

# finite category のオイラー標数とゼータ関数

Kazunori Noguchi

Shinshu University

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# Outline

- ① Euler characteristics of a finite category
- ② The zeta function of a finite category
- ③ A relationship between these two notions

# The Euler characteristic of categories

- $\chi_L$  The Euler characteristic of finite categories [Leinster, 2008]
- $\chi_\Sigma$  The series Euler characteristic [Berger-Leinster, 2008]
- $\chi^{(2)}$  The  $L^2$ -Euler characteristic [Fiore-Lück-Sauer, 2010]
- $\chi_{\text{fil}}$  The Euler characteristic of  $\mathbb{N}$ -filtered acyclic categories [2010]

# The definition of an adjacency matrix

Definition 1.1

$I$  :finite category

$$\text{Ob}(I) = \{x_1, x_2, \dots, x_n\}$$

The adjacency matrix of  $I$

$$A_I = (\#\text{Hom}(x_i, x_j))_{i,j}$$

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## Example 1.2

Let

$$I = x_1 \begin{array}{c} \nearrow \\[-1ex] \searrow \end{array} x_2$$

Then,

$$A_I = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

# The definition of the Euler characteristic of finite categories

Definition 1.3 (Leinster,2008)

Let  $\mathbf{w}, \mathbf{c}$  be row vectors of  $\mathbb{Q}^n$ . Then, we say  $\mathbf{w}$  is a weighting on  $I$  if

$$A_I {}^t \mathbf{w} = A_I \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

We say  $\mathbf{c}$  is a coweighting on  $I$  if

$$\mathbf{c} A_I = (c_1, c_2, \dots, c_n) A_I = (1, \dots, 1).$$

Definition 1.4 (Leinster,2008)

Define the Euler characteristic  $\chi_L(I)$  of  $I$  by

$$\chi_L(I) = \sum_{i=1}^n w_i \in \mathbb{Q}$$

if  $I$  has both a weighting  $\mathbf{w}$  and a coweighting  $\mathbf{c}$ .

# Möbius inversion

Definition 1.5 (Leinster,2008)

$I$  : finite category

$I$  has Möbius inversion  $\stackrel{\text{def}}{\Leftrightarrow} \exists A_I^{-1}$ .

Remark 1.6

$$\chi_L(I) = \text{sum}(A_I^{-1}).$$

# The Euler characteristic for a simplicial complex and Leinster's one

- A simplicial complex  $\Delta$  can be regarded as a poset by

$$P(\Delta) = (\{\text{Faces}\}, \subset).$$

- A poset  $P$  can be regarded as a category by  $\text{Ob} = P$ ,  
 $\text{Mor} = \{(x, y) \mid x \leq y\}$ .
- $\chi(\Delta) = \chi_L(P(\Delta))$ .

# The definition of acyclic categories

## Definition 1.7

$\mathcal{A}$  is an acyclic category if

- ① For  $\forall x, y : \text{objects}$  ( $x \neq y$ ) if  $\exists x \rightarrow y$ , then  $\nexists y \rightarrow x$ .
- ② For  $\forall x : \text{object}$ ,  $x \xrightarrow{i=1_x} x$ .

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## Example 1.8

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## Example 1.9

These are not acyclic categories.



## Theorem 1.10 (Leinster,2008)

*A:finite acyclic category*

$$\chi_L(A) = \sum_{n \geq 0}^M (-1)^n \# \overline{N}_n(A) = \chi(BA)$$

where  $\overline{N}_n(A) = \{ (x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} x_n) \text{ in } A \mid 0 \leq n, f_i \neq 1 \}$

*The R.H.S is defined by the alternating sum of the numbers of n-cells*

# The Euler characteristic of a finite group as a category

## Example 1.11

$G$ : finite group

$$\text{Cat}(G) = \begin{cases} \text{Ob} = * \\ \text{Mor} = G \end{cases}$$

$$\chi_L(\text{Cat}(G)) = \frac{1}{\#G}$$

since  $A_{\text{Cat}(G)} = (\#G)$ .

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For the fibration

$$G \rightarrow EG \rightarrow BG$$

$$\begin{aligned}\chi(EG) &= \chi(G)\chi(BG) \\ 1 &= \#G\chi(BG) \\ \chi(BG) &= \frac{1}{\#G}\end{aligned}$$

# An invariant for equivalence of categories

Theorem 1.12 (Leinster,2008)

$I_1, I_2$  : finite categories, they are equivalent

Then,

$$\exists \chi_L(I_1) \Leftrightarrow \exists \chi_L(I_2)$$

In that case,

$$\chi_L(I_1) = \chi_L(I_2)$$

---

Corollary 1.13 (Leinster,2008)

$$\chi_L(I_1) \neq \chi_L(I_2) \Rightarrow I_1 \not\sim I_2$$

# $\chi_{\Sigma}$ のアイディア

Fact 2.1

$I$ : small category

$$\overline{N}_n(I) \cong \{n\text{-cells in } BI\}$$

CW-complex のオイラー標数は  $n$ -cell の数の交代和で定義されているので、finite category  $I$  のオイラー標数を

$$\sum_{n=0}^{\infty} (-1)^n \# \overline{N}_n(I)$$

とするのが自然ではないか？

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とするのが自然ではないか？

問題は収束するケースが少ないとこと

# Berger-Leinster の解決策

$$\sum_{n=0}^{\infty} \# \overline{N_n}(I) t^n = \frac{\text{sum}(\text{adj}(E - (A_I - E)t))}{\det(E - (A_I - E)t)}$$

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## Example 2.2

$$1 + t + t^2 + \cdots = \frac{1}{1-t}$$

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$$1 + t + t^2 + \cdots = \frac{1}{1-t}$$

これにより

$$1 - 1 + 1 - 1 + \cdots = \frac{1}{2}$$

という解釈が可能になる。

# The series Euler characteristic $\chi_\Sigma$

Definition 2.3 (Berger-Leinster, 2008)

$I$  : finite category

Define

$$f_I(t) = \frac{\text{sum}(\text{adj}(E - (A_I - E)t))}{\det(E - (A_I - E)t)}$$

$$\chi_\Sigma(I) = f_I(-1) \in \mathbb{Q}$$

if it exists.

## Example 2.4

$G$ : finite group

$$\begin{aligned} f_{\text{Cat}(G)}(t) &= \sum_{n=0}^{\infty} (\#G - 1)^n t^n \\ &= \frac{1}{1 - (\#G - 1)t} \\ \chi_{\Sigma}(\text{Cat}(G)) &= \frac{1}{\#G}. \end{aligned}$$

# Properties of $\chi_{\Sigma}$

Theorem 2.5 (Berger-Leinster, 2008)

$I$  :finite category

$I_0$ : skeleton of  $I$  and  $\exists A_{I_0}^{-1}$

Then,

- ①  $\exists \chi_{\Sigma}(I)$
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Fact 1

$\chi_{\Sigma}$  is not an invariant for equivalence of categories.

# The zeta function of a finite category

Definition 3.1 (N,2012)

$I$ :finite category

Define

$$\zeta_I(z) = \exp \left( \sum_{m=1}^{\infty} \frac{\#N_m(I)}{m} z^m \right)$$

as a formal power series where

$$N_n(I) = \{ (x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} x_n) \text{ in } I \}$$

# The simplest example

## Example 3.2

Let  $*$  be the one-point category.

$$\begin{aligned}\zeta_*(z) &= \exp\left(\sum_{m=1}^{\infty} \frac{\#N_m(*)}{m} z^m\right) \\ &= \exp\left(\sum_{m=1}^{\infty} \frac{1}{m} z^m\right) \\ &= \exp(-\log(1-z)) \\ &= \frac{1}{1-z}.\end{aligned}$$

# A covering of small categories

## Definition 3.3

*A category  $C$  is connected  $\Leftrightarrow$  for  $\forall x, y$ : objects in  $C$*

$$\exists x \longrightarrow x_1 \longleftarrow x_2 \longrightarrow \cdots \longleftarrow y$$

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## Definition 3.4

*A functor  $P : E \rightarrow B$  between small categories is a covering if*

- ①  $B$  is connected
- ②

$$P : S(x) \rightarrow S(P(x))$$

$$P : T(x) \rightarrow T(P(x))$$

*are bijective for  $\forall x \in \text{Ob}(E)$  where*

$$S(x) = \{f : x \rightarrow * \in \text{Mor}(E)\}, T(x) = \{g : * \rightarrow x \in \text{Mor}(E)\}.$$

# A covering of finite categories and $\zeta$

## Theorem 3.5

- $P : E \rightarrow B$ : a covering of finite categories
- $b \in \text{Ob}(B)$
- $n = \#P^{-1}(b)$ : the number of sheet of  $P$

$$P^{-1}(b) = \{e \in \text{Ob}(E) \mid P(e) = b\}$$

Then, we have

$$\zeta_E(z) = \zeta_B(z)^n.$$

## Example 3.6

$$A_I = \begin{pmatrix} 2 & 3 & 2 \\ 1 & 2 & 6 \\ 1 & 1 & 2 \end{pmatrix}$$

$\text{sum}(A_I^m) = \#N_m(I)$  なので計算可能

$$\zeta_I(z) = \frac{1}{(1 - 6z)^{\frac{125}{37}} (1 - iz)^{\frac{-7+5i}{37}} (1 + iz)^{\frac{-7-5i}{37}}}$$

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- ② 6,  $i$ ,  $-i$  are algebraic integers.

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②  $6, i, -i$  are algebraic integers.

③  $\frac{1}{6} \frac{125}{37} + \frac{1}{i} \frac{-7+5i}{37} + \frac{1}{-i} \frac{-7-5i}{37} = \text{sum}(A_I^{-1}) = \chi_L(I) = \chi_{\Sigma}(I).$

# Main theorem

Theorem 3.7 (N,2012)

$I$ : finite category,  $\exists \chi_{\Sigma}(I)$



$$\zeta_I(z) = \prod_{k=1}^n \frac{1}{(1 - a_k z)^{b_{k,0}}} \exp \left( \sum_{j=1}^{e_k-1} \frac{b_{k,j} z^j}{j(1 - a_k z)^j} \right)$$

for some  $a_k, b_{k,j} \in \mathbb{C}$  and  $n, e_k \in \mathbb{N}$

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- Each  $a_k$  is an eigen value of  $A_I$  and it is an algebraic integer.

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$$\sum_{k=1}^n \sum_{j=0}^{e_k-1} (-1)^j \frac{b_{k,j}}{a_k^{j+1}} = \chi_{\Sigma}(I)$$

# これからの課題

## ① Möbius function とゼータ関数

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} (\operatorname{Re}(s) > 1)$$

where  $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ : Riemann zeta function,  $\mu$ : classical Möbius function

## ② 関数等式

$$\zeta(s) = 2^s \pi^{s-1} \sin \frac{1}{2} s \pi \Gamma(1-s) \zeta(1-s)$$

$$Z\left(\frac{1}{q^n t}\right) = \pm q^{n \frac{E}{2}} t^E Z(t)$$

## ③ リーマン予想の類似物

$$\zeta(s) = 0, s \notin \mathbb{R}, 0 < \operatorname{Re} s < 1 \implies \operatorname{Re}(s) = \frac{1}{2}$$