GEOMETRY OF CLOSED KINEMATIC CHAIN

Mathematics in Interface, Dislocation and Structure of Crystals
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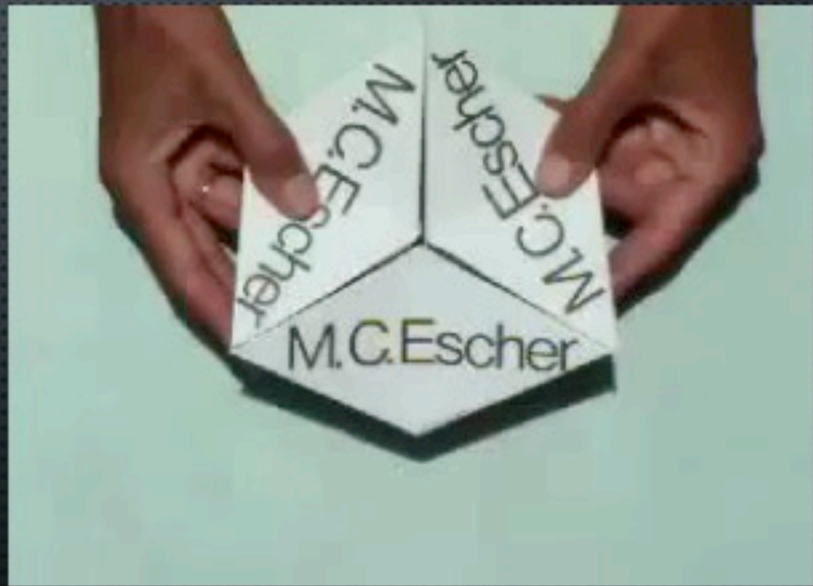
OKINAWA INST. SCIENCES AND TECHNOLOGY

STANDARD KALEIDOCYCLE

- Recreational maths
  e.g., Rouse Ball, W. W. 1939
  "Mathematical recreations and essays"
- Pure Maths
  A flexible polyhedral object,
  which can be made from a sheet of paper
  (c.f. Cauchy's rigidity theorem)
- Kinematics/Robotics
  it is a Bricard 6R linkage, which violates
  Maxwell's mobility formula



https://www.youtube.com/watch?v=V_fdN3HIlsI

HOW IS KALEIDOCYCLE RELEVANT TO CRYSTAL DISLOCATION ?

COMMON KEYWORDS (EXCUSE)

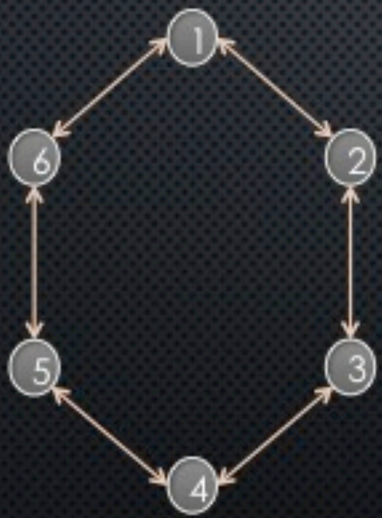
- REPRESENTATION BY GRAPH OF LOCAL FRAMES
- MONODROMY ON GRAPH
- ENERGY EQUILIBRIUM
- CONFIGURATION SPACE
  (REAL ALGEBRAIC GEOMETRY)

GRAPHICAL MODEL OF KALEIDOCYCLE

We model Kaleidocycles using graphs:

- vertices: shared edges of tetrahedra ("hinge")
- directed edges: tetrahedra
- each edge labelled by an element of the group SE(3) of Euclidean motion in R^3 (this accounts for the shape of the tetrahedron)

the change in local frame is recorded by $A_{kj} \in SE(3)$



Underlying graph of 6-Kleidocycle
Notice the directed
(doubled headed) edges

More generally, we call the pair of
the graph G=(V,E) and
the edge labelling E → SE(3)
a *kinematic chain*

STATE SPACE OF A KINEMATIC CHAIN

Fix a kinematic chain: G=(V,E), A: E→ SE(3)

A *STATE* IS A MAP Θ : V → R SATISFYING THE *CLOSING CONDITION*

$$A_{i_1 i_2} R_{\theta_{i_2}} A_{i_2 i_3} \cdots R_{\theta_{i_k}} A_{i_k i_1} \{p^{\pm}\} = \{p^{\pm}\}$$

for any cycle $(i_1 i_2, i_2 i_3, \cdots, i_k i_1)$

where $p^{\pm} = (0, 0, \pm\epsilon, 1)^T \quad (\epsilon > 0 \text{ small})$

$R_\theta \in SE(3)$ is the rotation around z-axis by magnitude θ

We say the state is oriented when + always goes + in the closing condition

BASIC QUESTIONS

- WHAT IS THE TOPOLOGY OF THE STATE SPACE?
  - WHEN DOES IT HAVE NON-TRIVIAL TOPOLOGY?
  - ➔KALEIDOCYCLES HAVE INTERESTING STATE SPACES
- WHAT STATES ARE STABLE WITH RESPECT TO A CERTAIN ENERGY?
  - ➔SOME KALEIDOCYCLES HAVE A CONSTANT ENERGY

KEY OBSERVATION

The state space (the *configuration space*) is the space of the real solutions to a system of polynomials defined by the closing condition.

AN INTERESTING FAMILY OF KINEMATIC CHAINS

SETTING

FROM NOW ON, WE FOCUS ON THE CASE WHEN THE UNDERLYING GRAPH G=(V,E) IS A CYCLE:

V = {1,2,3,…,N}, WHERE WE REGARD N+1=1

$$E = \{(i, i+1) \mid i = 1, 2 \ldots, N\}$$

$$A_{i,i+1} = \begin{pmatrix} 1 & 0 & 0 & \sin\phi \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Edge label is same everywhere
This means all tetrahedra are congruent

We call this type of kinematic chain and its states Kaleidocycles and denote it by KN

DENAVIT-HARTENBERG FORMULATION

For this special system, there is a classical and a neat parametrisation

Let $B_i \in S^2$ be hinge direction vectors.
We define the local frames similar to Frenet-Serret

$$A_i = \langle T_i, N_i, B_i \rangle, \quad T_i = \frac{B_i \times B_{i+1}}{|B_i \times B_{i+1}|}, \; N_i = B_i \times T_i.$$

$$A_{i,i+1} = A_i^{-1} A_{i+1}$$

The congruency and the closing conditions are given by

★ $$\langle B_i, B_{i+1} \rangle = \cos\phi, \qquad \sum_{i=1}^{n} B_i \times B_{i+1} = 0$$

we set $B_{N+1} = \pm B_1$
according to orientability

BREAK: HANDICRAFT SESSION

Try gluing the ends in different ways by twisting

EXTREME KALEIDOCYCLES

- THE SOLUTION TO THE POLYNOMIAL SYSTEM ★ PROJECTED ON THE Φ AXIS GIVES A (UNION OF) INTERVALS
- BY NUMERICAL EXPERIMENTS, WE FOUND THEY ARE OF THE FOLLOWING FORM:
  - WHEN N:ODD
    FOR ORIENTED SYSTEM: Φ IN $[0, \pi - C_N]$
    FOR NON-ORIENTED SYSTEM: Φ IN $[C_N, \pi]$
  - WHEN N:EVEN
    FOR ORIENTED SYSTEM: Φ IN $[0, \pi]$
    FOR NON-ORIENTED SYSTEM: Φ IN $[C_N, \pi - C_N]$

> - There is no solution for N<6
> - N C_N converges monotonously to a constant

Kaleidocycles with the non-trivial extreme twist angles exhibit interesting properties, which we call extreme Kaleidocycles

HOW TO FIND?
-- REAL ALGEBRAIC GEOMETRY --

- STUDY OF THE REAL SOLUTIONS OF A SYSTEM OF POLYNOMIAL EQUATIONS
- REAL SOLUTIONS ARE HARD: E.G.

$$x^2 + bx + c = 0 \text{ has a solution } \Leftrightarrow b^2 - 4c \geq 0$$

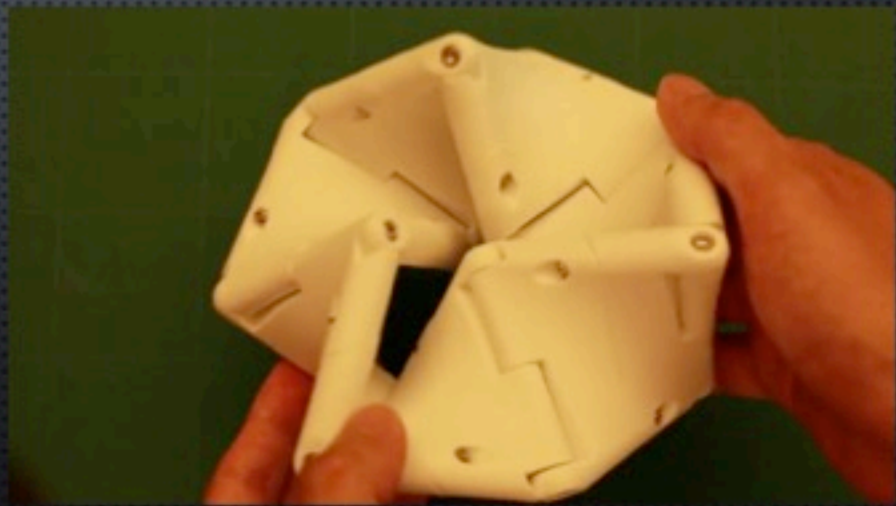
- APPLIED IN: ROBOTICS, COMPUTER VISION, CONFIGURATION SPACE
- MILNOR'S RESULT ON THE BETTI NUMBERS
- SOFTWARE
  - MAPLE, MATHEMATICA, MATLAB (NON LINEAR PROGRAMING)
  - BERTINI, PHCPACK (NUMERICAL)
  - REGULAR CHAINS, QEPCAD (EXACT)

Interesting properties of extreme Kn

- Single DOF
- Constant torsional energy (minimum for K7)
  Constant dipole energy (for oriented Kn)
- Falling cat motion
- Mobius strip
- Duality

MOBILITY ANALYSIS

- Maxwell's law counts the number of constraints and freedom, giving a rough estimation of the dimension of the configuration space of the system(dimension counting)
- In our case,
  Each hinge has one DOF
  Closing condition kills five DOF
  So the system's DOF "=" (N-1)-5
  (-1 comes from global orientation)



Only numerically confirmed

The state space of any extreme Kaleidocycle is the circle (everting motion)
thus violate Maxwell's law! (except for K=7)

ENERGY EQUILIBRIUM

ONE OF THE MAIN TOPICS IN MATERIALS SCIENCE IS TO FIND EQUILIBRIUM STATES OF THE MODEL WITH RESPECT TO A CERTAIN ENERGY

The shape of a Möbius strip, E. L. Starostin & G. H. M. van der Heijden, 2007

TORSIONAL ENERGY

- IMAGINE THAT EACH HINGE IS EQUIPPED WITH A WINDING SPRING
- THE ENERGY OF THE STATE IS DEFINED BY

$$E(\theta) = \sum_i \phi^2$$

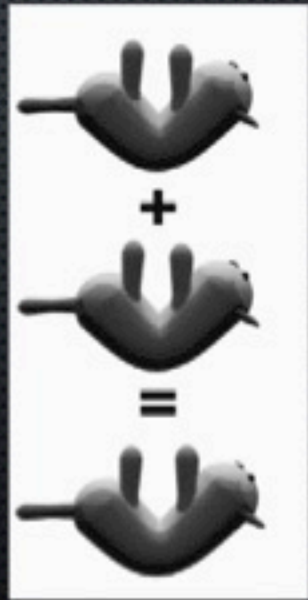
Only numerically confirmed

- Any extreme Kaleidocycle takes a constant energy for all state θ.
  When N=7, the value is minimum among all K7.
- Also, any extreme oriented KN takes a (almost) constant dipole energy

$$\sum_{i<j} \frac{B_i B_j}{|O_i - O_j|^3} - 3\frac{(B_i(O_i - O_j))(B_j(O_i - O_j))}{|O_i - O_j|^5}$$

FALLING CAT

- NON-RIGID OBJECT CAN CHANGE ORIENTATION WHILE PRESERVING MOMENTUM
- THE EVERTING MOTION OF KN EXHIBITS THE PHENOMENON



LINKING NUMBER AND MOBIUS STRIP

- WE ARE INTERESTED IN HOW MANY TIMES THE STRIP IS TWISTED
- THIS IS MADE PRECISE BY LOOKING AT THE *LINKING NUMBER* OF THE FOLLOWING POLYLINES]
  - ONE CONNECTING THE ORIGINS OF HINGES
  - THE OTHER CONNECTING THE HEAD OF HINGES

Calugareanu's theorem
Total twisting = 2(Twist + Writhe)

Twist = $N\Phi/2\pi$ (so it depends only on the shape of the tetrahedron)
Writhe depends only on the center curve (it is related to the *torsion*)

Observation: Extreme ones have Total Twisting = 3 or N-3

Question: What happens if we look at the components of the solution space consisting of KN's with a fixed TT?

CONTINUOUS LIMIT

WE CAN CONSIDER THE CONTINUOUS LIMIT OF EXTREME K_N ($N \to \infty$).

WITH THE FRENET-SERRET FRAME, HINGE DIRECTION CORRESPONDS TO THE BINORMAL.

THE CORE CURVE SEEMS TO CONVERGE TO A ~~CONSTANT CURVATURE AND~~ CONSTANT TORSION CURVE.

OPEN PROBLEM: WHAT IS THE LIMIT BAND?

DUALITY

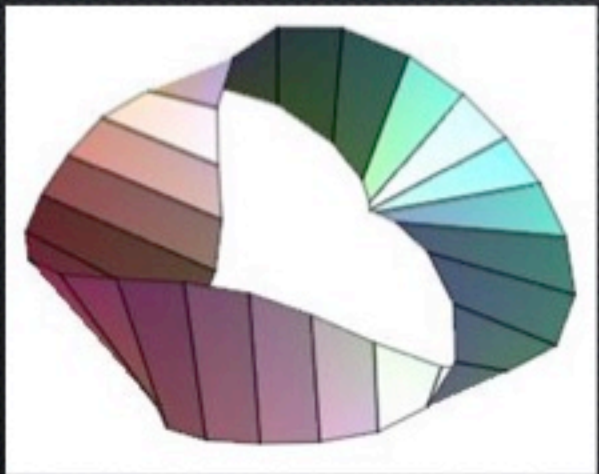
- EXTREME KN'S HAVE AN ACTION OF Z/2Z BY INVERTING THE DIRECTION OF EVERY OTHER HINGE
  - $\Phi \rightarrow \Pi - \Phi$
  - TOTAL TWISTING ⇔ N – TOTAL TWISTING
  - IN PARTICULAR
    ORIENTED ⇔ NON ORIENTED
    WHEN N: ODD

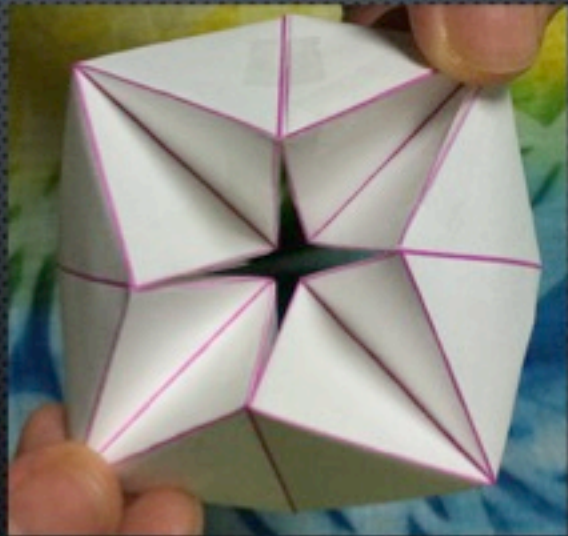
K13 with total twisting = 3

K13 with total twisting = 10

THREE INCARNATIONS OF KN

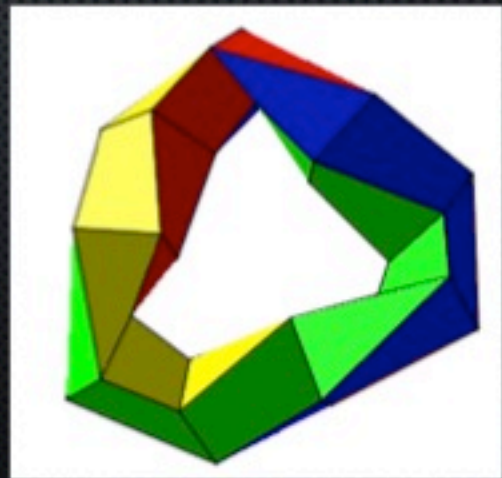
Mobius strip





Ring of Tetrahedra

Twisted Torus



POSSIBLE APPLICATIONS

- toy, ornament: use a motored gimbal to fixture one hinge and hang it.
  Print interesting things on the surface
  For example, Musicbox with transcription printed on it (c.f. Bach's Crab canon)
- Escher-like maze, where connection of paths changes during the everting motion
- ``developable'' chain/arm/curtain by allowing free slides at hinges (c.f. folding umbrella)
- musical instrument: hit and rotate to make sound with Doppler's effect (rotary speaker)
- screw, mixer, valve, pasta maker (which makes patterns on noodle and dough)
- Robot arm, Snake-like self-propulsion robot (c.f. Origami rotor)
- Chemistry: micro molecule structure (c.f. Boerdijk–Coxeter helix and alpha helix)

OPEN QUESTIONS

- THE CONTINUOUS LIMIT OF K_N $(N \to \infty)$
- EXTREME K_N'S WITHIN A SPECIFIED TOTAL TWISTING
- PROVE PROPERTIES OF K_N RIGOROUSLY
  (CURRENTLY, EVERYTHING IS NUMERICAL)
- MORE GENERALLY, DEVELOP METHODS TO ANALYSE KINEMATIC CHAINS AND THEIR STATE SPACES
- FIND INTERESTING EXAMPLES AND APPLICATIONS