

Multiplicative structure of mod 2 ring spectra and its applications to algebraic K -theory

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Background

Throughout in this talk, we consider in the category of spectra.

E : ring spectrum with

- $\iota : S^0 \rightarrow E$ (unit map),
- $m : E \wedge E \rightarrow E$ (multiplication).

Remark.

In this talk, we don't suppose that E is commutative or associative.

id_X : identity map of X

$T_{X,Y} : X \wedge Y \rightarrow Y \wedge X$: switching map

If E and F are (commutative, associative) ring spectra, then so is $E \wedge F$ with the structure below.

$$\begin{aligned} S^0 &= S^0 \wedge S^0 \xrightarrow{\iota_E \wedge \iota_F} E \wedge F, \\ (E \wedge F) \wedge (E \wedge F) &\xrightarrow{\text{id}_E \wedge T_{F,E} \wedge \text{id}_F} E \wedge E \wedge F \wedge F \xrightarrow{m_E \wedge m_F} E \wedge F. \end{aligned}$$

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When E is a commutative associative ring spectrum, $\pi_*(E)$ can be a (graded) commutative ring.

p : prime number

- $M_p := S^0 \cup_p e^1$: mod p Moore spectrum,
- $i_p : S^0 \rightarrow M_p$: inclusion to the bottom cell,
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$$S^0 \xrightarrow{P} S^0 \xrightarrow{i_p} M_p \xrightarrow{j_p} S^1 : \text{cofiber sequence of spectra}$$

Fact

- If $p > 3$, then M_p is a commutative associative ring spectrum.
- At the prime 3, M_3 is a commutative ring spectrum, but it is not associative. (Toda, '71)

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- $\pi_0(M_p) = \mathbb{Z}/p\mathbb{Z} \cdot i_p.$

If M_2 has a ring spectrum structure, then i_2 is the unit map.

$$M_2 \xrightarrow{2\text{id}_{M_2}} M_2 \xrightarrow{i_2 \wedge \text{id}_{M_2}} M_2 \wedge M_2 \xrightarrow{j_2 \wedge \text{id}_{M_2}} \Sigma M_2$$
$$\exists_m$$

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When does $E/2 := M_2 \wedge E$ become a commutative associative ring spectrum again ?

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Unitality

Lemma

If E is a ring spectrum, then

$$E/2 \text{ is a ring spectrum} \iff 2\text{id}_{E/2} = 0.$$

Assume that $2\text{id}_{E/2} = 0$.

$$\begin{array}{ccccc} E/2 & \xrightarrow{2\text{id}_{E/2}=0} & E/2 & \xrightarrow{i_2 \wedge \text{id}_{E/2}} & M_2 \wedge (E/2) \\ & & \swarrow & & \xrightarrow{j_2 \wedge \text{id}_{E/2}} \\ & & \exists \mu_0 & & \Sigma(E/2) \end{array}$$

For $x \in [M_2 \wedge (E/2), E/2]$, we define $m(x) \in [(E/2) \wedge (E/2), E/2]$ to be

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$$[\Sigma E, E/2] \xrightarrow{\begin{array}{c} (j_2 \wedge \text{id}_E)^* \\ \exists \xi_0 \end{array}} [E/2, E/2] \xrightarrow{\begin{array}{c} (\mu_0 \circ (\text{id}_{M_2} \wedge i_2 \wedge \text{id}_E))^* \\ \text{id}_{E/2} - \mu_0(\text{id}_{M_2} \wedge i_2 \wedge \text{id}_E) \end{array}} [E, E/2] \xrightarrow{\begin{array}{c} (i_2 \wedge \text{id}_E)^* \\ \mapsto 0 \end{array}}$$

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$$\mu_1 := \mu_0 + \xi_1(j_2 \wedge \text{id}_{E/2}) \in [M_2 \wedge (E/2), E/2].$$

Then we have $m_1 := m(\mu_1) \in [(E/2) \wedge (E/2), E/2]$.

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$$[\Sigma E, E/2] \xrightarrow{\begin{array}{c} (j_2 \wedge \text{id}_E)^* \\ \exists \xi_0 \end{array}} [E/2, E/2] \xrightarrow{\begin{array}{c} (i_2 \wedge \text{id}_E)^* \\ \text{id}_{E/2} - \mu_0(\text{id}_{M_2} \wedge i_2 \wedge \text{id}_E) \end{array}} [E, E/2] \xrightarrow{\begin{array}{c} (i_2 \wedge \text{id}_E)^* \\ \mapsto \\ 0 \end{array}}$$

$$[\Sigma(E/2), E/2] \xrightarrow{\begin{array}{c} (\Sigma i_2 \wedge \text{id}_E)^* \\ \exists \xi_1 \end{array}} [\Sigma E, E/2] \xrightarrow{\begin{array}{c} 2=0 \\ \xi_0 \end{array}} [\Sigma E, E/2]$$

$$\mu_1 := \mu_0 + \xi_1(j_2 \wedge \text{id}_{E/2}) \in [M_2 \wedge (E/2), E/2].$$

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Commutativity

- η : stable complex Hopf map.

$$\begin{array}{ccc} \Sigma M_2 & & \\ \uparrow \Sigma i_2 & \searrow \bar{\eta} & \\ S^1 & \xrightarrow{\eta} & S^0. \end{array}$$

- $[\Sigma M_2, S^0] = \mathbb{Z}/4\mathbb{Z} \cdot \bar{\eta}$ with

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Hereafter we assume that a ring spectrum E we consider satisfies that $2\text{id}_{E/2} = 0$.

Lemma

If E is a commutative ring spectrum, then the statements below are equivalent.

- ① The pair $(i_2 \wedge \iota, m_1)$ is a commutative ring spectrum structure on $E/2$.
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Associativity

$$\begin{array}{ccccccc}
 E/2 & \xrightarrow{2\text{id}_{E/2}=0} & E/2 & \xrightarrow{i_2 \wedge \text{id}_{E/2}} & M_2 \wedge (E/2) & \xrightarrow{j_2 \wedge \text{id}_{E/2}} & \Sigma(E/2) \\
 & & \swarrow \mu_1 & & \swarrow \exists! \hat{\mu}_1 & & \\
 & & & & & &
 \end{array}$$

such that

- ① $(j_2 \wedge \text{id}_{E/2})\hat{\mu}_1 = \text{id}_{\Sigma(E/2)}$,
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The map $\alpha \in [\Sigma(E/2), E/2]$ is given by

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 \Sigma(E/2) & \xrightarrow{\text{id}_{\Sigma M_2} \wedge \iota \wedge \text{id}_E} & \Sigma(E/2) \wedge E & \xrightarrow{\hat{\mu}_1 \wedge \text{id}_E} & M_2 \wedge (E/2) \wedge E \\
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 \mu_2 &:= \mu_1 + \alpha(j_2 \wedge \text{id}_{E/2}) \in [M_2 \wedge (E/2), E/2], \\
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Lemma

Let E be a commutative associative ring spectrum such that $2\text{id}_{E/2} = 0$. Then, if the pair $(i_2 \wedge \iota, m_1)$ is a commutative ring spectrum structure, then $(i_2 \wedge \iota, m_2)$ gives a commutative associative ring spectrum structure on $E/2$.

Theorem

When E is a commutative associative ring spectrum, the spectrum $E/2$ can be a commutative associative ring spectrum again if E satisfies $2\text{id}_{E/2} = 0$ and the following equivalent conditions.

- ① $\widetilde{\Sigma}\bar{\eta} \wedge \text{id}_E = 0 \in [\Sigma^2 E, E/2]$.
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$$\pi_2(E) \xrightarrow{2} \pi_2(E) \xrightarrow{(\Sigma j_2)^*} [\Sigma M_2, E] \xrightarrow{(\Sigma i_2)^*} \pi_1(E) \xrightarrow{2} \pi_1(E)$$

↓↓

$$0 \rightarrow \pi_2(E)/2\pi_2(E) \xrightarrow{(\Sigma j_2)^*} [\Sigma M_2, E] \xrightarrow{(\Sigma i_2)^*} \pi_1(E)[2] \rightarrow 0$$

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Associativity of m_2 is not depend on $\pi_3(E)$!!

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Applications to Algebraic K -theory

Algebraic K -theory

$$\begin{array}{ccc} \{\text{Waldhausen categories}\} & \xrightarrow{K(-)} & \{\text{Spectra}\} \\ \mathcal{C} & \mapsto & K(\mathcal{C}) \end{array}$$

- $K(\mathcal{C})$ is called the algebraic K -theory of \mathcal{C} .
- Let R be a ring and \mathcal{P}_R be the category of finitely generated projective R -modules. Then $K(R) := K(\mathcal{P}_R)$ and it is called the algebraic K -theory of R .

Topological Hochschild homology

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$K_*(\mathcal{C}) := \pi_*(K(\mathcal{C}))$ and $THH_*(\mathcal{C}) := \pi_*(THH(\mathcal{C}))$.

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$B : K(\mathcal{C}) \rightarrow THH(\mathcal{C})$

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$THH_*(\mathbb{Z})$ is already determined by Bökstedt-Madsen, but $K_*(\mathbb{Z})$ still has many open problems.

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$THH_*(\mathbb{Z})$ is already determined by Bökstedt-Madsen, but $K_*(\mathbb{Z})$ still has many open problems.

Applications to Algebraic K -theory

Let p be a prime.

- $THH(\mathcal{C})$ is an S^1 -spectrum. (Bökstedt)
- TR -spectrum of \mathcal{C} is given by

$$TR^n(\mathcal{C}; p) := THH(\mathcal{C})^{C_{p^{n-1}}}$$

where $C_{p^{n-1}}$ is the cyclic group of order p^{n-1} .

- For any $n > 1$, maps $F, R : TR^n(\mathcal{C}; p) \rightarrow TR^{n-1}(\mathcal{C}; p)$, Frobenius and restriction, are defined.
- $TR(\mathcal{C}; p) := \text{holim}_R TR^n(\mathcal{C}; p)$.
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Cycrotomic trace map

Böksted-Hsiang-Madsen constructed the map

$$tr : K(\mathcal{C}) \rightarrow TC(\mathcal{C}; p),$$

cycrotomic trace map, such that

$$\begin{array}{ccc} & tr & \rightarrow \\ K(\mathcal{C}) & \xrightarrow{B} & TC(\mathcal{C}; p) \\ & \curvearrowleft & \downarrow \\ & & TR^n(\mathcal{C}; p) \\ & & \downarrow \\ & & THH(\mathcal{C}) = TR^1(\mathcal{C}; p) \end{array}$$

is commutative. So $tr_* : K_*(\mathcal{C}) \rightarrow TC_*(\mathcal{C}; p) := \pi_*(TC(\mathcal{C}; p))$ has more information of $K_*(\mathcal{C})$ than B_* .

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Applications to Algebraic K -theory

- K : complete discrete valuation field of char.= 0 with the perfect residue field of char.= $p > 0$.
- V : valuation ring of K .

In this situation, the Waldhausen category $\mathcal{C}_{V|K}$ is given by

Objects	=	bounded chain complexes in \mathcal{P}_V ,
Arrows	=	chain maps,
w -cofibrations	=	degree-wise monomorphisms,
w -weak equivalences	=	$\{f : K \otimes_V f \text{ is a quasi-iso.}\}$.

$$THH(V|K) := THH(\mathcal{C}_{V|K}).$$

$TR^n(V|K; p)$ and $TC(V|K; p)$ are given as above.

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Applications to Algebraic K-theory

Hereafter we will consider at $p = 2$.

Let $W_n\Omega_{(V,M)}^*$ be the 2-typical de Rham-Witt complex over the canonical log ring (V, M) of length n .

Facts

- The canonical map

$$W_n\Omega_{(V,M)}^q \rightarrow TR_q^n(V|K; 2) := \pi_q(TR^n(V|K; 2))$$

is isomorphic for $q \leq 2$. (Hesselholt-Madsen)

- $TR_2^n(V|K; 2)$ is uniquely divisible. (Hesselholt-Madsen)

Hence

$$\begin{array}{ccccc} TR_2^n(V|K; 2) & \xrightarrow{\text{2=iso.}} & TR_2^n(V|K; 2) & & \\ & \xrightarrow{(\Sigma j_2)^*=0} & [\Sigma M_2, TR^n(V|K; 2)] & \xrightarrow{(\Sigma i_2)^*=\text{inj.}} & TR_1^n(V|K; 2) \end{array}$$

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- If $\sqrt{-1} \in K$, then $(TR^n(V|K; 2)/2 \xrightarrow{2\text{id}} TR^n(V|K; 2)/2) = 0$.

Theorem

If $\sqrt{-1} \in K$ and $d\log[-1]_n = 0$ in $W_n\Omega_{(V, M)}^1$, then

$TR^n(V|K; 2)/2 = M_2 \wedge TR^n(V|K; 2)$ is a commutative associative ring spectrum. In particular, at $n = 1$, if $\sqrt{-1} \in K$, then

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If this problem is settled, then we can handle $TR^n(V|K; 2)/2$ as a ring spectrum. So we can expect that 2-primary $K(K)$ can be investigated by the cyclotomic trace map.

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Thank you for your attention!