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AMS Summer Institute in Algebraic Geometry

Constructing mirrors via shift operators

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(based on arXiv:1411.6840, arXiv:1503.02919)

Related joint works:

[González-I] *Seidel elements and mirror transformations*, Selecta Math 18 (3): 557–590, arXiv:1103.4171

[Coates–Corti–I–Tseng] *A mirror theorem for toric stacks*, arXiv:1310.4163

[Coates–I–Jiang] *The crepant transformation conjecture for toric complete intersections*, arXiv:1410.0024

Talk Plan:

1. (Equivariant) toric mirror symmetry
2. Seidel elements and shift operators
3. Construction of big & equivariant mirrors
   — a tautological proof of mirror symmetry
   — need not to assume that X is projective or that $-K_X$ is nef
4. Gamma integral structure

§1. Mirrors of toric varieties X (Givental '94, Hori-Vafa '00) are functions f on the torus $(\mathbb{C}^\times)^n$ ($n = \dim X$):

$$f(x) = Q^{\beta_1} x^{b_1} + \cdots + Q^{\beta_m} x^{b_m}$$

where

- $b_1, \ldots, b_m \in \mathbb{Z}^n$ are generators of 1-dimensional cones of the fan of X;
- $x = (x_1, \ldots, x_n) \in (\mathbb{C}^\times)^n$, $x^{b_j} = \prod_{i=1}^n x_i^{b_{ji}}$;
- Q is the Novikov variable, $\beta_i \in H_2(X, \mathbb{Z})$.

Remark: $f(x)$ is called the Landau-Ginzburg potential.

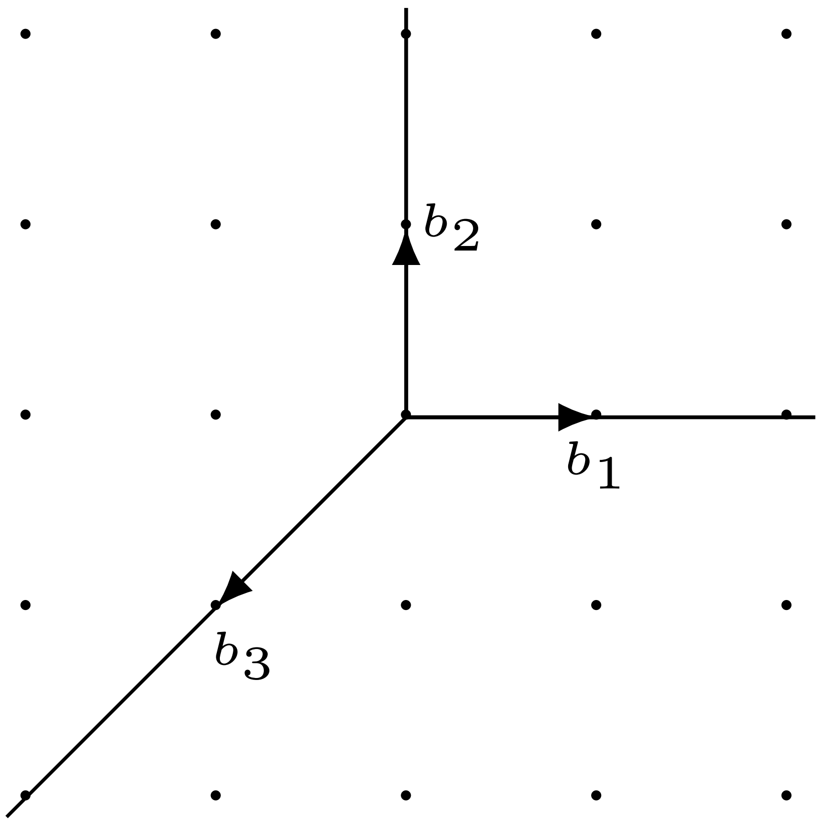
Equivariant mirrors of X (Givental, Hori-Vafa) are multi-valued functions f_λ on the torus $(\mathbb{C}^\times)^n$:

$$f_\lambda(x) = Q^{\beta_1} x^{b_1} + \cdots + Q^{\beta_m} x^{b_m}$$
$$-\lambda_1 \log x_1 - \cdots - \lambda_n \log x_n .$$

where

- $b_1, \ldots, b_m \in \mathbb{Z}^n$ are generators of 1-dimensional cones of the fan;
- $x = (x_1, \ldots, x_n) \in (\mathbb{C}^\times)^n$, $x^{b_j} = \prod_{i=1}^n x_i^{b_{ji}}$;
- Q is the Novikov variable, $\beta_i \in H_2(X, \mathbb{Z})$;
- $\lambda_1, \ldots, \lambda_n$ are T-equivariant parameters (T is the open dense torus of X).

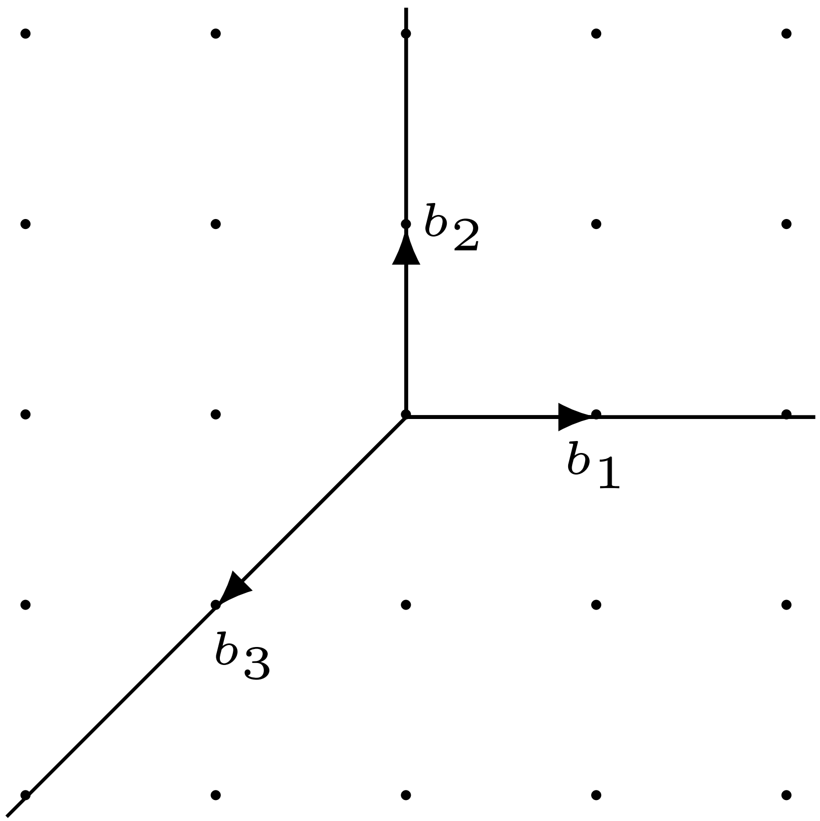
Example: $X = \mathbb{P}^2$



Fan of $\mathbb{P}^2$

$$f(x) = x_1 + x_2 + \frac{Q}{x_1 x_2}$$

Example: $X = \mathbb{P}^2$ with $(\mathbb{C}^\times)^2$-action



Fan of $\mathbb{P}^2$

$$f_\lambda(x) = x_1 + x_2 + \frac{Q}{x_1 x_2} - \lambda_1 \log x_1 - \lambda_2 \log x_2$$

Mirror symmetry:

— an isomorphism between the small quantum cohomology and the Jacobi ring:

$$QH^*(X) \cong \mathrm{Jac}(f)$$

— an isomorphism between the small quantum D-module and the twisted de Rham cohomology:

$$QDM(X) \cong H^n(\Omega^\bullet_{(\mathbb{C}^\times)^n}[z], zd + df\wedge)$$

Mirror symmetry (equivariant case):

— an isomorphism between the small quantum cohomology and the Jacobi ring:

$$QH_T^*(X) \cong \mathrm{Jac}(f_\lambda)$$

— an isomorphism between the small quantum D-module and the twisted de Rham cohomology:

$$QDM_T(X) \cong H^n(\Omega^\bullet_{(\mathbb{C}^\times)^n}[z], zd + df_\lambda \wedge)$$

Remark: df_λ is an algebraic differential form (i.e. it does not contain $\log x_i$).

Twisted de Rham cohomology

Cohomology for "exponential periods": one can consider the oscillating integral

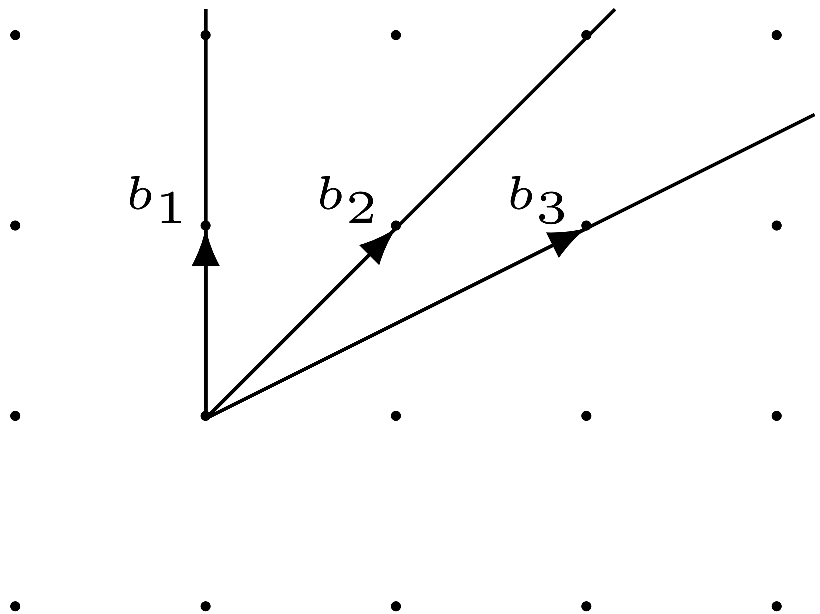
$$\int_\Gamma e^{f(x)/z}\phi(x)\frac{dx_1\cdots dx_n}{x_1\cdots x_n}$$

for a cohomology class

$$\left[\phi(x)\frac{dx_1\cdots dx_n}{x_1\cdots x_n}\right]\in H^n(\Omega^\bullet_{(\mathbb{C}^\times)^n}[z], zd+df\wedge)$$

and a Lefschetz thimble Γ (a Morse cycle for $\mathrm{Re}(f)$).
Note: the variable z is the Laplace-dual of $f(x)$. Later it will be an equivariant variable.

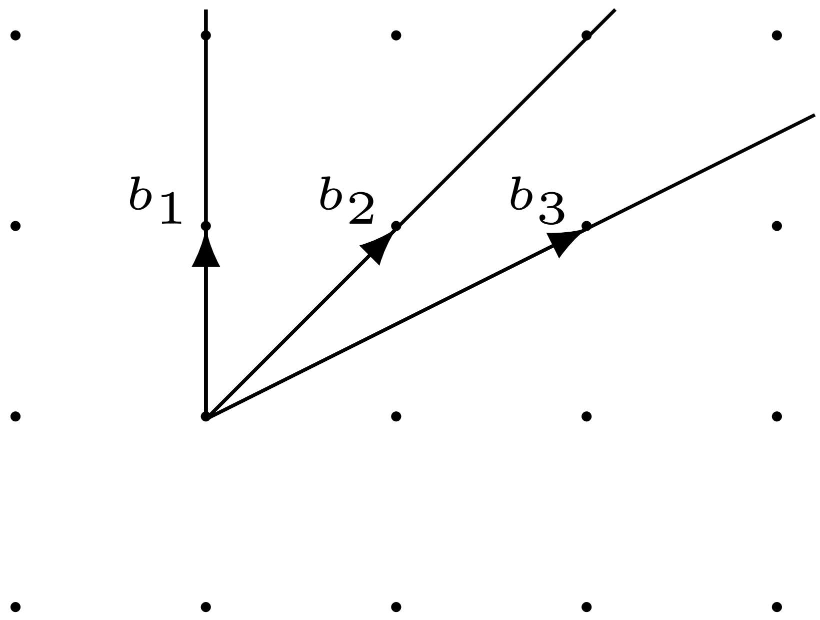
Example: $X = \mathcal{O}_{\mathbb{P}^1}(-2)$ with $(\mathbb{C}^\times)^2$-action (resolution of the A_1 singularity $\mathbb{C}^2/\mu_2$).



$$f_\lambda(x) = x_2(1 + x_1 + Qx_1^2) \\ -\lambda_1 \log x_1 - \lambda_2 \log x_2$$

Fan of $\mathcal{O}_{\mathbb{P}^1}(-2)$

Example: $X = \mathcal{O}_{\mathbb{P}^1}(-2)$ with $(\mathbb{C}^\times)^2$-action (resolution of the A_1 singularity $\mathbb{C}^2/\mu_2$).



$$f_\lambda(x) = x_2(1 + (1+Q)x_1 + Qx_1^2) - \lambda_1 \log x_1 - \lambda_2 \log x_2$$

There is a mirror map correction

Fan of $\mathcal{O}_{\mathbb{P}^1}(-2)$

Remark:

The coefficients of the mirror $f(x)$ are conjectured to be open Gromov-Witten invariants (counting holomorphic discs with boundary in a Lagrangian torus orbit $\cong (S^1)^n$).

[Cho-Oh '03]:
Fano case, the same as Givental-Hori-Vafa mirror.

[Fukaya-Oh-Ohta-Ono '08-'10]:
General case, an isomorphism between the quantum cohomology and the Jacobi ring of the 'FOOO potential'.

[Chan-Lau-Leung-Tseng '12]:
Computation of the FOOO potential in the nef ($c_1(X) \geq 0$) case. They showed that the potential matches with Givental-Hori-Vafa's potential under Givental's mirror map.

Mirrors for big quantum cohomology

(Barannikov '00, Douai-Sabbah '02): consider the miniversal deformation of $f(x)$:

$$f(x;t) = f(x) + \sum_{i=1}^{N} t^i \phi_i(x)$$

where $\phi_1(x), \ldots, \phi_N(x)$ are representatives of a basis of the Jacobi ring $\mathrm{Jac}(f)$.

$\Longrightarrow$ isomorphism of Frobenius manifolds (flat structures) between the A-model and the B-model

Remark:

[Fukaya-Oh-Ohta-Ono '08]: bulk deformation of the potential in Lagrangian Floer theory.

[Gross '09]: construction of a big mirror for $\mathbb{P}^2$ via tropical disc counting.

§2. Seidel elements and shift operators

[Seidel '95] associated an invertible element of $QH^*(X)$ to a loop in $\mathrm{Ham}(X,\omega)$.

[Okounkov-Pandharipande '09, Braverman-Maulik-Okounkov '10, Maulik-Okounkov '13] introduced a shift operator on $QH_T^*(X)$ for each homomorphism $\mathbb{C}^\times \to T$.

- A shift operator induces a shift of equivariant parameters by an integer multiple of z.
- The non-equivariant limit ($z \to 0$) of shift operators are Seidel elements.

B-model shift operator (very simple!)

Shifting $\lambda_1, \ldots, \lambda_n$ by $kz = (k_1 z, \cdots, k_n z)$ yields:

$$\int_\Gamma e^{f_\lambda(x)/z} \phi(x) \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n}$$

$$\longrightarrow \int_\Gamma e^{f_{\lambda - kz}(x)/z} \phi(x) \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n}$$

$$= \int_\Gamma e^{f_\lambda(x)/z} x_1^{k_1} \cdots x_n^{k_n} \phi(x) \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n}$$

Recall:

$$f_\lambda(x) = f(x) - \lambda_1 \log x_1 - \cdots - \lambda_n \log x_n$$

A-model shift operator

T: an algebraic torus

X: a semi-projective (i.e. projective over affine) T-variety such that T-weights of $H^0(X, \mathcal{O})$ are contained in a strictly convex cone in $\mathfrak{t}^*$.

Choose a one-parameter subgroup $k \colon \mathbb{C}^\times \to T$ (which pairs non-positively with all the T-weights of $H^0(X, \mathcal{O})$).

This defines an X-bundle E_k over $\mathbb{P}^1$ (Seidel space):

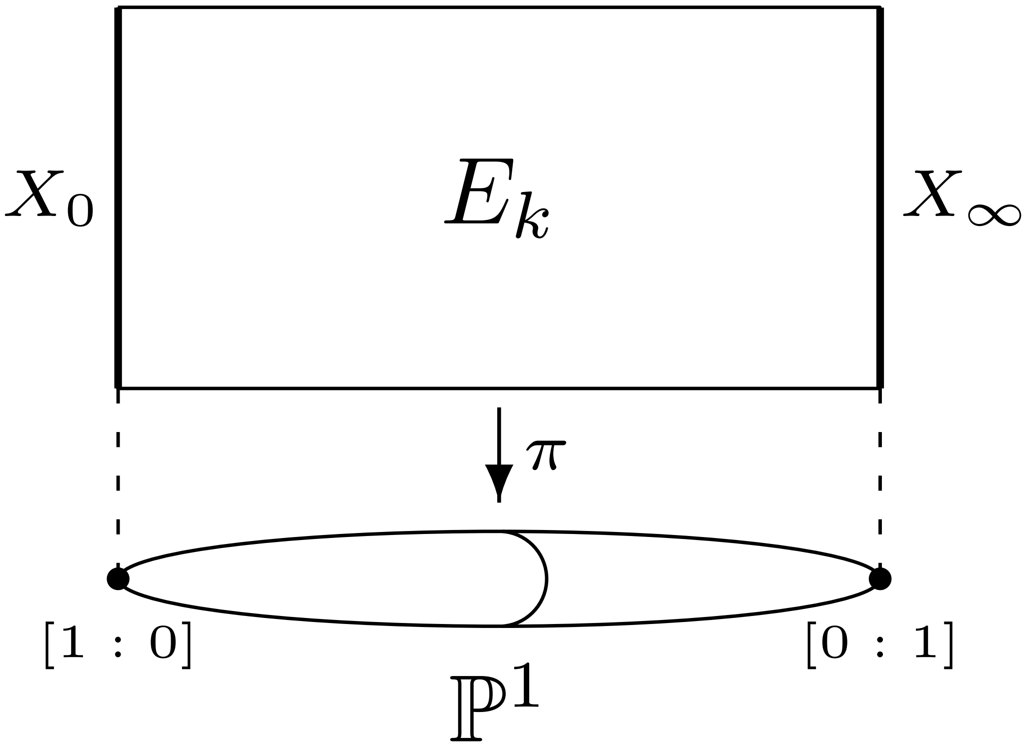
$$E_k = \left(X \times (\mathbb{C}^2 \setminus \{0\})\right)/\mathbb{C}^\times$$

$$\downarrow \pi$$

$$\mathbb{P}^1 = (\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^\times$$

count sections

where $\mathbb{C}^\times$ acts as $s \cdot (x, (v_1, v_2)) = (s^k \cdot x, (s^{-1} v_1, s^{-1} v_2))$.



$\widehat{T} := T \times \mathbb{C}^\times$ acts on E_k by

$$(t,u) \cdot [x, (v_1, v_2)]$$
$$= [t \cdot x, (v_1, uv_2)]$$

for $(t, v) \in \widehat{T} = T \times C^\times$

The induced $\widehat{T}$-action on the fibers X_0, X_∞:

$(t,u) \cdot x = t \cdot x$ for $x \in X_0$

$(t,u) \cdot x = tu^k \cdot x$ for $x \in X_\infty$ ← shift of the T-action

The shift operator $\widetilde{\mathbb{S}}_k : H^*_{\widehat{T}}(X_0) \to H^*_{\widehat{T}}(X_\infty)$ is defined by the correspondence:

$$X_0 \xleftarrow[\mathrm{ev}_0]{} \overline{\left\{ \begin{array}{c} \text{moduli space of} \\ \text{hol. sections of } E_k \end{array} \right\}} \xrightarrow[\mathrm{ev}_\infty]{} X_\infty$$

$$\left(\widetilde{\mathbb{S}}_k \alpha, \beta\right) = \sum_{\substack{n \geq 0, \\ \hat{d}:\ \text{section class} \\ \text{of } E_k}} \frac{Q^{\hat{d}-\sigma_{\mathrm{min}}}}{n!} \langle \iota_{0*}\alpha, \hat{\tau}, \dots, \hat{\tau}, \iota_{\infty*}\beta \rangle^{E_k, \widehat{T}}_{0,n+2,\hat{d}}$$

$\tau \in H^*_T(X)$ is a bulk parameter and $\hat{\tau} \in H_{\widehat{T}}(E_k)$ is a lift of τ

Composing $\widetilde{\mathbb{S}}_k$ with the standard isomorphism

$$\Phi_k^{-1} : H^*_{\widehat{T}}(X_\infty) \xrightarrow{\cong} H^*_{\widehat{T}}(X_0)$$

we get a self-map $\mathbb{S}_k = \Phi_k^{-1} \circ \widetilde{\mathbb{S}}_k : H^*_{\widehat{T}}(X_0) \to H^*_{\widehat{T}}(X_0)$.

General properties

1. $\mathbb{S}_k$ yields a shift of equivariant parameters:

$$\mathbb{S}_k \circ \lambda_i = (\lambda_i - k_i z) \circ \mathbb{S}_k$$

(note: Φ_k^{-1} is not an isomorphism of $H^*_{\widehat{T}}(\mathrm{pt})$-modules).

2. $\mathbb{S}_k$ commutes with the equivariant quantum connection

$$\mathbb{S}_k \circ \nabla_\alpha = \nabla_\alpha \circ \mathbb{S}_k,$$

where $\nabla_\alpha = \partial_\alpha + z^{-1}(\alpha \bullet_\tau)$ is the equivariant quantum connection in the direction of $\alpha \in H^*_T(X)$.

3. $\mathbb{S}_k$ defines an action of the lattice $\mathrm{Hom}(\mathbb{C}^\times, T)$:

$$\mathbb{S}_k \circ \mathbb{S}_l = Q^{d(k,l)} \mathbb{S}_{k+l}$$

for $k, l \in \mathrm{Hom}(\mathbb{C}^\times, T)$ (Seidel representation).

Remark:

The case $k = 0$ was originally studied by Givental ('96 IMRN). The relevant moduli spaces are called graph spaces. Givental derived the equivariant quantum connection and the J-function from graph spaces. The proof of the above properties is similar to (or an extension of) his argument.

§3. Construction of a big and equivariant mirror

X: a smooth semi-projective toric variety, $T \curvearrowright X$
$\mathbf{N} := \mathrm{Hom}(\mathbb{C}^\times, T) \cong \mathbb{Z}^n$ $(n = \dim X)$
$\Sigma :=$ the fan of X defined on the vector space $\mathbf{N} \otimes \mathbb{R}$.

Introduce a "universal function":

$$F(x; \mathbf{y}) = \sum_{k \in \mathbf{N} \cap |\Sigma|} y_k x^k Q^{\beta(k)}$$

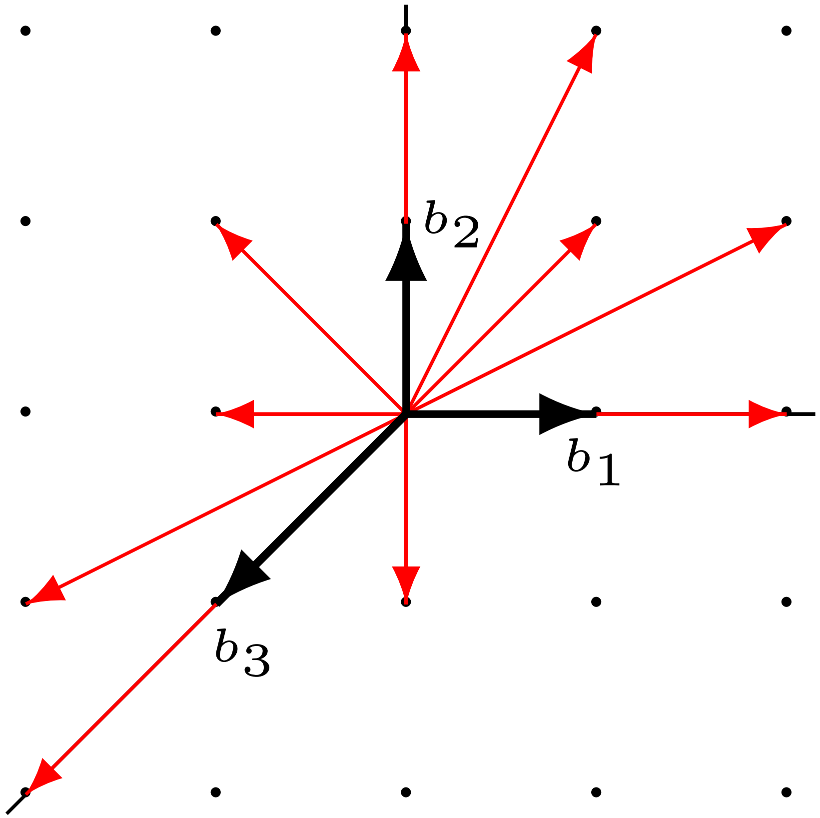
where $\mathbf{y} = \{y_k : k \in \mathbf{N} \cap |\Sigma|\}$ are infinitely many parameters and $\beta(k)$ is a curve class determined by choosing a splitting of the fan exact sequence. Define

$$F_\lambda(x; \mathbf{y}) := F(x; \mathbf{y}) - \lambda_1 \log x_1 - \cdots - \lambda_n \log x_n.$$

$F(x; \mathbf{y})$ is a deformation of the original mirror potential $f(x)$:

$$F(x; \mathbf{y}) \to f(x) \quad \text{as } y_k \to \begin{cases} 1 & \text{if } k = b_i \text{ for some } i \\ 0 & \text{otherwise} \end{cases}$$

Example

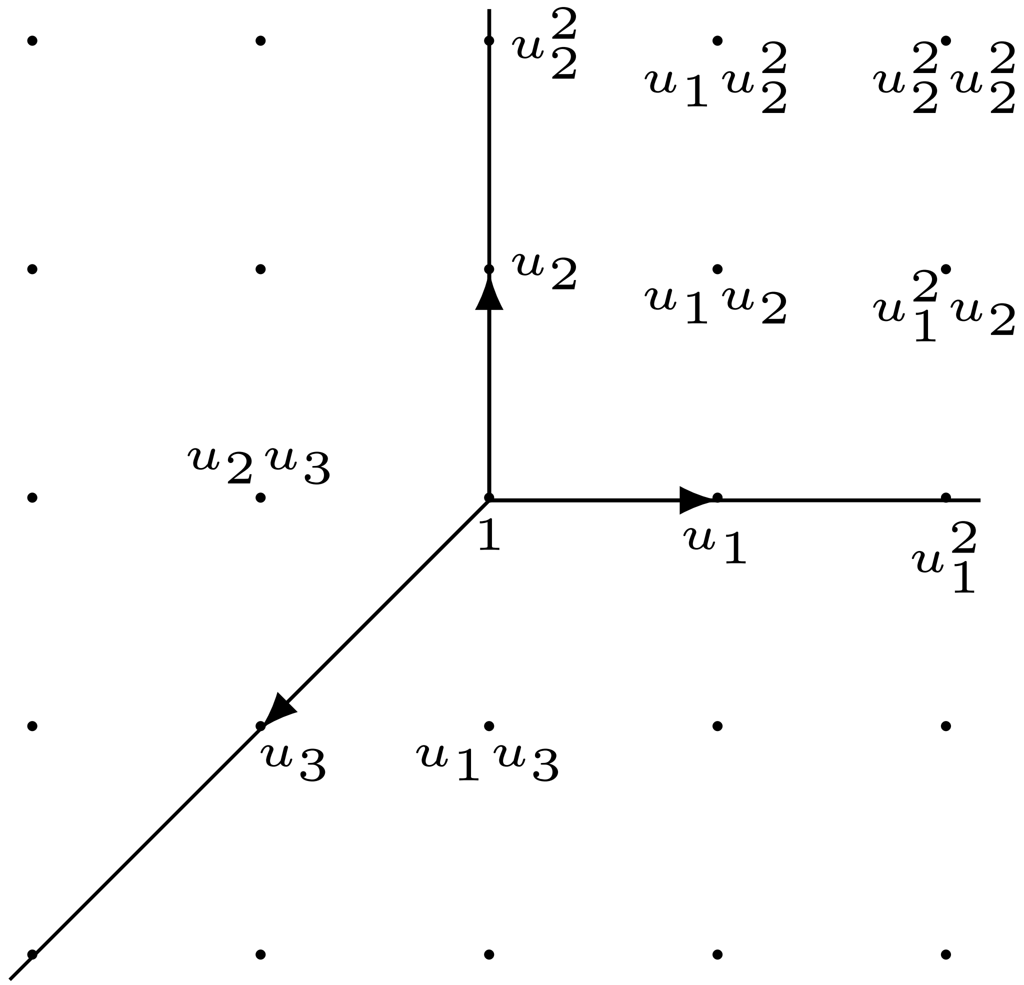


Fan of $\mathbb{P}^2$

$$
\begin{aligned}
F(x) = y_0 &+ y_{b_1} x_1 + y_{b_2} x_2 + y_{b_3} \frac{Q}{x_1 x_2} \\
&+ y_{(1,1)} x_1 x_2 + y_{(-1,0)} \frac{Q}{x_1} \\
&+ y_{(0,-1)} \frac{Q}{x_2} + y_{(2,0)} x_1^2 \\
&+ y_{(0,2)} x_2^2 + y_{(-2,-2)} \frac{Q^2}{(x_1 x_2)^2} \\
&+ y_{(1,2)} x_1 x_2^2 + y_{(2,1)} x_1^2 x_2 \\
&+ y_{(-1,1)} \frac{Q x_2}{x_1} + y_{(-2,-1)} \frac{Q^2}{x_1^2 x_2} \\
&+ \cdots
\end{aligned}
$$

universal function

Remark: lattice points in $\mathbb{Z}^2 \Longleftrightarrow \mathbb{C}$-basis of $H_T^*(\mathbb{P}^2)$.



u_1, u_2, u_3 are the classes of toric divisors

Theorem: There exists an isomorphism

$$\left(\begin{array}{l}\text{the twisted de Rham coh.}\\ \text{of } zd + dF_\lambda \wedge\end{array}\right) \overset{\Theta}{\cong} \left(\begin{array}{l}\text{the big and equivariant}\\ \text{quantum connection of } X\end{array}\right).$$

1. We have an invertible change of variables (mirror map) between the parameters:

$$\mathbf{y} = \{y_k\} \quad \longleftrightarrow \quad \tau \in H_T^*(X)$$

2. Θ intertwines the Gauss-Manin connection with the quantum connection

$$\Theta \circ \nabla^{\mathrm{GM}} = \nabla^{\mathrm{QC}} \circ \Theta$$

3. Θ intertwines a monomial multiplication with the shift operator

$$\Theta \circ Q^{\beta(k)} x^k = \mathbb{S}_k \circ \Theta$$

Construction of the isomorphism Θ

The isomorphism Θ and the change of variables $\tau = \tau(\mathbf{y})$ are almost determined by the properties in the theorem.
Consider the following "universal oscillatory form" (defining an element of the twisted de Rham cohomology):

$$\omega(\mathbf{y}, \mathbf{y}^+) = \exp\left(\sum_{k \in \mathbf{N} \cap |\Sigma|} \sum_{i=0}^{\infty} y_{k,i} x^k Q^{\beta(k)} z^{i-1} \right) x^{-\lambda/z} \frac{dx}{x}$$

$$= \exp\left(\sum_{k \in \mathbf{N} \cap |\Sigma|} \sum_{i=1}^{\infty} y_{k,i} x^k Q^{\beta(k)} z^{i-1} \right) e^{F_\lambda(x;\mathbf{y})/z} \frac{dx}{x}$$

where we set $y_{k,0} := y_k$ and

$$\mathbf{y}^+ = \{ y_{k,i} : k \in \mathbf{N} \cap |\Sigma|, i \geq 1 \}$$

is a new set of variables.

Commuting flows on the Givental cone

By the requirements in the theorem, the image of $\omega(\mathbf{y}, \mathbf{y}^+)$ under Θ has to satisfy the following differential equations:

$$\frac{\partial \tau(\mathbf{y})}{\partial y_k} = S_k(\tau(\mathbf{y})) \qquad \leftarrow \text{generalizes [González-I '11]}$$

$$\frac{\partial \Theta(\omega(\mathbf{y}, \mathbf{y}^+))}{\partial y_{k,i}} = \left[z^{i-1} \mathbb{S}_k \right]_+ \Theta(\omega(\mathbf{y}, \mathbf{y}^+))$$

where $S_k(\tau) = \lim_{z \to 0} \mathbb{S}_k \cdot 1$ is the Seidel element. These equations define infinitely many commuting flows on

$$\Big(\tau(\mathbf{y}), \Theta(\omega(\mathbf{y}, \mathbf{y}^+)) \Big) \in \underbrace{H_T^*(X) \times H_{\widehat{T}}^*(X)}_{\text{can be identified with the Givental cone}}$$

$\Rightarrow$ τ and Θ are given by solving these differential equations.

- We can identify $H^*_T(X) \times H^*_{\widehat{T}}(X)$ with Givental's Lagrangian cone $\mathcal{L} \subset H^*_{\widehat{T}}(X)_{\mathrm{loc}}$ using a standard fundamental solution of the quantum connection.
- The flow $\partial/\partial y_{k,i}$ is identified with the following vector field on the cone $\mathcal{L}$:

$$\mathcal{L} \ni \mathbf{f} \longmapsto z^{i-1} \mathcal{S}_k \mathbf{f} \in T_{\mathbf{f}} \mathcal{L}$$

  where $\mathcal{S}_k$ is a constant shift operator which acts on the fixed point basis $\{\delta_x : x \in X^T\}$ as follows:

$$\mathcal{S}_k(\delta_x) = Q^{\sigma_x - \sigma_{\mathsf{min}}} \prod_{i=1}^{n} \frac{\prod_{c \le 0} \rho_i + cz}{\prod_{c \le -\rho_i \cdot k} \rho_i + cz} \delta_x$$

  where $\rho_1, \dots, \rho_n$ are T-weights at the tangent space $T_x X$.
- The image of $\omega(\mathbf{y}, \mathbf{y}^+)$ in $\mathcal{L}$ is identified with the extended I-function [CCIT '13] $I(\mathbf{y}, \mathbf{y}^+)$, which is an integral curve of the above vector field.

- Moreover the I-function flow gives a co-ordinatization of the whole Givental cone, that is, the map

$$(\mathbf{y}, \mathbf{y}^+) \longmapsto I(\mathbf{y}, \mathbf{y}^+) \in \mathcal{L}$$

  is an isomorphism (of formal schemes).
- By pulling back 1 by Θ, we obtain an primitive form (in the sense of K. Saito):

$$\zeta = \Theta^{-1}(1)$$

  This gives a canonical *cochain-level* primitive form in the equivariant theory.
- As a corollary, we obtain an isomorphism between the Jacobi ring of $F_\lambda(x, \mathbf{y})$ and the equivariant quantum cohomology of X:

$$QH_T^*(X) \cong \mathrm{Jac}(F_\lambda)$$

Non-equivariant limit and reparametrization group

The group $J\mathcal{G}$ of formal reparametrizations of the form:

$$x_i \mapsto x_i \exp\left(\sum_{k\in\mathbf{N}\cap|\Sigma|, j\geq 0} \epsilon_{k,i,j} x^k Q^{\beta(k)} z^j \right)$$

acts on the "universal" oscillatory form $\omega(\mathbf{y}, \mathbf{y}^+)$, and therefore acts on the equivariant Givental cone.

Theorem: The non-equivariant limit map

$$\mathcal{L}_{\text{equiv}} \to \mathcal{L}_{\text{noneq}}$$

identifies $\mathcal{L}_{\text{noneq}}$ with the quotient $\mathcal{L}_{\text{equiv}}/J\mathcal{G}$.

Other approaches to toric mirror symmetry

[González-Woodward '10]: quantum Kirwan map
[Ciocan-Fontanin–Kim '13,
Cheong–Ciocan-Fontanine–Kim '14]: stable quasimaps

Question:

The inverse mirror map $y_k = y_k(\tau)$ could be interpreted as a generating function of open Gromov-Witten invariants (with bulk insertions)? How about $\mathbf{y}^+(\tau)$ corresponding to the primitive form?

§4 A relation to the Gamma structure

The (equivariant) $\widehat{\Gamma}$-class (Libgober '98)

$$\widehat{\Gamma}_X = \widehat{\Gamma}(TX) = \prod_{i=1}^{n} \Gamma(1 + \delta_i)$$

where $\delta_1, \ldots, \delta_n$ are (equivariant) Chern roots of TX. This can be expressed in terms of the (equivariant) Chern classes $\mathrm{ch}_k(TX)$ and zeta-values $\zeta(k)$:

$$\widehat{\Gamma}_X = \exp\left(-\gamma c_1(TX) + \sum_{k\geq 2} (-1)^k (k-1)! \zeta(k) \,\mathrm{ch}_k(TX)\right)$$

and lies in $H^*(X, \mathbb{R})$ (resp. $H_T^{**}(X, \mathbb{R})$).

The Gamma integral structure is an embedding of the K-group into the space of homogeneous flat sections of the quantum connection (I '07, Katzarkov-Kontsevich-Pantev '08):

$$K_T^0(X) \hookrightarrow \text{(the space of homogeneous flat sections)}$$

given by the formula

$$\mathfrak{s}(E)(\tau) = (2\pi)^{-\frac{n}{2}} L(\tau) z^{-\mu} z^{c_1(TX)} \left(\widehat{\Gamma}_X (2\pi \mathtt{i})^{\frac{\deg}{2}} \operatorname{ch}(E) \right)$$

where

$$L(\tau)\alpha = \alpha + \sum_{d,n,i} \frac{Q^d}{n!} \left\langle \phi_i, \tau, \dots, \tau, \frac{\alpha}{-z-\psi} \right\rangle_{0,n+2,d}^{X,T} \phi^i$$

is a standard fundamental solution for the quantum connection and $\mu \in \operatorname{End}(H_T^{**}(X))$ is the grading operator.

Properties

1. pairing preserving

$$\big(\mathfrak{s}(E)(\tau, e^{-\pi\mathbf{i}} z), \mathfrak{s}(F)(\tau, z)\big) = \chi_T(E, F)\Big|_{\lambda \to 2\pi\mathbf{i}\lambda/z}$$

2. invariant under shift operator:

$$\mathbb{S}_k \mathfrak{s}(E)|_{Q=1} = \mathfrak{s}(E)|_{Q=1}$$

- Part 1 follows from the identity $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin \pi z}$ and Hirzebruch-Riemann-Roch
- Part 2 follows from the identity $\Gamma(1+z) = z\Gamma(z)$.

Recall that $K_T^0(X)$ is a module over $\mathbb{Z}[e^{\lambda_1}, \ldots, e^{\lambda_n}]$.
Shift operator exhibits the symmetry $\lambda \to \lambda + 2\pi\mathbf{i}$ of equivariant K-theory in quantum cohomology.

Motivation: mirror symmetry

Theorem ('07): The Gamma integral structure of quantum cohomology of a toric orbifold matches with the natural integral structure on the B-side given by Lefschetz thimbles.

Global nature of the Gamma structure

Mirror symmetry suggests that the Gamma structure is global over Kähler moduli.

Theorem (Borisov-Horja '05 for part 3, Coates-I-Jiang '14) $\mathfrak{X}_1$, $\mathfrak{X}_2$: toric orbifolds related by a crepant birational transformation (i.e. K-equivalent).

1. The equivariant quantum connection of $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are analytically continued to each other over a "global Kähler moduli space" (crepant transformation conjecture).

2. The analytic continuation matches up the Gamma integral structures of $\mathfrak{X}_1$ and $\mathfrak{X}_2$.

3. Moreover, it is induced by a Fourier-Mukai transformation $D^b_T(\mathfrak{X}_1) \to D^b_T(\mathfrak{X}_2)$.