QUANTUM D-MODULES OF TORIC VARIETIES AND OSCILLATORY INTEGRALS

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ABSTRACT. We review mirror symmetry for the quantum cohomology D-module of a compact weak-Fano toric manifold. We also discuss the relationship to the GKZ system, the Stanley-Reisner ring, the Mellin-Barnes integrals, and the $\widehat{\Gamma}$-integral structure.

1. INTRODUCTION

The purpose of the present notes is to give a concise review on mirror symmetry for the quantum D-modules of toric varieties, as proposed by Givental [26]. Our goal will be very modest: we restrict to weak-Fano compact toric manifolds and describe their mirror symmetry concretely and explicitly. We will also discuss the $\widehat{\Gamma}$-integral structure of the quantum D-modules and its role in mirror symmetry.

A mirror of a Fano manifold X is given by a Landau-Ginzburg model, that is, a complex manifold Y equipped with a holomorphic function (potential) $W: Y \to \mathbb{C}$ (see [3, 12, 26, 28, 39]). Under mirror symmetry, it is expected that the quantum cohomology of X is isomorphic to the Jacobi ring of W, and that the quantum cohomology D-module of X is isomorphic to the mirror D-module whose solutions are oscillatory integrals associated with W (see Table 1). For a (weak-)Fano toric manifold X, the mirror W is given by a family of Laurent polynomials whose Newton polytopes are given by the fan diagram of X, and the mirror correspondence as in Table 1 is by now well-established. We will explain the relationship between the mirror D-module and the Gelfand-Kapranov-Zelevinsky (GKZ) system and how the (quantum) Stanley-Reisner ring arises as a limit of the mirror D-module. The GKZ system or the Stanley-Reisner rings can be regarded as intermediate objects between quantum cohomology of toric varieties and their mirrors.

TABLE 1. Mirror correspondence.

Fano manifold X	Landau-Ginzburg model $W: Y \to \mathbb{C}$	toric case
quantum cohomology ring	Jacobi ring (or $\mathbb{H}(Y, (\Omega_Y^\bullet, dW))$)	quantum Stanley-Reisner ring
quantum D-module $zQ\frac{\partial}{\partial Q} + p\star$	D-module of oscillatory integrals $\int e^{W/z}\Omega$	GKZ system $\Box_d I = 0$

Even for Fano manifolds other than toric varieties, the mirror space Y is often given by (a partial compactification of) the algebraic torus $(\mathbb{C}^\times)^n$ and the potential W is given by a Laurent polynomial on it; however, the coefficients of the mirror

Laurent polynomial W are very special (see e.g. [13, 48]). Many people have observed (see [6]) that, when a Fano manifold X admits a flat degeneration to a (possibly singular) toric variety X', the mirror of X is often given as a special subfamily of the Laurent polynomials that are mirror to a crepant resolution of X'. Such a subfamily corresponds to a certain stratum in the discriminant loci of the GKZ system for X'. Therefore, toric mirror symmetry can be viewed as a generic phenomenon whereas mirror symmetry for more general Fano manifolds can be viewed as its specialization. In this article, we do not delve into mirror symmetry for non-toric Fano manifolds; we refer the reader to the article of Golyshev [29] in the same volume for the differential equations aspects of the story[1].

In the last two sections, we will discuss the $\widehat{\Gamma}$-integral structure [36, 38] in the context of toric mirror symmetry. The $\widehat{\Gamma}$-integral structure is a certain integral local system underlying the quantum D-module, which is defined by the $\widehat{\Gamma}$-class and the topological K-group. Conjecturally this corresponds to the natural integral structure on the mirror. The $\widehat{\Gamma}$-class involves transcendental numbers such as the Riemann ζ-values and its origin is quite mysterious from a viewpoint of curve counting or symplectic topology. On the other hand, the existence of such a structure had been suggested since the beginning of mirror symmetry. Candelas-de la Ossa-Green-Parkes [8] already observed that $\chi(X)\zeta(3)$ appears in the asymptotics of periods of mirror Calabi-Yau threefolds near the large complex structure limit (LCSL). Hosono-Klemm-Theisen-Yau [35] observed "remarkable identities" that relate certain characteristic numbers of complete intersection Calabi-Yau manifolds with hypergeometric solutions of the Picard-Fuchs equation of the mirror family. This observation led Libgober [42] to introduce the (inverse) $\widehat{\Gamma}$-class of a complex manifold. Later, Hosono [34] made this connection more precise in terms of central charges and homological mirror symmetry.

When a Fano manifold X is mirror to the Landau-Ginzburg model $W\colon Y \to \mathbb{C}$, the compatibility of the $\widehat{\Gamma}$-integral structure and mirror symmetry would imply the following "mirror symmetric Gamma conjecture" (cf. [23, 24], see §7)

$$\int_{\Gamma\subset Y} e^{-tW}\Omega \sim \int_X t^{-c_1(X)}\cdot \widehat{\Gamma}_X \quad \text{as } t\to 0.$$

This says that the asymptotics of the mirror oscillatory integral is described in terms of the $\widehat{\Gamma}$-class of X. The integration cycle Γ here should be a Lagrangian section of the Strominger-Yau-Zaslow fibration [53] (i.e. mirror to the structure sheaf of X). In the last section, we will introduce Mellin-Barnes integral representations for the mirror oscillatory integrals (in the toric case) and explain these $\widehat{\Gamma}$-phenomena for toric varieties.

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2. Preliminaries on toric varieties

A toric variety is a GIT quotient of a vector space $\mathbb{C}^m$ by a torus $K \cong (\mathbb{C}^\times)^k$, where the torus K acts on $\mathbb{C}^m$ via an injective group homomorphism $K \to (\mathbb{C}^\times)^m$. Let $D_1, \dots, D_m \in \operatorname{Hom}(K, \mathbb{C}^\times)$ denote the characters defining the homomorphism $K \to (\mathbb{C}^\times)^m$. A GIT quotient is given by choosing a stability condition $\omega \in \mathfrak{k}_\mathbb{R}^*$, where $\mathfrak{k}_\mathbb{R}$ denotes the Lie algebra of the maximally compact subgroup $K_\mathbb{R}$ of K. We assume that

(a) ω lies in the cone $\sum_{i=1}^m \mathbb{R}_{\ge 0} D_i$;
(b) if ω lies in $\sum_{i \in I} \mathbb{R}_{>0} D_i$ for a subset $I \subset \{1, \dots, m\}$, then $\{D_i\}_{i \in I}$ spans $\mathfrak{k}_\mathbb{R}^*$ over $\mathbb{R}$;
(c) the cone $\sum_{i=1}^m \mathbb{R}_{\ge 0} D_i$ is strictly convex.

Then the toric variety corresponding to ω is defined by

$$X_\omega := U_\omega / K$$

where $U_\omega := \mathbb{C}^m \setminus \bigcup_I \mathbb{C}^I$ with I ranging over subsets of $\{1, \dots, m\}$ such that $\omega \notin \sum_{i \in I} \mathbb{R}_{>0} D_i$. The conditions (a), (b), (c) respectively ensure that X_ω is non-empty, that X_ω has at worst orbifold singularities, and that X_ω is compact. The space $\mathfrak{k}_\mathbb{R}^*$ of stability conditions has a fan structure, called the GKZ fan. A maximal cone of the GKZ fan is given by the closure of a connected component of the set of $\omega \in \mathfrak{k}_\mathbb{R}^*$ satisfying the conditions (a)–(c). The toric variety X_ω depends only on the maximal cone A_ω to which ω belongs.

We assume that X_ω is weak-Fano, or equivalently, that $c_1(X_\omega)$ is nef. For simplicity of notation and exposition, we will also assume that X_ω is a smooth manifold (without orbifold singularities) and that $\{z_i = 0\} \cap U_\omega$ is non-empty for all $i = 1, \dots, m$, where z_i is the ith co-ordinate on $\mathbb{C}^m$. With these assumptions, we have $\mathfrak{k}_\mathbb{R}^* \cong H^2(X_\omega, \mathbb{R})$ and the closure of the ample cone of X_ω corresponds to the maximal cone A_ω of the GKZ fan. Under this isomorphism, the character D_i corresponds to the class of the toric divisor $\{z_i = 0\}$ in X_ω.

Remark 1. Although we restrict to smooth toric manifolds in this article, all the results discussed in this paper can be extended to toric orbifolds [10, 11, 14, 36, 44]. In the orbifold case, it is important to allow $\{z_i = 0\} \cap U_\omega = \emptyset$ for some i as such indices correspond to twisted sectors.

A toric variety can be also described in terms of a fan. Let $\mathfrak{k}_\mathbb{Z} = \operatorname{Hom}(\mathbb{C}^\times, K)$ denote the cocharacter lattice of K. Consider the natural map $\mathfrak{k}_\mathbb{Z} \to \mathbb{Z}^m$ induced by the inclusion $K \to (\mathbb{C}^\times)^m$ and complete it to a short exact sequence

$$0 \to \mathfrak{k}_\mathbb{Z} \to \mathbb{Z}^m \to N \to 0$$

where $N := \mathbb{Z}^m / \mathfrak{k}_\mathbb{Z}$ is a free abelian group of rank $n = m - k$. The fan Σ_ω of X_ω is defined on the vector space $N_\mathbb{R} = N \otimes \mathbb{R}$: the image $b_i \in N$ of $e_i \in \mathbb{Z}^m$ gives a generator of a 1-dimensional cone of Σ_ω, and the cone $\sum_{i \in I} \mathbb{R}_{\ge 0} b_i$ belongs to Σ_ω if and only if $\omega \in \sum_{i \notin I} \mathbb{R}_{>0} D_i$.

3. Quantum D-modules

The cocharacter lattice $\mathfrak{k}_{\mathbb{Z}} = \mathrm{Hom}(\mathbb{C}^\times, K)$ is identified with $H_2(X_\omega, \mathbb{Z})$ and the dual cone $A_\omega^\vee \subset \mathfrak{k}_{\mathbb{R}}$ of A_ω is identified with the cone of curves. The (small) quantum product $\star$ is a commutative and associative product on the space $H^*(X_\omega) \otimes \mathbb{C}[\![A_\omega^\vee \cap \mathfrak{k}_{\mathbb{Z}}]\!]$ defined by

$$(\alpha \star \beta, \gamma) = \sum_{d \in A_\omega^\vee \cap \mathfrak{k}_{\mathbb{Z}}} \langle \alpha, \beta, \gamma \rangle_{0,3,d} Q^d \qquad \text{for all } \alpha, \beta, \gamma \in H^*(X_\omega),$$

where $\langle \alpha, \beta, \gamma \rangle_{0,3,d}$ is the genus-zero, three-points, degree-d Gromov-Witten invariant and Q^d denotes the element of $\mathbb{C}[\![A_\omega^\vee \cap \mathfrak{k}_{\mathbb{Z}}]\!]$ corresponding to d. The Dubrovin connection is a flat connection on the trivial $H^*(X_\omega)$-bundle over $\mathrm{Spec}\,\mathbb{C}[\![A_\omega^\vee \cap \mathfrak{k}_{\mathbb{Z}}]\!]$ given by

$$\nabla_{\xi Q \frac{\partial}{\partial Q}} = \xi Q \frac{\partial}{\partial Q} + \frac{1}{z}(\xi \star), \qquad \xi \in H^2(X_\omega) \tag{1}$$

where z is a formal parameter and $\xi Q \frac{\partial}{\partial Q}$ is a derivation of $\mathbb{C}[\![A_\omega^\vee \cap \mathfrak{k}_{\mathbb{Z}}]\!]$ defined by $(\xi Q \frac{\partial}{\partial Q}) Q^d = \langle \xi, d \rangle Q^d$. This is a flat connection with logarithmic singularities. The Dubrovin connection $z\nabla$ (multiplied by z) acts on the space

$$\mathrm{QDM}(X_\omega) = H^*(X_\omega) \otimes \mathbb{C}[z][\![A_\omega^\vee \cap \mathfrak{k}_{\mathbb{Z}}]\!]$$

which we call the quantum D-module. It is not known in general whether the quantum product $\star$ converges or not. For toric varieties, it is known that the quantum product converges and hence the quantum D-module extends to an actual analytic neighbourhood of the origin "$Q = 0$" in $\mathrm{Spec}\,\mathbb{C}[A_\omega^\vee \cap \mathfrak{k}_{\mathbb{Z}}]$. The point $Q = 0$ is called the large radius limit point.

Remark 2. The Dubrovin connection can be also extended in the z-direction. The connection in the z-direction is given by

$$\nabla_{z\partial_z} = z \frac{\partial}{\partial z} - \frac{1}{z} c_1(X_\omega) \star + \mu$$

where $\mu \in \mathrm{End}(H^*(X_\omega))$ is defined by $\mu(\alpha) = (p - \frac{n}{2})\alpha$ for $\alpha \in H^{2p}(X_\omega)$ with $n = \dim_{\mathbb{C}} X_\omega$.

4. Mirror D-modules

We have the exact sequence of tori $1 \to K \to (\mathbb{C}^\times)^m \to T \to 1$ where $T := (\mathbb{C}^\times)^m / K \cong N \otimes \mathbb{C}^\times$ is a torus acting on X_ω with an open dense orbit. Consider the exact sequence $1 \to \check{T} \to (\mathbb{C}^\times)^m \to \check{K} \to 1$ of dual tori. The mirror Landau-Ginzburg model of a toric variety X_ω is given by the family of tori $\mathrm{pr}\colon (\mathbb{C}^\times)^m \to \check{K}$ together with a potential function $W\colon (\mathbb{C}^\times)^m \to \mathbb{C}$ defined by $W = u_1 + \cdots + u_m$, where u_i denotes the ith co-ordinate on $(\mathbb{C}^\times)^m$.

$$\begin{tikzcd} (\mathbb{C}^\times)^m \arrow[r, "W = u_1 + \cdots + u_m"] \arrow[d, "\mathrm{pr}"] & \mathbb{C} \\ \check{K} & \end{tikzcd}$$

Choosing a splitting of the sequence, we can also write $W_q = W|_{\mathrm{pr}^{-1}(q)}$ as

$$W_q = q^{l_1} x^{b_1} + \cdots + q^{l_m} x^{b_m}$$

where $q \in \check{K}$, $x \in \check{T}$ and recall that b_i are generators of 1-dimensional cones of the fan Σ_ω. By varying $q \in \check{K}$, W_q can represent any Laurent polynomial in $x \in \check{T}$ having $b_1, \ldots, b_m$ as exponents. Hence the mirror of a toric variety can be thought of as generic Laurent polynomials.

Remark 3. We denote the B-model co-ordinates by q and the A-model co-ordinates by Q. These co-ordinates are related by the mirror map ψ below.

Using the GKZ fan on $\mathfrak{k}^*_{\mathbb{R}}$ and its 'preimage fan' on $\mathbb{R}^m$ (whose maximal cones are $(\mathbb{R}_{\geq 0})^m \cap \pi^{-1}(A)$, where A is a maximal cone the GKZ fan and $\pi \colon \mathbb{R}^m \to \mathfrak{k}^*_{\mathbb{R}}$ is the natural map given by $D_1, \ldots, D_m$), we can partially compactify the family $\mathrm{pr} \colon (\mathbb{C}^\times)^m \to \check{K}$ to a map between toric varieties $\mathrm{pr} \colon \mathcal{Y} \to \mathcal{M}$ and W extends to a regular function $\mathcal{Y} \to \mathbb{C}$.

$$\begin{tikzcd} (\mathbb{C}^\times)^m \arrow[r, hook] \arrow[d] & \mathcal{Y} \arrow[r, "W"] \arrow[d, "\mathrm{pr}"] & \mathbb{C} \\ \check{K} \arrow[r, hook] & \mathcal{M} & \end{tikzcd}$$

The maximal cone A_ω defines a torus-fixed point $\mathbf{0}_\omega \in \mathcal{M}$ which is mirror to the large-radius limit point $Q = 0$ of the quantum cohomology of X_ω. Let $\mathcal{Y}_q$ denote the fibre of $q \in \mathcal{M}$ and write $W_q = W|_{\mathcal{Y}_q}$. Givental introduced oscillatory integrals

$$\int_{\Gamma \subset \mathcal{Y}_q} e^{W_q/z} \Omega \tag{2}$$

as mirrors of the quantum D-module, where Γ is a non-compact Morse cycle for $\Re(W_q/z)$ and Ω is a holomorphic volume form on the fibre $\mathcal{Y}_q$. Introduce the log structures on $\mathcal{Y}$ and $\mathcal{M}$ given by their toric boundaries and let $\Omega^\bullet_{\mathcal{Y}/\mathcal{M}}$ denote the relative logarithmic de Rham complex. The integrands Ω of (2) can be naturally viewed as elements of the twisted de Rham cohomology:

$$\mathrm{GM}(W) = \mathrm{pr}_* H^{\mathrm{top}}(\Omega^\bullet_{\mathcal{Y}/\mathcal{M}}[z], zd + dW \wedge).$$

This is equipped with the (logarithmic) Gauss-Manin connection and the higher residue pairing; such structures were introduced by K. Saito [50] in singularity theory. The Gauss-Manin connection is a map

$$\nabla \colon \mathrm{GM}(W) \to \frac{1}{z} \mathrm{GM}(W) \otimes_{\mathcal{O}_\mathcal{M}} \Omega^1_\mathcal{M} \oplus \mathrm{GM}(W) \frac{dz}{z^2}$$

which has the same pole structure along $z = 0$ as the Dubrovin connection (here $\Omega^1_\mathcal{M}$ denotes the sheaf of logarithmic 1-forms). When we choose a splitting of the sequence $1 \to \check{T} \to (\mathbb{C}^\times)^m \to \check{K} \to 1$ and choose co-ordinates $x = (x_1, \ldots, x_n)$ on $\check{T} \cong (\mathbb{C}^\times)^n$ and $q = (q_1, \ldots, q_k)$ on $\check{K} \cong (\mathbb{C}^\times)^k$, the connection ∇ is given by[2]

$$\nabla[f\Omega_0] = \sum_{a=1}^k \left[\left(\partial_a f + \frac{\partial_a W}{z} f \right) \Omega_0 \right] \frac{dq_a}{q_a} + \left[\left(z\partial_z f - \frac{W}{z} f - \frac{n}{2} f \right) \Omega_0 \right] \frac{dz}{z}$$

[2]Here we twist the Gauss-Manin connection by $-\frac{n}{2}\frac{dz}{z}$ so that it is compatible with the Dubrovin connection.

where $\partial_a = q_a \frac{\partial}{\partial q_a}$, $f = f(x,q,z) \in \mathcal{O}_{\mathcal{Y}}[z]$ and Ω_0 is the standard relative volume form of the family $\mathrm{pr}\colon \mathcal{Y} \to \mathcal{M}$:

$$\Omega_0 = \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}. \tag{3}$$

The higher residue pairing is a map

$$(-)^* \operatorname{GM}(W) \otimes_{\mathcal{O}_{\mathcal{M}}[z]} \operatorname{GM}(W) \to \mathcal{O}_{\mathcal{M}}[\![z]\!]$$

which coincides with the residue pairing when restricted to $z = 0$, where $(-)$ denotes the automorphism of $\mathcal{O}_{\mathcal{M}}[z]$ sending $f(q,z)$ to $f(q,-z)$. For a concrete description of the higher residue pairing, we refer the reader to [11, §6]. There exists a Zariski open subset U of $\mathcal{M}$ containing $\mathbf{0}_\omega$ such that $\operatorname{GM}(W)$ is locally free and coherent (as an $\mathcal{O}$-module) over $U \times \mathbb{C}_z$ [36, 44, 47].

Theorem 4 ([11, 16, 26, 27, 36, 37, 44, 46, 47])**.** *Over an analytic open neighbourhood of $\mathbf{0}_\omega$, the mirror D-module $\operatorname{GM}(W)$ is isomorphic to the pull-back of the quantum D-module $\operatorname{QDM}(X_\omega)$ by a map $\psi \in \operatorname{Aut}(\mathbb{C}[\![A_\omega^\vee \cap \mathfrak{k}_{\mathbb{Z}}]\!])$, called the mirror map. Under this isomorphism, the higher residue pairing corresponds to the Poincaré pairing.*

The mirror map ψ above extends to a local isomorphism between analytic neighbourhoods of $\mathbf{0}_\omega$, and therefore the pull-back by ψ makes sense over an analytic neighbourhood. If $\mathcal{M}$ contains two different large radius limit points $\mathbf{0}_{\omega_1}$, $\mathbf{0}_{\omega_2}$ such that $X_{\omega_1}, X_{\omega_2}$ are weak Fano, the theorem implies that the quantum D-modules of X_{ω_1} and X_{ω_2} are isomorphic under analytic continuation; this is an example where the crepant transformation conjecture holds [9, 15].

Remark 5. The above theorem is essentially due to Givental [26, 27]. Givental introduced the mirror potential function W (see also Hori-Vafa [33]) and expressed solutions of the quantum D-module in terms of oscillatory integrals. Givental's mirror theorem [27] (see also Lian-Liu-Yau [40, 41]) is stated as the equality between the J-function and the I-function, where the J-function is a solution to the quantum D-module and the I-function is a solution to the mirror D-module (see the next section). Mirror symmetry as an isomorphism of D-modules has been studied in details by [36, Proposition 4.8], [47, Theorem 4.11], [16, Theorem 5.1.1], [46, Theorem 7.43], [37, Theorem 4.2], [44, Theorem 6.4], [11, Theorem 1.1]. The logarithmic extension across the boundary divisors of $\mathcal{M}$ has been studied by [16, 44, 46, 47] in terms of (GKZ) D-modules; the logarithmic extension of the mirror Landau-Ginzburg model itself was discussed in [11, 37, 43]. In this paper, we restrict to the small quantum cohomology, but we also have mirror symmetry for the big quantum cohomology, see [4, 11, 17, 18, 37].

Example 6. The mirror family of $\mathbb{P}^n$ is given by the diagram

$$\begin{tikzcd} \mathbb{C}^{n+1} \arrow[r, "W=u_0+\cdots+u_n"] \arrow[d, "\mathrm{pr}"] & \mathbb{C} \\ \mathbb{C} & \end{tikzcd}$$

with $q = \mathrm{pr}(u_0, \dots, u_n) = u_0 u_1 \cdots u_n$, where $\mathcal{Y} = \mathbb{C}^{n+1}$ and $\mathcal{M} = \mathbb{C}$. We can write $W_q = W|_{\mathrm{pr}^{-1}(q)}$ as $W_q = x_1 + \cdots + x_n + \frac{q}{x_1 \cdots x_n}$ by setting $x_i := u_i$ for $1 \le i \le n$.

5. The GKZ system and hypergeometric solutions

The mirror D-module $\mathrm{GM}(W)$ can be described in terms of the Gelfand-Kapranov-Zelevinsky (GKZ) system [25]. To describe the GKZ system explicitly, we introduce a basis $p_1, \dots, p_k$ of $\mathfrak{k}_{\mathbb{Z}}^* \cong H^2(X_\omega, \mathbb{Z})$ and write $D_i = \sum_{a=1}^k m_{ia} p_a$. The elements $p_1, \dots, p_k$ define co-ordinates $q_1, \dots, q_k$ on $\check{K} \cong (\mathbb{C}^\times)^k$. We write $q^d = \prod_{a=1}^k q_a^{p_a \cdot d}$ for $d \in \mathfrak{k}_{\mathbb{Z}}$ and set $\partial_a = q_a \frac{\partial}{\partial q_a}$. Over the open torus $\check{K} \subset \mathcal{M}$, the mirror D-module $\mathrm{GM}(W)$ is generated by the standard relative volume form Ω_0 (see (3)) as a D-module, or more precisely, as a module over the ring $\mathcal{O}_{\check{K}}[z]\langle z\partial_1, \dots, z\partial_k\rangle$, where $z\partial_a$ acts by the Gauss-Manin connection $z\nabla_{\partial_a}$. All the D-module relations of Ω_0 are generated by $\Box_d[\Omega_0] = 0$ with $d \in \mathfrak{k}_{\mathbb{Z}}$, where

$$(4)\quad \Box_d := \prod_{i: D_i \cdot d > 0} \prod_{j=0}^{D_i \cdot d - 1} \left(\sum_{a=1}^k m_{ia} z\partial_a - jz \right) - q^d \prod_{i: D_i \cdot d < 0} \prod_{j=0}^{-D_i \cdot d - 1} \left(\sum_{a=1}^k m_{ia} z\partial_a - jz \right).$$

This system of differential equations is called the GKZ system. Givental's I-function [27] is a cohomology-valued function annihilated by $\Box_d$:

$$I(q,z) = \sum_{d \in A_\omega^\vee \cap \mathfrak{k}_{\mathbb{Z}}} q^{d + p/z} \prod_{i=1}^m \frac{\prod_{j=-\infty}^0 (D_i + jz)}{\prod_{j=-\infty}^{D_i \cdot d} (D_i + jz)}$$

where $q^{p/z} = \prod_{a=1}^k e^{p_a \log q_a / z}$. The components of the I-function form a basis of solutions to the GKZ system near $\mathbf{0}_\omega$ [36, 44, 47]. The mirror map ψ is determined by the z^{-1}-expansion of the I-function

$$I(q,z) = 1 + \frac{1}{z} \sum_{a=1}^k p_a \log \psi_a(q) + O(z^{-2})$$

where we write the mirror map in the form $Q_a = \psi_a(q)$, $a = 1, \dots, k$ using the A-model co-ordinates $Q_1, \dots, Q_k$ dual to $p_1, \dots, p_k$. We have $\psi_a(q) = q_a +$ higher order terms; in the Fano case the mirror map is trivial $\psi_a(q) = q_a$.

Remark 7 ([2, 25, 36])**.** The rank of the GKZ system (at a generic point in $\check{K}$) is equal to the normalized volume of the fan polytope, that is, the convex hull of $b_1, \dots, b_m$ in $N_{\mathbb{R}}$; this is also equal to $\dim H^*(X_\omega)$.

Example 8 (continuation of Example 6)**.** In the case of $\mathbb{P}^n$, the mirror oscillatory integral satisfies the differential equation:

$$\left((zq\partial_q)^{n+1} - q \right) \int_\Gamma e^{\left(x_1 + \cdots + x_n + \frac{q}{x_1 \cdots x_n}\right)/z} \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n} = 0.$$

The I-function is given by

$$I(q,z) = \sum_{d=0}^\infty \frac{q^{p/z + d}}{\prod_{j=1}^d (p + jz)^{n+1}}$$

(with $p = c_1(\mathcal{O}(1))$) and the mirror map is trivial: $Q = q$. The relationship between these two solutions will be discussed in §7-8, see Remark 11 and Example 14.

6. The Jacobi ring and the Stanley-Reisner ring

A ring structure arises from these D-modules in the $z \to 0$ limit. The $z \to 0$ limit of the Dubrovin connection $z\nabla_{\xi Q\frac{\partial}{\partial Q}}$ is the quantum product by $\xi \in H^2(X_\omega)$ (see (1)), and the quantum multiplication by divisors generate the quantum cohomology ring. On the other hand, the $z \to 0$ limit of the mirror D-module is given by the (logarithmic) Jacobi ring of W:

$$\operatorname{Jac}(W) = \operatorname{pr}_* \left(\mathcal{O}_{\mathcal{Y}} \Big/ \left\langle x_1 \frac{\partial W}{\partial x_1}, \dots, x_n \frac{\partial W}{\partial x_n} \right\rangle\right)$$

where we choose $\mathbb{C}^\times$-co-ordinates $(x_1, \dots, x_n)$ of $\check{T} \cong (\mathbb{C}^\times)^n$ and $x_i \frac{\partial}{\partial x_i}$ defines a relative tangent vector field of the family $\mathcal{Y} \to \mathcal{M}$. Theorem 4 induces an isomorphism

$$\psi^* QH^*(X_\omega) \cong \operatorname{Jac}(W)$$

between the quantum cohomology ring and the Jacobi ring. This isomorphism has been studied by many people, see [5,20–22,31,45,49,52] together with the references in Remark 5.

Over the affine open chart $\mathcal{M}_\omega := \operatorname{Spec}(\mathbb{C}[A_\omega^\vee \cap \mathfrak{k}_\mathbb{Z}])$ of $\mathcal{M}$, the Jacobi ring is a quotient of the quantum Stanley-Reisner ring associated with the fan Σ_ω. For $v \in N$, choose a cone $\sum_{i \in I} \mathbb{R}_{\ge 0} b_i$ of the fan Σ_ω containing v and write $v = \sum_{i \in I} v_i b_i$. We set $v_i = 0$ for $i \notin I$. Set $w_v := \prod_{i=1}^m u_i^{v_i}$. Then $H^0(\mathcal{M}_\omega, \operatorname{pr}_* \mathcal{O}_{\mathcal{Y}})$ is a free $\mathbb{C}[A_\omega^\vee \cap \mathfrak{k}_\mathbb{Z}]$-module generated by w_v with $v \in N$. The product structure is given by

$$w_v w_{v'} = q^{\ell(v,v')} w_{v+v'} \tag{5}$$

where $\ell(v,v') \in A_\omega^\vee \cap \mathfrak{k}_\mathbb{Z}$ is the element given by the linear relation $(v_i + v_i' - (v + v')_i)_{i=1}^m$ among b_i's via the exact sequence $0 \to \mathfrak{k}_\mathbb{Z} \to \mathbb{Z}^m \to N \to 0$. This ring $H^0(\mathcal{M}_\omega, \operatorname{pr}_* \mathcal{O}_{\mathcal{Y}})$ is called the quantum Stanley-Reisner ring. The Jacobian ideal gives additional linear relations among $u_1, \dots, u_m$ given by:

$$\sum_{i=1}^m u_i b_i = 0. \tag{6}$$

The relations (5), (6) define the Jacobi ring $\operatorname{Jac}(W)$; they are also known as the relations of Batyrev's quantum ring [5]. At the large-radius limit $q \to \mathbf{0}_\omega$, the quantum Stanley-Reisner relations (5) reduce to

$$w_v w_{v'} = \begin{cases} w_{v+v'} & \text{if } v \text{ and } v' \text{ lie in a common cone of } \Sigma_\omega; \\ 0 & \text{otherwise.} \end{cases}$$

These relations define the Stanley-Reisner ring. As is well-known in toric geometry, the Stanley-Reisner ring modulo the linear relations (6) is isomorphic to $H^*(X_\omega)$.

Remark 9. Over an analytic neighbourhood of $\mathbf{0}_\omega$, $\operatorname{pr}_* \mathcal{O}_{\mathcal{Y}}$ is isomorphic to the T-equivariant quantum cohomology of X_ω and the left-hand side of the linear relations (6) correspond to the T-equivariant parameters. Moreover, the multiplication by the co-ordinates $x_1, \dots, x_n$ of $\check{T}$ corresponds to the Seidel representation [51] on the equivariant quantum cohomology. McDuff-Tolman [45] used the Seidel representation to determine a presentation of the quantum cohomology ring of a toric manifold. In [37], we showed how equivariant mirror symmetry for toric manifolds follows almost tautologically from the Seidel representation and shift operators [7].

7. $\widehat{\Gamma}$-integral structure

The $\widehat{\Gamma}$-class [42] of an almost complex manifold X is defined to be the characteristic class

$$\widehat{\Gamma}_X = \prod_{i=1}^{n} \Gamma(1+\delta_i)$$

where $\delta_1, \dots, \delta_n$ are the Chern roots of the tangent bundle (so that $c(TX) = \prod_{i=1}^n (1+\delta_i)$) and $\Gamma(x)$ is Euler's Γ-function. The Γ-function $\Gamma(1+\delta)$ here should be expanded in Taylor series at $\delta = 0$. For a toric variety $X = X_\omega$, it is given by

$$\widehat{\Gamma}_{X_\omega} = \prod_{j=1}^{m} \Gamma(1+D_j) = \prod_{j=1}^{m} \exp\left(-\gamma D_j + \sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{k} D_j^k\right)$$

where γ is the Euler constant, $\zeta(s)$ is the Riemann ζ-function and recall that D_j is the class of a prime toric divisor.

The $\widehat{\Gamma}$-integral structure [36,38] is an integral lattice in the space of (multi-valued) flat sections of the Dubrovin connection. For an element $E \in K^0(X_\omega)$ of the topological K-group, there exists a unique flat section $s_E(Q,z)$ of the Dubrovin connection (which is flat also in the z-direction, see Remark 2) such that

$$s_E(Q,z) \sim \frac{1}{(2\pi)^{n/2}} Q^{-p/z} z^{-\mu} z^{c_1(X_\omega)} \left(\widehat{\Gamma}_{X_\omega} \cup (2\pi\sqrt{-1})^{\deg/2} \operatorname{ch}(E)\right)$$

as $Q \to 0$, where $Q^{-p/z} = \prod_{a=1}^k e^{-p_a \log Q_a/z}$ (we choose co-ordinates $Q_1, \dots, Q_k$ dual to $p_1, \dots, p_k$ as in §5) and μ is as in Remark 2. These flat sections $s_E(Q,z)$ span the $\widehat{\Gamma}$-integral structure. Its important properties are as follows:

(i) this lattice is invariant under local monodromy around the large radius limit point; therefore it defines a $\mathbb{Z}$-local system underlying the quantum D-module;

(ii) the Poincaré pairing of these flat sections coincide with the Euler pairing on the derived category: $(s_E(Q, e^{-\pi\sqrt{-1}}z), s_F(Q,z)) = \chi(E,F)$.

The second property follows from the identity $\Gamma(1+x)\Gamma(1-x) = \pi x/\sin(\pi x)$ (which implies that the $\widehat{\Gamma}_X$ is the "half" of the Todd class) and the Hirzebruch-Riemann-Roch formula.

The mirror D-module $\mathrm{GM}(W)$ from §4 has a natural integral structure[3] dual to the relative homology $H_n(\mathcal{Y}_q, \{\Re(W_q/z) \ll 0\}; \mathbb{Z})$ via the oscillatory integral (2). Here note that an element of the relative homology gives an integration cycle in (2). These two integral structures coincide under mirror symmetry.

Theorem 10 ([36]). *The $\widehat{\Gamma}$-integral structure coincides with the natural integral structure on the mirror D-module under the mirror isomorphism in Theorem* 4.

This theorem follows from the following identity of periods of both sides:

$$\int_{X_\omega} s_E(\psi(q), z) = \frac{1}{(2\pi z)^{n/2}} \int_{\Gamma(E)} e^{-W_q/z}\Omega_0 \tag{7}$$

where $E \mapsto \Gamma(E)$ is an isomorphism between $K^0(X_\omega)$ and $H_n(\mathcal{Y}_q, \{\Re(W_q/z) \gg 0\}; \mathbb{Z})$; this correspondence should be a shadow of homological mirror symmetry,

[3] To be more precise, we twist the local system of relative cohomology by $(-2\pi z)^{-n/2}$ so that it is compatible with the Dubrovin connection.

i.e. an equivalence between the derived category of coherent sheaves on X_ω and the Fukaya-Seidel category of W_q (see [19]):

$$D^b_{\mathrm{coh}}(X_\omega) \cong FS(Y, W_q).$$

When E is the structure sheaf $\mathcal{O}$ on X_ω, the corresponding cycle $\Gamma(\mathcal{O})$ is given by the positive real locus $(\mathbb{R}_{>0})^n$ in $\mathcal{Y}_q \cong \check{T} \cong (\mathbb{C}^\times)^n$ (when q lies in the positive real locus $\mathrm{Hom}(\mathfrak{k}_\mathbb{Z}, \mathbb{R}_{>0})$ of $\check{K} = \mathrm{Hom}(\mathfrak{k}_\mathbb{Z}, \mathbb{C}^\times)$ and $z > 0$). In this case, (7) yields the following asymptotics:

$$\int_{(\mathbb{R}_{>0})^n} e^{-W_q/z}\Omega_0 \sim \int_{X_\omega} q^{-p} z^{c_1(X_\omega)} \cup \widehat{\Gamma}_{X_\omega} \tag{8}$$

as $q \to \mathbf{0}_\omega$ in the positive real locus and for $z > 0$. Namely, the $\widehat{\Gamma}$-class appears in the $q \to \mathbf{0}_\omega$ asymptotics of the exponential period of the mirror. It would be very interesting to study if such asymptotics hold for more general Fano manifolds and their mirrors.

Remark 11. The equality (7) can be restated in terms of the I-function as follows:

$$\int_{X_\omega} \left(z^{c_1(X_\omega)} z^{\frac{\deg}{2}} I(q, -z) \right) \cup \widehat{\Gamma}_{X_\omega} (2\pi\sqrt{-1})^{\frac{\deg}{2}} \operatorname{ch}(E) = \int_{\Gamma(E)} e^{-W_q/z}\Omega_0.$$

This formula expresses the oscillatory integral as an explicit linear combination of components of the I-function.

Remark 12. There is another version of "Gamma conjecture" for quantum cohomology of Fano manifolds formulated by Galkin, Golyshev and the author [23], which does not involve mirror symmetry. The quantum D-module of a Fano manifold has irregular singularities at "$Q = \infty$", and the conjecture is about the Stokes structure at $Q = \infty$ and the connection of solutions between $Q = 0$ and $Q = \infty$. This version of the conjecture, if true, implies that the $\widehat{\Gamma}$-class can be recovered from the quantum cohomology of a Fano manifold. See also [24, 30].

Remark 13. Abouzaid, Ganatra, Sheridan and the author [1] recently proposed an approach to proving the asymptotics (8) (for Calabi-Yau mirror pairs) using Strominger-Yau-Zaslow picture [53] and tropical geometry.

8. Mellin-Barnes integral representation

The oscillatory integrals (2) give an integral representation of solutions to the GKZ system. There is another integral representation, Mellin-Barnes integral representation, which is "Gale dual" to (2). We shall regard $p_1, \dots, p_k \in \mathfrak{k}^*_\mathbb{Z}$ as co-ordinates on $\mathfrak{k}_\mathbb{C} = \mathrm{Lie}(K)$ which are Mellin-dual to $q_1, \dots, q_k$. Under the Mellin transformation $I(q) \mapsto \widehat{I}(p) = \int I(q) q^p \frac{dq}{q}$, the differential operator $\partial_a = q_a \frac{\partial}{\partial q_a}$ corresponds to the multiplication by $-p_a$ and the multiplication by q_a corresponds to the shift operator $T_a \colon p_b \mapsto p_b + \delta_{a,b}$. Thus the GKZ equations $\Box_d I(q, z) = 0$ (see (4)) with $d \in \mathfrak{k}_\mathbb{Z}$ are transformed into the difference equations:

$$\left(\prod_{i: D_i\cdot d > 0} \prod_{j=0}^{D_i\cdot d - 1} (-D_i - j)z - T^d \prod_{i: D_i\cdot d < 0} \prod_{j=0}^{D_i\cdot d - 1} (-D_i - j)z \right) \widehat{I}(p, z) = 0$$

where $D_i = \sum_{a=1}^k m_{ia} p_a \in \mathfrak{k}_{\mathbb{Z}}^*$ is regarded as the pull-back of the standard coordinates on $\mathbb{C}^m$ via the inclusion $\mathfrak{k}_{\mathbb{C}} \hookrightarrow \mathbb{C}^m$ and $T^d = \prod_{a=1}^k T_a^{p_a \cdot d}$. It is easy to check that this system has the following simple solution:

$$\widehat{I}(p,z) = \prod_{i=1}^m (-z)^{D_i} \Gamma(D_i).$$

In fact, $\widehat{I}$ is (the restriction of) the Mellin transform of $e^{W/z} = \prod_{i=1}^m e^{u_i/z}$; here we recall $\int_0^\infty e^{u/z} u^D \frac{du}{u} = (-z)^D \Gamma(D)$ with $z < 0$ and $D > 0$. By the inverse Mellin transformation, we get a solution to the GKZ system:

$$\frac{1}{(2\pi\sqrt{-1})^k} \int_{C \subset \mathfrak{k}_{\mathbb{C}}} q^{-p} \left(\prod_{i=1}^m (-z)^{D_i} \Gamma(D_i) \right) dp_1 \cdots dp_k \tag{9}$$

where $C \subset \mathfrak{k}_{\mathbb{C}}$ is a suitable (non-compact) k-cycle so that the integral converges. For a suitable choice of C, (9) should coincide with an oscillatory integral $\int_\Gamma e^{W_q/z} \Omega_0$ (see Figure 1 below), but the author does not know a precise choice of cycles in general. Via the residue calculation, such a formula would explain the $\widehat{\Gamma}$-class appearing in the leading asymptotics (8), see Example 14 below.

$$\begin{tikzcd}
e^{W/z} = \prod_{i=1}^m e^{u_i/z} \text{ on } (\mathbb{C}^\times)^m \arrow[r, "\mathrm{pr}_*"] \arrow[d, leftrightarrow, "\text{Mellin}"] & \int e^{W/z}\Omega \text{ on } \check{K} \arrow[d, leftrightarrow, "\text{Mellin}"] \\
\prod_{i=1}^m (-z)^{D_i}\Gamma(D_i) \text{ on } \mathbb{C}^m \arrow[r, "\text{restriction}"] & \widehat{I}(p,z) \text{ on } \mathfrak{k}_{\mathbb{C}}
\end{tikzcd}$$

FIGURE 1. Oscillatory integral and its Mellin transform

Example 14 (continuation of Example 8)**.** We consider the mirror oscillatory integral of $\mathbb{P}^n$ again. The following method is borrowed from [38]. The Mellin transform of the oscillatory integral $\mathcal{I}(q) = \int_{(\mathbb{R}_{>0})^n} e^{-(x_1 + \cdots + x_n + \frac{q}{x_1 \cdots x_n})/z} \frac{dq}{q}$ (with $q, z > 0$) gives

$$\begin{aligned}
\widehat{\mathcal{I}}(p) &= \int_0^\infty q^p \mathcal{I}(q) \frac{dq}{q} \\
&= \int_{(\mathbb{R}_{>0})^{n+1}} (u_0 \cdots u_n)^p e^{-(u_0 + \cdots + u_n)/z} \frac{du_0}{u_0} \wedge \cdots \wedge \frac{du_n}{u_n} \\
&= z^{(n+1)p} \Gamma(p)^{n+1}.
\end{aligned}$$

This coincides with $\widehat{I}(p, -z)$ as expected. Then the Mellin inversion formula gives

$$\mathcal{I}(q) = \frac{1}{2\pi\sqrt{-1}} \int_{c - \sqrt{-1}\infty}^{c + \sqrt{-1}\infty} q^{-p} \widehat{\mathcal{I}}(p) dp$$

with $c > 0$. By closing the integration contour to the left, we can express the right-hand side as the sum over residues at $p = 0, -1, -2, \ldots$, arriving at the asymptotics in (8).

$$\begin{aligned}\mathcal{I}(q) &= \sum_{d=0}^{\infty} \mathrm{Res}_{p=-d}\left(q^{-p} z^{(n+1)p}\Gamma(p)^{n+1} dp\right) \\ &\sim \mathrm{Res}_{p=0}\left(q^{-p} z^{(n+1)p}\Gamma(1+p)^{n+1}\frac{dp}{p^{n+1}}\right) = \int_{\mathbb{P}^n} q^{-p} z^{c_1(\mathbb{P}^n)} \cup \widehat{\Gamma}_{\mathbb{P}^n}.\end{aligned}$$

Remark 15. The Mellin-Barnes integral representations (9) appear in physics literature as hemisphere partition functions (studied for more general gauged linear sigma models), see, e.g. [32].

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