A delooped "$+ = S$" theorem in algebraic K-theory

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In this talk, I shall report the following:

Theorem 0.1 (delooped "$+ = S$" theorem). *For a ring R with the invariant basis property (which is satisfied by any commutative ring), there is a fibration-sequence-up-to-homotopy*

$$B\left(\coprod_{n\geqq 0} BGL_n(R)\right) \to \left|N_\bullet\left(iS_\bullet\mathbf{P}(R)\right)\right| \to B\left(K_0(R)/\mathbb{Z}\right), \tag{1}$$

which is natural with respect to ring homomorphisms between rings with invariant basis property.

The map $B\left(\coprod_{n\geqq 0} BGL_n(R)\right) \to \left|N_\bullet\left(iS_\bullet\mathbf{P}(R)\right)\right|$ in (1) is an explicitly constructed very simple map, and the proof is given in the framework of Waldhausen K-theory in a straight forward fashion. Such is the case, we do not have to resort to the Quillen "$+ = Q$" theorem in our proof. To the contrary , Theorem 0.1 immediately implies the following Waldhausen analogue of "$+ = Q$ theorem":

Corollary 0.2 ("$+ = S$" theorem). *For a ring R with the invariant basis property, there is a (not natural) homotopy equivalence*

$$K_0(R) \times BL_\infty(R)^+ \xrightarrow{\simeq} K(R), \tag{2}$$

where $K(R)$ is the Waldhausen K-theory.

Of course, using the Waldhausen equivalence between the Waldhausen K-theory and the QUillen K-theory for exact categories, Corollary 0.2 implies the original Quillen "$+ = Q$" theorem.

Theorem 0.1 is used elsewhere to supply some details in the Morel-Voevodsky proof of their K-theory representability theorem.

It appears that Theorem 0.1 can be generalized under more general setting.