \bigoplus_{i=1}^{m} k_{I_i} \cong \bigoplus_{i=1}^{m'} k_{I'_i}$

Consider the composition

$$q_j : k_{I_1} \xhookrightarrow{i_1} \bigoplus_{i=1}^{m} k_{I_i} \xrightarrow{f} \bigoplus_{i=1}^{m'} k_{I'_i} \overset{\pi_j}{\twoheadrightarrow} k_{I'_j} \xhookrightarrow{i_j} \bigoplus_{i=1}^{m'} k_{I'_i} \xrightarrow{f^{-1}} \bigoplus_{i=1}^{m} k_{I_i} \overset{\pi_1}{\twoheadrightarrow} k_{I_1}.$$

Then, $\sum q_j = Id$ so some q_j is an isomorphism.

For an induction argument on m to work, we need a “cancellation” lemma.

Lemma
If there exists an isomorphism $f: V \oplus V' \to W \oplus W'$ whose restriction gives an isomorphism $V \to W$,
there exists an isomorphism $g: V' \to W'$

A proof is given essentially by a block diagonalisation.

Easier proof?

Notation

$$0 = V_0 \xrightarrow{h_0} V_1 \xrightarrow{h_1} \cdots \xrightarrow{h_{M-1}} V_M \xrightarrow{h_M} V_{M+1} = 0$$

- For $v \in V_r$
  - Write $|v| = r$
  - Write $h^j v = h_{r+j-1} \circ h_{r+j-2} \circ \cdots \circ h_r v$
  - Define $e(v)$ is the minimum j s.t. $h^j v = 0$
- For $v_1, \dots, v_m \in \bigcup_{r=0} V_r$ (v_i are homogeneous)
  - Let $\langle v_1, \dots, v_m \rangle$ be the submodule generated by $\{h^j v_i\}$
  - In particular, $\langle v \rangle = k_{[|v|,|v|+e(v))}$

$e(v)$: "life expectancy"

In the view of $k[t]$-module
$Ann(v) = (t^{e(v)})$

Note that the following are equivalent

1. $\langle v_1, \dots, v_m \rangle = \langle v_1 \rangle \oplus \langle v_2 \rangle \oplus \cdots \oplus \langle v_m \rangle$
2. $\exists r, \exists J \subset \{i \mid |v_i| \le r\}, \exists \{c_i \in k \mid i \in J\}$ s.t. $\sum_{i \in J} c_i h^{r-|v_i|} v_i = 0$
   $\Rightarrow \forall i \in J, c_i h^{r-|v_i|} v_i = 0$

"No non-trivial relation"

An elementary and concise proof: existence

Lemma: Let $S = \{v_1, \dots, v_m\}$ s.t. $V = \langle S \rangle$.

If $V \not\cong \langle v_1 \rangle \oplus \langle v_2 \rangle \oplus \cdots \oplus \langle v_m \rangle$, there exists another generating set S'

with $\sum_{v \in S'} e(v) < \sum_{v \in S} e(v)$

Proof of Theorem: Since $\sum_{v \in S} e(v)$ is a non-negative integer, the process terminates after finite iterations.

Proof of Lemma: Assume $\exists r, \exists J \subset \{i \mid |v_i| \le r\}, \exists \{c_i \in k \mid i \in J\}$ s.t. $\sum_{i \in J} c_i h^{r-|v_i|} v_i = 0$ and $\exists i \in J, c_i h^{r-|v_i|} v_i \neq 0$.

Let v_j be one with $c_i h^{r-|v_i|} v_i \neq 0$ having the largest $|v_j|$.

v_j is the youngest among those who constitute a non-trivial relation

Put $\overline{v_j} = \sum_{i \in J} c_i h^{|v_j|-|v_i|} v_i = c_j v_j + \sum_{i \in J \setminus \{j\}} c_i h^{|v_j|-|v_i|} v_i$

$\overline{v_j}$=0 may happen

Since $c_j \neq 0$, we see $S \cup \{\overline{v_j}\} \setminus \{v_j\}$ generates V.

Since $c_j h^{r-|v_j|} v_j \neq 0$, we have $e(v_j) > r - |v_i|$.

Since $h^{r-|v_j|} v_j = \sum_{i \in J} c_i h^{r-|v_i|} v_i = 0$, we have $e\left(\overline{v_j}\right) \le r - |v_i|$. So $e(\overline{v_j}) < e(v_j)$.

There exists a unique minimum due the the uniqueness

Generators with the minimum total life expectancy give the decomposition!
(cover the barcodes efficiently with no overlaps)

Essentially the same as one of the well-known proofs

An elementary and concise proof: uniqueness

Assume $V \cong \bigoplus_{i=1}^{m} k_{[a_i,b_i)}$. We prove the uniqueness of the multiset $\{(a_i, b_i)\}$ by counting the multiplicity of (a_i, b_i) in terms of invariants of V.

Let $V_r^i = \{v \in V_r | e(v) \leq i\}$

Idea: Count the number of intervals in terms of $\dim(V_r^i)$

Since $\#\{(a_i, b_i) | a_i \leq r, b_i \leq r + i\} = \dim(V_r^i)$

we have $\#\{(a_i, b_i) | a_i \leq r, b_i = r + i\} = \dim(V_r^i) - \dim(V_r^{i-1})$

And $\#\{(a_i, b_i) | a_i = r, b_i = r + i\} = \left(\dim(V_r^i) - \dim(V_r^{i-1})\right) - \left(\dim(V_{r-1}^{i+1}) - \dim(V_{r-1}^{i})\right)$

Elder rule revisited

Proposition:

Let $v \in V$ s.t. $e(v)$ is the largest.

Then, $\langle v \rangle$ splits off from V.

That is, there exists V' such that $V = \langle v \rangle \oplus V'$

Proof: extend $\{v\}$ to a generating set.
Recall that in the proof of Lemma, the youngest v_j
(the one with the largest $|v_j|$) in the relation is replaced or removed
to form a new generating set.
So v is kept intact in the iterative process.

“Youngest rule”

A similar argument shows

Proposition:

Let $v \in V$ s.t. $e(v)$ is the smallest among those which constitutes a minimal generating set of V.

Then, $\langle v \rangle$ splits off from V.
That is, there exists V' such that $V = \langle v \rangle \bigoplus V'$

Iterative applications of this Lemma yields the interval decomposition as well.

Remarks

- The proof is not fully constructive unlike the matrix reduction.
  Can we make it into an algorithm?
- How far can we extend the argument to more general cases?

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