

素数3でのある2型環スペクトラム のピカール群を次数とするホモトピー群について

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\mathcal{L}_n のピカール群

S_p : p -local spectra の安定ホモトピー圏

(p : prime)

$E(n)$: n -th Jonson-Wilson spectrum

\mathcal{L}_n : $E(n)$ -local spectra からなる S_p の充満部分圏

この時, Bousfield localization functor $L_n : S_p \rightarrow \mathcal{L}_n$ が得られる.

Definition.

\mathcal{L}_n において,

$$\mathcal{L}_n \ni X : \text{invertible} \iff \exists Y \in \mathcal{L}_n \text{ s.t. } X \wedge Y = L_n S^0$$

\mathcal{L}_n のピカール群

$\text{Pic}(\mathcal{L}_n)$ を \mathcal{L}_n の invertible spectrum の同値類全体の集まりとすると $\text{Pic}(\mathcal{L}_2)$ は群である.

実際, $a = [A], b = [B] \in \text{Pic}(\mathcal{L}_n)$,

$$a + b := [A \wedge B]$$

で定義する.

$L_n S^m$ は invertible であるから, $m = [L_n S^m], m' = [L_n S^{m'}] \in \text{Pic}(\mathcal{L}_n)$ に対して,

$$m + m' = [L_n S^m \wedge L_n S^{m'}] = [L_n S^{m+m'}].$$

従って, $0 = [L_n S^0]$ は単位元で $\mathbb{Z} \subset \text{Pic}(\mathcal{L}_n)$ がわかる.

\mathcal{L}_n のピカール群

Theorem.(Hovey-Sadofsky)

$$n^2 + n < 2p - 2 \implies \text{Pic}(\mathcal{L}_n) \cong \mathbb{Z}$$

$$p = 2, n = 1 \implies \text{Pic}(\mathcal{L}_1) \cong \mathbb{Z} \oplus \mathbb{Z}/2$$

Theorem.(Kamiya-Shimomura)

$$p = 3, n = 2 \implies \text{Pic}(\mathcal{L}_2) \cong \mathbb{Z} \oplus \mathbb{Z}/3 \text{ or } \mathbb{Z} \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3$$

Theorem.(Goerss-Henn-Mahowald-Rezk, Shimomura)

$$p = 3, n = 2 \implies \text{Pic}(\mathcal{L}_2) \cong \mathbb{Z} \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3$$

ピカール群を次数とするホモトピー群

$\lambda (= [S^\lambda]) \in \text{Pic}(\mathcal{L}_n)$ を次元とした spectrum X のホモトピー群は

$$\pi_\lambda(X) := [S^\lambda, X] \cong [S^0, X \wedge S^{-\lambda}] \cong \pi_0(X \wedge S^{-\lambda})$$

で定義される。

ここからは, $p = 3, n = 2$ のときを考え, $\mathcal{S}_3 \supset \mathcal{L}_2$ 内での話とする.

$$\text{Pic}(\mathcal{L}_2) \cong \mathbb{Z} \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3 \cong E_2^{5,4}(S^0)$$

$P : \text{Pic}(\mathcal{L}_2)$ の左側の $\mathbb{Z}/3$ の生成元を与える invertible spectrum
($\omega_1 \in E_2^{5,4}(S^0)$ に対応するもの)

$Q : \text{Pic}(\mathcal{L}_2)$ の右側の $\mathbb{Z}/3$ の生成元を与える invertible spectrum
($\omega_2 \in E_2^{5,4}(S^0)$ に対応するもの)

$V(1)$ のホモトピー群

M : mod 3 Moore spectrum

α : Adams map ($BP_*(\alpha) = v_1$)

$V(1) : \Sigma^4 M \xrightarrow{\alpha} M$ のコファイバー

Theorem. Ichigi-Shimomura

$V(1) \wedge P \cong \Sigma^{48} V(1)$ であり, $\pi_*(V(1) \wedge P) \cong \pi_*(V(1))$ である.

また, $\pi_*(V(1)) \not\cong \pi_*(V(1) \wedge Q) \cong \pi_*(V(1) \wedge Q^{\wedge 2})$

$\pi_*(V(1)), \pi_*(V(1) \wedge Q)$ ($* \in \mathbb{Z}$) の構造は Ichigi-Shimomura によって
次のように決定されている.

$$\pi_*(V(1)) \cong K \otimes (A \oplus B \oplus C \oplus D) \otimes E(\zeta_2)$$

$$K = \mathbb{Z}/3[v_2^9, v_2^{-9}]$$

$$P_k = \mathbb{Z}/3[b_0]/(b_0^k)$$

$$A = P_2\{v_2^i h_1, v_2^j b_1 \xi \mid i = 0, 1, 2, 5, 6, j = 1, 2, 3, 5, 6, 7\}$$

$$B = P_3\{v_2^i h_0, v_2^j \xi \mid i = 1, 2, 5, 6, 7, j = 0, 2, 3, 4, 7, 8\}$$

$$C = P_5\{v_2^i, v_2^j \psi_1 \mid i = 0, 1, 5, j = 2, 6, 7\}$$

$$D = P_4\{v_2^i b_1, v_2^j \psi_0 \mid i = 0, 3, 4, j = 2, 3, 7\}$$

$$\pi_*(V(1) \wedge Q) \cong K \otimes ((A' \oplus B' \oplus C' \oplus D') \otimes E(\zeta_2) \oplus M \oplus N \oplus O)$$

$$A' = P_2\{v_2^i h_1, v_2^j b_1 \xi \mid i = 0, 1, 2, 5, 6, j = 1, 5, 6\}$$

$$B' = P_3\{v_2^i h_0, v_2^j \xi \mid i = 1, 2, 5, 6, 7, j = 2, 3, 7\}$$

$$C' = P_5\{v_2^i \psi_1 \mid i = 2, 6, 7\}$$

$$D' = P_4\{v_2^j \psi_0 \mid i = 2, 3, 7\}$$

$$M = P_3\{b_0^2, v_2 b_0^2, v_2^5 b_0^2, v_2^3 b_1 b_0, v_2^4 b_1 b_0, v_2^8 b_1 b_0, \xi, v_2^4 \xi, v_2^8 \xi\}$$

$$N = P_1\{v_2^2 b_1 \xi \zeta_2, v_2^3 b_1 \xi \zeta_2, v_2^7 b_1 \xi \zeta_2\}$$

$$O = \mathbb{Z}/3\{1, v_2, v_2^5\} \otimes (P_2\{v_2^2 b_1 \xi\} \oplus P_4\{v_2^3 b_1 \zeta_2\} \oplus P_5\{\zeta_2\})$$

Vのホモトピー群

$$V(1) : \Sigma^4 M \xrightarrow{\alpha} M \xrightarrow{i_1} V(1) \xrightarrow{j_1} \Sigma^5 M$$

$$V : \Sigma^8 M \xrightarrow{\alpha^2} M \xrightarrow{i_2} V \xrightarrow{j_2} \Sigma^9 M,$$

Then the 3×3 Lemma shows the cofiber sequence

$$\Sigma^4 V(1) \xrightarrow{\bar{\alpha}} V \xrightarrow{\kappa} V(1) \xrightarrow{\lambda} \Sigma^5 V(1),$$

which induces the short exact sequence of $E(2)_*$ -homology

$$0 \rightarrow E(2)_*/(3, v_1) \xrightarrow{v_1} E(2)_*/(3, v_1^2) \xrightarrow{\kappa_*} E(2)_*/(3, v_1) \rightarrow 0.$$

This gives rise to the long exact sequence

$$\begin{aligned} 0 \rightarrow E_2^0(V(1)) &\xrightarrow{v_1} E_2^0(V) \xrightarrow{\kappa_*} E_2^0(V(1)) \xrightarrow{\delta} E_2^1(V(1)) \rightarrow \dots \\ &\rightarrow E_2^s(V(1)) \xrightarrow{v_1} E_2^s(V) \xrightarrow{\kappa_*} E_2^s(V(1)) \xrightarrow{\delta} E_2^{s+1}(V(1)) \rightarrow \dots \end{aligned}$$

of the Adams-Novikov E_2 -terms.

Consider the module

$$\begin{aligned} P &= K(2)_*[b_0] \\ F &= K(2)_*\{1, h_0, h_1, b_1, \xi, \psi_0, \psi_1, b_1\xi\}. \end{aligned}$$

Put $E_2 = E_2(V(1))$, then

$$\begin{aligned} v_2 &\in E_2^{0,16}, \quad h_0 = [v_2^3 t_1] \in E_2^{1,52}, \quad h_1 = [v_2^{-2} t_1^3] \in E_2^{1,-20}, \\ b_0 &= [b_{10}] \in E_2^{2,12}, \quad b_1 = [v_2^3 b_{11}] \in E_2^{2,84}, \quad \xi = [v_2^{-4} x] \in E_2^{2,-56}, \\ \psi_0 &= [v_2^3 f_0] \in E_2^{3,48}, \quad \psi_1 = [v_2^{-3} f_1] \in E_2^{3,-24}. \end{aligned}$$

Then,

$$E_2^*(V(1)) = P \otimes_{K(2)_*} F \otimes_{K(2)_*} E(\zeta_2)$$

as a P -module with relation

$$\begin{aligned} h_0 h_1 &= 0, \quad h_0 \xi = 0, \quad h_1 \xi = 0, \\ h_0 b_0 &= h_1 b_1, \quad v_2^9 h_1 b_0 = -h_0 b_1, \\ b_1 \xi &= h_0 \psi_1 = h_1 \psi_0, \quad v_2^9 b_0 \xi = -h_0 \psi_0 = v_2^9 h_1 \psi_1, \\ v_2^9 b_0^2 &= -b_1^2, \quad v_2^9 b_0 \psi_1 = -b_1 \psi_0, \quad b_0 \psi_0 = b_1 \psi_1. \end{aligned}$$

$$K(2)_*^{(3^i)} := \mathbb{Z}/3[v_2^{\pm 3^i}]$$

Consider the algebras

$$\begin{aligned} L^{(3^i)} &= K(2)_*^{(3^i)}[b_0, b_1]/(b_1^2 + v_2^9 b_0^2) \\ &\cong K(2)^{(3^i)}[b_0] \otimes E(b_1) \text{ as a module if } i \leq 2. \end{aligned}$$

Put $L = L^{(1)}$ then, the module $E_2^*(V(1))$ is expressed as

$$E_2^*(V(1)) = L \otimes E(h_1, \psi_1, \zeta_2) \oplus K(2)_*\{\xi, h_0, \psi_0, b_1\xi\},$$

since

$$K(2)_* \otimes E(b_1, h_1, \psi_1) = K(2)_*\{1, h_1, \psi_1, \xi b_0, b_1, h_0 b_0, \psi_0 b_0, b_1 \xi b_0\}.$$

The connecting homomorphism $\delta : H^s K(2)_* \rightarrow H^{s+1} K(2)_*$ acts trivially on b_i ($i = 0, 1$) and the generators of the exterior algebra, and the action of δ is obtained by

$$\delta(v_2^s) = sv_2^{s+1}h_1.$$

Lemma.

$$E_2^*(V) = M^* \otimes E(\psi_1, \zeta_2) \oplus R^*$$

Here,

$$\begin{aligned} M^* &= L^{(3)} \otimes E(v_1 v_2 h_1) \oplus L\{v_1, h_1\} \\ R^* &= (K(2)_*\{\xi, b_1 \xi\} \oplus K(2)_*^{(3)}\{\psi_0\}) \otimes E(\zeta_2) \\ &\quad \oplus h_0 K(2)_*. \end{aligned}$$

Lemma.

The differential d_5 acts trivially except for

$$\begin{aligned} d_5(v_2^{3t}) &= -tv_2^{3t} h_1 b_0^2 \\ d_5(v_1 v_2^{3t+i}) &= \begin{cases} -tv_1 v_2^{3t+1} h_1 b_0^2 & i = 1 \\ 0 & \text{otherwise} \end{cases} \\ d_5(v_2^{3t+j} h_1) &= \begin{cases} tv_1 v_2^{3t-6} b_1 b_0^2 & j = 2 \\ (t+1)v_1 v_2^{3t-7} b_1 b_0^2 & j = 1 \\ 0 & j = 0 \end{cases} \end{aligned}$$

Put

$$\omega_2 = v_2^3 h_1 \psi_1 \zeta_2,$$

and consider the ideal $A(\omega_2) = \{x \in E_2^*(V) : x\omega_2 = 0\}$.

Lemma. Let I be the submodule of $E_2^*(V)$ given by

$$I = M^* \otimes \overline{E}(\psi_1, \zeta_2) \oplus R^*.$$

Then, $I \subset A(\omega_2)$ and I is a differential submodule with the differential d_5 . Here, $\overline{E}(\psi_1, \zeta_2) = \mathbb{Z}/p\{\psi_1, \zeta_2, \psi_1 \zeta_2\}$.

Lemma. $H^*(M^*; d_5)$ is the direct sum of $L^{(9)}$ -modules

$$\begin{aligned} & L^{(9)}\{1; h_1, v_2^2 h_1, v_2^7 h_1; v_1 v_2, v_1 v_2^3, v_1 v_2^8; v_1 v_2 h_1\} \\ & L^{(9)}/(b_0^2)\{v_2^3 h_1, v_2^6 h_1; v_1 v_2^4 h_1, v_1 v_2^7 h_1\} \\ & L^{(9)}/(b_1 b_0^2)\{v_1, v_1 v_2^2, v_1 v_2^5, v_1 v_2^6\} \end{aligned}$$

The short exact sequence

$$0 \rightarrow I \hookrightarrow V^* \rightarrow E_2^*(V)/I = M^* \rightarrow 0$$

for $V^* = E_2^*(V)$ induces the long exact sequence

$$0 \rightarrow H^k(I; d_5) \rightarrow H^k(V^*; d_5) \rightarrow H^k(M^*; d_5) \xrightarrow{\delta} H^{k+5}(I; d_5) \rightarrow \dots$$

for $0 \leq k < 5$.

Lemma. For the generator of $g_i \in E_2^{0,0}(V \wedge Q^{\wedge i})$, $d_5(g^i) = i\omega_2 g_i$, $H^k(I; d_5) = H^k(Ig_i; d_5)$ and $H^k(M^*; d_5) = H^k(M^*g_i; d_5)$.

Lemma. $H^*(I; d_5) = H^*(M^*; d_5) \otimes \overline{E}(\psi_1, \zeta_2) \oplus R^*$, which is isomorphic to

$$S^* \oplus H^*(M^*; d_5) \otimes \mathbb{Z}/3\{\psi_1, \zeta_2\} \oplus R^*$$

where

$$\begin{aligned} S^* &= \omega_2 L^{(9)} \{v_2^{-3}, v_2^{-1}, v_2^4; v_1 v_2^{-2}\} \oplus \omega_2 L^{(9)} / (b_0^2) \{1, v_2^3; v_1 v_2, v_1 v_2^4\} \\ &\quad \oplus \psi_1 \zeta_2 L^{(9)} \{1; v_1 v_2, v_1 v_2^3, v_1 v_2^8\} \\ &\quad \oplus \psi_1 \zeta_2 L^{(9)} / (b_1 b_0^2) \{v_1, v_1 v_2^2, v_1 v_2^5, v_1 v_2^6\}. \end{aligned}$$

Proposition. $E_6^*(V) = H^*(I; d_5) \oplus H^*(M^*; d_5) = H^*(M^*; d_5) \otimes E(\psi_1, \zeta_2) \oplus R^*$ and $E_6^*(Vg_i) \cong A^* \oplus B^*$

Proposition. $E_6^*(V) = H^*(I; d_5) \oplus H^*(M^*; d_5) = H^*(M^*; d_5) \otimes E(\psi_1, \zeta_2) \oplus R^*$ and $E_6^*(Vg_i) \cong A^* \oplus B^*$

Here,

$$\begin{aligned} A^* &= H^*(I; d_5)/(\omega_2, v_1v_2\omega_2) \\ &= \overline{S}^* \oplus H^*(M^*; d_5) \otimes \mathbb{Z}/3\{\psi_1, \zeta_2\} \oplus R^* \\ B^* &= \text{Ker}\delta \\ &= L^{(9)}\{b_0^2; h_1, v_2^2h_1b_0^2, v_2^7h_1b_0^2; v_1v_2b_0^2, v_1v_2^3, v_1v_2^8; v_1v_2h_1\} \\ &\quad \oplus L^{(9)}/(b_0^2)\{v_2^3h_1, v_2^6h_1; v_1v_2^4h_1, v_1v_2^7h_1\} \\ &\quad \oplus L^{(9)}/(b_1b_0^2)\{v_1, v_1v_2^2, v_1v_2^5, v_1v_2^6\}. \end{aligned}$$

Here,

$$\begin{aligned} \overline{S}^* &= \omega_2 L^{(9)}\{v_2^{-3}, v_2^{-1}, v_2^4; v_1v_2^{-2}\} \oplus \omega_2 L^{(9)}/(b_0^2)\{v_2^3; v_1v_2^4\} \\ &\quad \oplus \psi_1\zeta_2 L^{(9)}\{1; v_1v_2, v_1v_2^3, v_1v_2^8\} \\ &\quad \oplus \psi_1\zeta_2 L^{(9)}/(b_1b_0^2)\{v_1, v_1v_2^5\} \oplus \psi_1\zeta_2 K(2)^{(9)}\{v_1v_2^2, v_1v_2^6\} \end{aligned}$$

Lemma. On E_9 -term, d_9 acts on $E_2^*(V)$ with internal degree in the $\text{Pic}(\mathcal{L}_2)$ as follows:

$$\begin{aligned} d_9(v_2^9 h_1) &= b_1 b_0^4 \\ d_9(v_1 v_2 h_1) &= v_1 v_2^{-8} b_1 b_0^4 \\ d_9(v_1 v_2^3) &= v_2^2 h_1 b_0^4 \\ d_9(v_1 v_2^8) &= v_2^7 h_1 b_0^4 \end{aligned}$$

up to sign.

Lemma. The differential d_9 acts on $H^*(M^*; d_5)$, and $C^* = H^*(H^*(M^*; d_5); d_9)$

$$\begin{aligned} C^* = & \ L^{(9)}/(b_1 b_0^4) \otimes E(v_1 v_2) \oplus L^{(9)}/(b_0^4)\{v_2^2 h_1, v_2^7 h_1\} \\ & \oplus L^{(9)}/(b_0^2)\{v_2^3 h_1, v_2^6 h_1; v_1 v_2^4 h_1, v_1 v_2^7 h_1\} \\ & \oplus L^{(9)}/(b_1 b_0^2)\{v_1, v_1 v_2^2, v_1 v_2^5, v_1 v_2^6\}. \end{aligned}$$

Similarly, $\tilde{B}^* = H^*(B^*; d_9)$ and $\tilde{S}^* = H^*(\bar{S}^*; d_9)$ are isomorphic to

$$\begin{aligned} \tilde{B}^* = & \ L^{(9)}/(b_1 b_0^2)\{b_0^2, v_1 v_2 b_0^2\} \oplus L^{(9)}/(b_0^2)\{v_2^2 h_1 b_0^2, v_2^7 h_1 b_0^2\} \\ & \oplus L^{(9)}/(b_0^2)\{v_2^3 h_1, v_2^6 h_1; v_1 v_2^4 h_1, v_1 v_2^7 h_1\} \\ & \oplus L^{(9)}/(b_1 b_0^2)\{v_1, v_1 v_2^2, v_1 v_2^5, v_1 v_2^6\} \\ \tilde{S}^* = & \ \omega_2 L^{(9)}/(b_0^4)\{v_2^{-1}, v_2^4\} \oplus \omega_2 L^{(9)}/(b_0^2)\{v_2^3; v_1 v_2^4\} \\ & \oplus \psi_1 \zeta_2 L^{(9)}/(b_1 b_0^4)\{1; v_1 v_2\} \\ & \oplus \psi_1 \zeta_2 L^{(9)}/(b_1 b_0^2)\{v_1, v_1 v_2^5\} \oplus K(2)^{(3^i)}\{v_1 v_2^2, v_1 v_2^6\}. \end{aligned}$$

Theorem. E_{10} -term for $\pi_*(V)(*\in \text{Pic}(\mathcal{L}_2))$ is given by

$$\begin{aligned}
E_{10}^*(V) &= C^* \otimes E(\psi_1, \zeta_2) \\
&\quad \oplus \bigoplus_{i=1,2} (\tilde{S}^* \oplus C^* \otimes \mathbb{Z}/3\{\psi_1, \zeta_2\} \oplus \tilde{B}^*)g^i \\
&= C^* \otimes \mathbb{Z}/3\{\psi_1, \zeta_2\}[g]/(g^3) \\
&\quad \oplus C^* \otimes E(\psi_1\zeta_2) \oplus \bigoplus_{i=1,2} (\tilde{S}^* \oplus \tilde{B}^*)g^i.
\end{aligned}$$