Mod p decompositions of gauge groups

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Mod p decompositions of Lie groups

Decompositions of Lie groups start with the rational case; for a (p-local) finite H-space X, $\exists$rational decomposition

$$X \simeq_{(0)} S^{2n_1+1} \times \cdots \times S^{2n_\ell+1}$$

- $\mathfrak{t}(X) = \{n_1, \ldots, n_\ell\}$ is called the type of X.
- Put $\mathfrak{t}_i(X) = \{k \in \mathfrak{t}(X) \mid k \equiv i \mod (p-1)\}$ for a prime p.

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Theorem (Mimura, Nishida & Toda '77, Wilkerson '74)

Let G be a compact, simply connected, simple Lie group such that $H_*(G;\mathbb{Z})$ is p-torsion free. Then $\exists B_1, \ldots, B_{p-1}$ satisfying

$$G_{(p)} \simeq B_1 \times \cdots \times B_{p-1}$$

where the type of B_i is $\mathfrak{t}_i(G)$. Moreover, if $G \neq \mathrm{Spin}(2n)$, each B_i is irreducible.

Our object and aim

Let G be a topological group and let P be a principal G-bundle over K. Our object is the gauge group $\mathcal{G}(P)$ which is the topological group of automorphisms of P.

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$\exists$two approaches to gauge groups; one uses mapping spaces via classifying spaces and the other is fiberwise homotopy theory via adjoint bundles.

$\exists$two preceding works; one is due to K & Kono using adjoint bundles and the other is due to Theriault using mapping spaces. This time, we take the mapping space approach and the result of Theriault is actually a special case of ours.

Main result

Theorem

Let G be a compact, simply connected, simple Lie group such that $H_*(G;\mathbb{Z})$ is p-torsion free and $G \neq \mathrm{Spin}(2n)$. Let P be a principal G-bundle over S^{2d+2} with $d \in \mathfrak{t}(G)$. Then $\exists \mathcal{B}_1^P, \ldots, \mathcal{B}_{p-1}^P$ satisfying

$$\mathcal{G}(P)_{(p)} \simeq \mathcal{B}_1^P \times \cdots \times \mathcal{B}_{p-1}^P$$

and a homotopy fiber sequence

$$\Omega(\Omega_0^{2d+1} B_i) \to \mathcal{B}_i^P \to B_{i-d-1}$$

for each $i \in \mathbb{Z}/(p-1)$.

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Remark

For convenience, we put $\mathcal{G}(P)_{(p)} = \Omega(B\mathcal{G}(P)_{(p)})$.

Remark

By the preceding result of K & Kono, the $\mathrm{Spin}(2n)$ case can be deduced from the $\mathrm{Spin}(2n-1)$ case for most bundles.

Outline of the proof

Let $\phi^u : BG_{(p)} \to BG_{(p)}$ be the unstable Adams operation of degree u with $p \nmid u$. The mod p decompositions of Lie groups are obtained by the homotopy colimit of the composite of

$$\Omega\phi^u - u^k : G_{(p)} \to G_{(p)}$$

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Lemma

For $d \in \mathfrak{t}(G)$, $\pi_{2d+2}(BG_{(p)}) \cong \mathbb{Z}_{(p)}$ and $\phi^u_* = u^{d+1}$ in π_{2d+2}.

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$\Rightarrow$ We get

$$\phi^u_* : \mathrm{map}(S^{2d+2}, BG; \alpha)_{(p)} \to \mathrm{map}(S^{2d+2}, BG; \alpha \circ \underline{u^{d+1}})_{(p)}$$

where $\underline{q} : S^n \to S^n$ is the degree q map.

Outline of the proof

$\Rightarrow$ $\exists$self-map

$$(\underline{u^{-d-1}})^* \circ \phi^u_* : \mathrm{map}(S^{2d+2}, BG; \alpha)_{(p)} \to \mathrm{map}(S^{2d+2}, BG; \alpha)_{(p)}$$

satisfying a commutative diagram

$$\begin{tikzcd}
\Omega(\Omega_0^{2d+1} G_{(p)}) \arrow[r, "\substack{u^{-d-1}\circ\Omega^{2d+2}\phi^u - u^k \\ \fallingdotseq u^{-d-1}\circ(\Omega^{2d+2}\phi^u - u^{k+d+1})}"] \arrow[d] & \Omega(\Omega_0^{2d+1} G_{(p)}) \arrow[d] \\
\Omega\mathrm{map}(S^{2d+2}, BG; \alpha)_{(p)} \arrow[r, "(\underline{u^{-d-1}})^*\circ\Omega\phi^u_* - u^k"] \arrow[d] & \Omega\mathrm{map}(S^{2d+2}, BG; \alpha)_{(p)} \arrow[d] \\
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Taking the homotopy colimit of each row, we get a homotopy fiber sequence

$$\Omega(\Omega_0^{2d+1} B_i) \to \mathcal{B}_i^P \to B_{i-d-1}$$

by the following lemma.

Outline of the proof

Lemma

Suppose $\exists$commutative diagram

$$\begin{tikzcd}
F_1 \arrow[r] \arrow[d] & F_2 \arrow[r] \arrow[d] & F_3 \arrow[r] \arrow[d] & \cdots \\
E_1 \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & E_3 \arrow[r] \arrow[d] & \cdots \\
B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & \cdots
\end{tikzcd}$$

in which each column is a homotopy fiber sequence. Then

$$\text{hocolim } F_n \to \text{hocolim } E_n \to \text{hocolim } B_n$$

is also a homotopy fiber sequence.

Principality

We investigate $\Omega(\Omega_0^{2d+1}B_i) \to \mathcal{B}_i^P \to B_{i-d-1}$ by proving its principality for (G, p) in the following table, a condition for each B_i being of rank $\leq p-2$.

$\mathsf{SU}(n)$	$(p-1)(p-2) \geq n-1$
$\mathsf{Sp}(n), \mathrm{Spin}(2n+1)$	$(p-1)(p-2) \geq 2n-1$
G_2, F_4, E_6	$p \geq 5$
E_7, E_8	$p \geq 7$

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The following result of Theriault is a special case of our result.

Let $\ell(G) = \max\{d \in \mathfrak{t}(G)\}$ and let δ_i^α be the composite

$$B_{i-d-1} \xrightarrow{\text{incl}} G_{(p)} \xrightarrow{\delta^\alpha_{(p)}} \Omega_0^{2d+1} G_{(p)} \xrightarrow{\text{proj}} \Omega_0^{2d+1} B_i$$

where $\delta^\alpha : G \to \Omega_0^{2d+1} G$ is the connecting map of the evaluation fibration $\mathrm{map}(S^{2d+2}, BG; \alpha) \to BG$.

Principality

Theorem (Theriault '10)

Let (G, p) be as in the table and let P be a principal G-bundle over S^4 classified by $\alpha \in \pi_4(BG)$.

1. *If $\ell(G) + 2 < p$,*
$$\mathcal{G}(P) \simeq_{(p)} \prod_{i \in \mathfrak{t}(G)} (S^{2i+1} \times \Omega_0^4 S^{2i+1}).$$
2. *Let $G = \mathrm{SU}(n)$ and let Z_i^P be the homotopy fiber of δ_i^α for $i = n, n+1$. Then*
$$\mathcal{G}(P)_{(p)} \simeq Z_n^P \times Z_{n+1}^P \times \prod_{\substack{i \not\equiv n-2, n-1 \\ \bmod (p-1)}} B_i \times \prod_{\substack{i \not\equiv n, n+1 \\ \bmod (p-1)}} \Omega_0^4 B_i.$$

Remark

Theriault's description of the second decomposition is more complicated. Our result simplifies it as above.

Decomposing δ^α

Let X be a homotopy commutative and homotopy associative H-space. X is called universal for $A \subset X$ if for $\forall f : A \to Y$ with a homotopy associative and homotopy commutative H-space Y, $\exists!$H-map $\bar{f} : X \to Y$ extending f, up to homotopy.

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Lemma

If homotopy associative and homotopy commutative H-spaces X_i are universal for A_i ($i = 1, 2$), then so is $X_1 \times X_2$ for $A_1 \vee A_2$.

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Theorem (Theriault '07)

Let (G, p) be as in the table. Then $\exists$homotopy commutative and homotopy associative H-structure μ_i of B_i and $A_i \subset B_i$ satisfying:

1. *B_i is universal for A_i by μ_i.*
2. *With μ_i, the inclusion $A_i \to B_i$ induces an isomorphism*

$$\Lambda(\overline{H}_*(A_i; \mathbb{Z}/p)) \xrightarrow{\cong} H_*(B_i; \mathbb{Z}/p).$$

Decomposing δ^α

Let (G, p) be as in the table and let $\alpha \in \pi_{2d+2}(BG)$ with $d \in \mathfrak{t}(G)$.

Lemma

$\delta^\alpha_{(p)} : G_{(p)} \to \Omega_0^{2d+1} G_{(p)}$ is an H-map with respect to $\mu_1 \times \cdots \times \mu_{p-1}$ on $G_{(p)}$ and the loop structure on $\Omega_0^{2d+1} G_{(p)}$.

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Proposition

If $d + \ell(G) + 1 \leq p(p-1)$, $\delta^\alpha_{(p)} \simeq \delta^\alpha_1 \times \cdots \times \delta^\alpha_{p-1}$.

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If $d + \ell(G) + 1 \leq p(p-1)$, $\delta^\alpha_{(p)} \simeq \delta^\alpha_1 \times \cdots \times \delta^\alpha_{p-1}$.

Proof.

$$\begin{aligned}\delta^\alpha_{(p)}|_{A_1 \vee \cdots \vee A_{p-1}} &\simeq \delta^\alpha_1|_{A_1} \vee \cdots \vee \delta^\alpha_{p-1}|_{A_{p-1}} \\ &= (\delta^\alpha_1 \times \cdots \times \delta^\alpha_{p-1})|_{A_1 \vee \cdots \vee A_{p-1}},\end{aligned}$$

implying the result by the universality of $\mu_1 \times \cdots \times \mu_{p-1}$. $\square$

Decomposing δ^α

Let P be as in the main theorem with the classifying map α.

Theorem

If $d + \ell(G) + 1 \leq p(p-1)$, the homotopy fiber sequence

$$\Omega(\Omega_0^{2d+1} B_i) \to \mathcal{B}_i^P \to B_{i-d-1}$$

is principal and is classified by $\delta_i^\alpha : B_{i-d-1} \to \Omega_0^{2d+1} B_i$.

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Proof.

If F_i^α is the homotopy fiber of δ_i^α, $\exists$commutative diagram

$$\begin{tikzcd}
\Omega(\Omega_0^{2d+1} B_i) \arrow[r] \arrow[d, "\mathrm{incl}"] \arrow[dd, bend right, equal] & F_i^\alpha \arrow[r] \arrow[d, "\exists"] & B_{i-d-1} \arrow[d, "\mathrm{incl}"] \arrow[dd, bend left, equal] \\
\Omega(\Omega_0^{2d+1} G_{(p)}) \arrow[r] \arrow[d, "\mathrm{proj}"] & \mathcal{G}(P) \arrow[r] \arrow[d, "\mathrm{proj}"] & G_{(p)} \arrow[d, "\mathrm{proj}"] \\
\Omega(\Omega_0^{2d+1} B_i) \arrow[r] & \mathcal{B}_i^P \arrow[r] & B_{i-d-1}.
\end{tikzcd}$$

The index $I_p(G,d)$

Put $I_p(G,d) = \{1 \leq i \leq p-1 \mid [A_{i-d-1}, \Omega_0^{2d+1} B_i] = 0\}$.

Corollary

If $d + \ell(G) + 1 \leq p(p-1)$,

$$\mathcal{G}(P)_{(p)} \simeq \prod_{i \notin I_p(G,d)} \mathcal{B}_i^P \times \prod_{i \in I_p(G,d)} (B_{i-d-1} \times \Omega(\Omega_0^{2d+1} B_i)).$$

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If $i \in I_p(G, d)$, $\delta_i^\alpha|_{A_{i-d-1}} \simeq *$, implying $\delta_i^\alpha \simeq *$. □

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Proposition

1. *If $\ell(G) + d + 1 < p$, $I_p(G,d) = \mathfrak{t}(G)$.*
2. *If $d + 1 < p$, $I_p(\mathrm{SU}(n), p)$ includes $1 \leq i \leq p-1$ with $i \not\equiv n, n+1, \ldots, n+d \mod (p-1)$.*

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$\Rightarrow$ The result of Theriault is the special case $d = 1$.

p-local homotopy types of gauge groups

Let P_k be a principal $\mathrm{SU}(n)$-bundle over S^4 classified by $k \in \mathbb{Z} \cong \pi_4(B\mathrm{SU}(n))$.

Theorem (Sutherland '92, Hamanaka & Kono '06)

If $\mathcal{G}(P_k) \simeq \mathcal{G}(P_\ell)$, then $(n(n^2-1), k) = (n(n^2-1), \ell)$.

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We prove the converse when localized at p.

Proposition (Whitehead '46)

For $\alpha \in \pi_d(X)$, δ^α corresponds to the Samelson product $\langle \bar{\alpha}, 1_{\Omega X} \rangle$ through

$$[\Omega X, \Omega_0^d X] \cong [S^{d-1} \wedge \Omega X, \Omega X]$$

where $\bar{\alpha} \in \pi_{d-1}(\Omega X)$ is the adjoint of α.

p-local homotopy types of gauge groups

Proposition

Let $G = \mathrm{SU}(n)$ and $n - 1 \leq (p-1)(p-2)$. For $i = n, n+1$, the order of δ_i^k is $\min\{\nu_p(i(i-1)), \nu_p(k)\}$.

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Proof.

Comparing $[A_{i-2}, \Omega_0^3\mathrm{SU}(n)_{(p)}]$ with $K^{-1}(A_{i-2})_{(p)}$, we can calculate that the order of δ_i^1 is $\nu_p(i(i-1))$ for $i = n, n+1$. By linearity of Samelson products, $\delta_i^k = k \cdot \delta_i^1$. □

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Theorem

Let $n-1 \leq (p-1)(p-2)$. Then $\mathcal{G}(P_k)_{(p)} \simeq \mathcal{G}(P_\ell)_{(p)}$ if and only if

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Proof.

$\mathcal{G}(P_k)_{(p)} \simeq \mathcal{G}(P_\ell)_{(p)} \Leftrightarrow |\delta_i^k| = |\delta_i^\ell|$ for $i = n, n+1$

$\Leftrightarrow$ $\min\{\nu_p(i(i-1)), \nu_p(k)\} = \min\{\nu_p(i(i-1)), \nu_p(\ell)\}$ for $i = n, n+1$.

$\square$

Adjoint bundles

Let G be a topological group and let P be a principal G-bundle over K. The adjoint bundle $\operatorname{ad} P$ is defined as

$$\operatorname{ad} P = (P \times G)/(x, g) \sim (x \cdot h, h^{-1}gh).$$

$\Rightarrow$ $\operatorname{ad} P$ is a fiberwise topological group over K.
$\Rightarrow$ The set of sections $\Gamma(\operatorname{ad} P)$ is a topological group.

$\exists$isomorphism of topological groups

$$\mathcal{G}(P) \cong \Gamma(\operatorname{ad} P).$$

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$\Rightarrow$ $\mathrm{ad}\, P$ is a fiberwise topological group over K.
$\Rightarrow$ The set of sections $\Gamma(\mathrm{ad}\, P)$ is a topological group.

$\exists$isomorphism of topological groups

$$\mathcal{G}(P) \cong \Gamma(\mathrm{ad}\, P).$$

For fiberwise spaces $X_1, \ldots, X_n$ over K,

$$\Gamma(X_1 \times_K \cdots \times_K X_n) = \Gamma(X_1) \times \cdots \times \Gamma(X_n).$$

$\Rightarrow$ A fiberwise mod p decomposition of $\mathrm{ad}\, P$ induces a mod p decomposition of $\mathcal{G}(P)$.

Adjoint bundles

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For fiberwise spaces $X_1, \ldots, X_n$ over K,

$$\Gamma(X_1 \times_K \cdots \times_K X_n) = \Gamma(X_1) \times \cdots \times \Gamma(X_n).$$

$\Rightarrow$ A fiberwise mod p decomposition of $\operatorname{ad} P$ induces a mod p decomposition of $\mathcal{G}(P)$.

Question

Is every mod p decomposition of $\mathcal{G}(P)$ induced from a fiberwise mod p decomposition of $\operatorname{ad} P$?

Uniqueness of mod p decompositions of Lie groups

Theorem (Wilkerson '75)

Let X be a simply connected, finite H-space. If

$$X_{(p)} \simeq X_1 \times \cdots \times X_n,$$

where each X_i is irreducible, then the homotopy types of $X_1, \ldots, X_n$ are unique up to permutations.

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Corollary

Let G be as in the main theorem. For any decomposition

$$G_{(p)} \simeq X_1 \times \cdots \times X_n,$$

there is a partition $I_1 \sqcup \cdots \sqcup I_n = \{1, \ldots, p-1\}$ such that

$$X_k \simeq \prod_{i \in I_k} B_i$$

for $k = 1, \ldots, n$.

Cohomology of free loop spaces

K & Kono constructed the following map in cohomology.

Theorem (K & Kono '10)

$\exists$*homomorphism*

$$\hat{\sigma} : H^*(X; R) \to H^{*-1}(\mathcal{L}X; R)$$

satisfying:

1. $\hat{\sigma}$ *restricts to* $\sigma : H^*(X; R) \to H^{*-1}(\Omega X; R)$.
2. $\hat{\sigma}$ *is a derivation.*
3. $\hat{\sigma}$ *commutes with cohomology operations.*

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Corollary

Let X be a simply connected space. If $H^*(X;R) = R[x_1, \dots, x_\ell]$,

$$H^*(\mathcal{L}X;R) = R[x_1, \dots, x_\ell] \otimes \Delta(\hat{\sigma}(x_1), \dots, \hat{\sigma}(x_\ell)).$$

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Proof.

$H^*(\Omega X; R) = \Delta(\sigma(x_1), \ldots, \sigma(x_\ell))$ by the Borel transgression theorem. Then we apply the Leray-Hirsch theorem to the fiber sequence $\Omega X \to \mathcal{L}X \to X$. $\square$

Cohomology of adjoint bundles

For the universal bundle $EG \to BG$,

$$\operatorname{ad} EG \simeq_{BG} \mathcal{L}BG.$$

$\Rightarrow$ For the classifying map $\alpha : K \to BG$ of P,

$$\operatorname{ad} P \simeq_{K} \alpha^{-1}\mathcal{L}BG.$$

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Lemma

Let P be a principal G-bundle over S^4 classified by the inclusion $j : S^4 \to BG$. If $H^*(BG; \mathbb{Z}/p) = \mathbb{Z}/p[x_1, \ldots, x_\ell]$,

$$H^*(\operatorname{ad} P; \mathbb{Z}/p) = \Lambda(u_4, \hat{x}_1, \ldots, \hat{x}_\ell)$$

where $\hat{x}_i = \bar{j}^*(\hat{\sigma}(x_i))$ for the lift $\bar{j} : \operatorname{ad} P \to \mathcal{L}BG$ of j.

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Proof.

$\hat{x}_1, \ldots, \hat{x}_\ell$ restricts to generators of the cohomology of the fiber G of $\mathrm{ad}\, P \to S^4$. Apply the Leray-Hirsch theorem. □

Indecomposability of adjoint bundles

Let $r_p(G)$ be the number of $i \in \mathbb{Z}/(p-1)$ such that $\mathfrak{t}_i(G) \neq \emptyset$.

Recall that if $H_*(G;\mathbb{Z})$ is p-torsion free,

$$H^*(BG;\mathbb{Z}/p) = \mathbb{Z}/p[x_{2i+2} \mid i \in \mathfrak{t}(G)].$$

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Theorem

Let G be as in the main theorem. Let P be a principal G-bundle over S^4 classified by $1 \in \mathbb{Z} \cong \pi_4(BG)$. Suppose the following conditions.

1. $p > 3$ *and* $p - 2 \in \mathfrak{t}(G)$.
2. $\mathcal{P}^1 x_4 = a x_4 x_{2p-2} + \cdots$ *for* $a \in (\mathbb{Z}/p)^\times$.

Then if

$$(\mathrm{ad}\, P)^{\mathrm{fib}}_{(p)} \simeq_{S^4} X_1 \times_{S^4} \cdots \times_{S^4} X_{r_p(G)},$$

X_i is trivial for some i.

Indecomposability of adjoint bundles

Proof.

If none of X_i is trivial, we may assume the fiber of $X_i \to S^4$ is B_i. But since

$$\mathcal{P}^1 \hat{x}_3 = a u_4 \hat{x}_{2p-3} + \cdots$$

in $H^*(\mathrm{ad}\, P; \mathbb{Z}/p)$, $B_1 \times B_{p-2}$ is included in the fiber of $X_1 \to S^4$, a contradiction. $\square$

Indecomposability of adjoint bundles

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Corollary

Let G be as in the main theorem and $G \neq \mathrm{SU}(2), \mathrm{SU}(3)$. Let P be a principal G-bundle over S^4 classified by $1 \in \mathbb{Z} \cong \pi_4(BG)$. Then $\exists p$ such that if

$$(\mathrm{ad}\, P)^{\mathrm{fib}}_{(p)} \simeq_{S^4} X_1 \times_{S^4} \cdots \times_{S^4} X_{r_p(G)},$$

X_i is trivial for some i.

Corollary

Let G, P be as above. Then for some p, the mod p decomposition of $\mathcal{G}(P)$ is not induced from that of $\mathrm{ad}\, P$.