QUANTUM COHOMOLOGY AND BIRATIONAL TRANSFORMATION

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Abstract. A famous conjecture of Yongbin Ruan says that quantum cohomology of birational varieties becomes isomorphic after analytic continuation when the birational transformation preserves the canonical class (the so-called crepant transformation). When the transformation is not crepant, the quantum cohomology becomes non-isomorphic, but it is conjectured that one side is a direct summand of the other. In this talk, I will explain a conjecture that a semiorthogonal decomposition of topological $K$-groups (or derived categories) should induce a relationship between quantum cohomology. The relationship between quantum cohomology can be described in terms of solutions to a Riemann-Hilbert problem.

References: [arXiv:1906.00801] (particularly the last section 8), discussion with Galkin

1. Introduction

$X$: smooth projective variety

$QH(X) = (H^*(X), \star_{\tau})_{\tau \in H^*(X)}$: quantum cohomology;

a family of (super)commutative product structures parametrized by $\tau \in H^*(X)$. It is defined in terms of genus-zero Gromov-Witten invariants.

$$(\alpha \star_{\tau} \beta, \gamma) = \sum_{n \geq 0, d \in H_2(X, \mathbb{Z})} (\alpha, \beta, \gamma, \overbrace{\tau, \ldots, \tau}^{n})_{0,n+3,d} \frac{1}{n!}$$

where $(\cdot, \cdot)$ is the Poincaré pairing. We don’t know the convergence in general, but we will assume it.

We have

$$\star_{\tau} \to \cup$$

in the large radius limit

$$\tau \in H^2(X), \quad \Re \left( \int_d \tau \right) \to -\infty \quad \text{for all effective curve classes } d \in H_2(X, \mathbb{Z}) \setminus \{0\}$$

Crepant resolution conjecture (Y. Ruan)

A birational map $\phi: X_1 \dashrightarrow X_2$ is crepant (or $K$-equivalent) if there exist a smooth projective variety $Z$ and a commutative diagram

$\begin{tikzcd}
Z \arrow[swap]{d}{f} \arrow{r}{g} & X_2 \\
X_1 \arrow{r}{\phi} & X_2
\end{tikzcd}$

with $f, g$ birational morphisms, such that $f^*K_{X_1} = g^*K_{X_2}$.

Conjecture: Then, $QH(X_1) \cong QH(X_2)$ after analytic continuation in $\tau$.

Rem: The isomorphism would depend on the choice of a path connecting the large radius limit points.

Rem: $\exists$ an isomorphism as graded vector spaces: $H^*(X_1) \cong H^*(X_2)$. (Kontsevich, Batyrev, Yasuda, ...)

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Discrepant case: Suppose that $f^*K_{X_1} < g^*K_{X_2}$, i.e. $g^*K_{X_2} - f^*K_{X_1}$ is an effective divisor.

$\implies QH(X_1)$ would be a direct summand of $QH(X_2)$ (after analytic continuation).

Goal: Want to understand a precise relationship in terms of quantum differential equations and certain Betti data (coming from the topological $K$-group).

2. Structure of quantum connection

Fix $\tau \in H = H^*(X)$. Consider the meromorphic flat connection $\nabla^{(\tau)}$ on the trivial bundle $H \times \mathbb{C}_z \to \mathbb{C}_z$:

$$\nabla^{(\tau)}_{\partial_z} = \frac{\partial}{\partial z} - \frac{1}{z^2} E^* \tau + \frac{1}{z} \mu$$

where

$$E = c_1(X) + \sum_i (1 - \frac{i}{2} \deg \phi_i) \tau^i \phi_i$$

is the Euler vector field and

$$\mu \in \text{End}(H), \quad \mu(\phi_i) = \frac{1}{2} (\deg \phi_i - \dim_{\mathbb{C}} X) \phi_i$$

is the grading operator. Here $\{\phi_i\}$ is a homogeneous basis of $H = H^*(X)$ and $\tau = \sum_i \tau^i \phi_i$.

It is called quantum connection or Dubrovin connection.

Fact (Dubrovin): the family of flat connections $\{\nabla^{(\tau)}_{\partial_z}\}_{\tau \in H}$ is isomonodromic, i.e. can be extended to a flat connection on the bundle $H \times (H \times \mathbb{C}) \to H \times \mathbb{C} \ni (\tau, z)$.

A formula for the extended connection:

$$\nabla_{\partial_\tau} = \frac{\partial}{\partial \tau} + \frac{1}{z} \phi_i \star \tau.$$

Rem: $\nabla^{(\tau)}_{\partial_z}$ has regular singularity (or better, logarithmic singularity) at $z = \infty$;

but has irregular singularities at $z = 0$ (in general) since it has order two poles at $z = 0$.

Rem: the quantum connection is self-dual with respect to the Poincaré pairing between the fibers at $z$ and $-z$.

Conjecture/Expectation: Write $\text{QC}(X)_\tau := (H \times \mathbb{C}_z \to \mathbb{C}_z, \nabla^{(\tau)}_{\partial_z})$.

(1) (formal decomposition); this is expected from mirror symmetry. First introduced by Hertling-Sevenerheck under the name require no ramifications; later it is called of exponential type by Katzarkov-Kontsevich-Pantev.

Consider the restriction of $\text{QC}(X)_\tau$ to the formal neighbourhood

$$\text{QC}(X)_\tau := \text{QC}(X) \otimes_{\mathbb{C}[z]} \mathbb{C}[z]$$

Then it should admit the following orthogonal decomposition:

$$\text{QC}(X)_\tau \cong \bigoplus_{u \in \text{Spec}(E^*)} (e^{u/z} \otimes F_u) \otimes_{\mathbb{C}[z]} \mathbb{C}[z]$$

where $\text{Spec}(E^*)$ is the set of eigenvalues of $E^*$, and

- $e^{u/z}$ denotes the rank one connection $(\mathbb{C}[z], d + d(u/z))$;
- $F_u$ is a free $\mathbb{C}[z]$-module with regular singular connection (and a pairing)

Rem: in general, Hukuhara-Turrittin theorem says that we have a certain similar decomposition over $\mathbb{C}((z))$ after pulling back by a ramified covering $z = w^r$, $r \in \mathbb{Z}_{>0}$. 

(2) (analytic lift): this is a fact (Hukuhara-Turrittin theorem) when (1) holds. The above formal decomposition lifts uniquely to an analytic decomposition over a sector $S = S_{\phi}$ centered at the angle $\phi$ and of angle $> \pi$

$$\text{QC}(X)_\tau \big|_S \cong \bigoplus_{u \in \text{Spec}(E^{\star})} e^{u/z} \otimes \mathcal{F}_u \big|_S$$

with

$$S = S_{\phi} = \{z \in \mathbb{C}^x : |\arg z - \phi| < \frac{\pi}{2} + \epsilon\}$$

if the direction $e^{i\phi}$ is admissible (in the sense that $u_1 - u_2 \notin \mathbb{R}_{>0} e^{i\phi}$ for all $u_1, u_2 \in \text{Spec}(E^{\star})$).

(3) (SOD and Stokes data): this is also a general fact provided (1) holds. The above analytic decomposition induces a semiorthogonal decomposition (SOD) of the space $V_S$ of flat sections of the quantum connection over $S$:

$$V_S = \bigoplus_{u \in \text{Spec}(E^{\star})} V_u$$

If we equip $V_S$ with the pairing

$$[s_1, s_2] = (s_1(e^{-\pi z}), s_2(z))_{\text{Poincaré}}$$

the decomposition is semiorthogonal in the sense that

$$[V_{u_1}, V_{u_2}] = 0 \quad \text{if} \quad \Im(u_1/e^{i\phi}) < \Im(u_2/e^{i\phi}).$$

The analytic decomposition associated with the opposite sector $-S$ is dual to the above decomposition

$$V_{-S} = \bigoplus_{u \in \text{Spec}(E^{\star})} V'_u$$

with respect to the natural pairing $V_{-S} \times V_S \to \mathbb{C}$. Since the sectors $S$ and $-S$ overlap in two connected components, we have two analytic continuation maps $t_{\pm}$:

$$V_{-S} \xrightarrow{t_+} V_S \xleftarrow{t_-} V_S$$

given by $\langle t_{-}(\alpha), \beta \rangle = [\alpha, \beta]$ and $\langle t_{+}(\alpha), \beta \rangle = [\beta, \alpha]$. These maps constitute the Stokes data: the formal decomposition together with the Stokes data reconstructs the analytic germ of $\text{QC}(X)_\tau$ at $z = 0$.

(4) (Dubrovin/Gamma conjecture): This is due to Galkin-Golyshev-Iritani (in the semisimple case) and Sanda-Shamoto (in general case). Under analytic continuation from $z = 0$ to $z = \infty$, the above SOD of the space of flat sections induces an SOD of the topological $K$-group (or derived category) via the $\Gamma$-integral structure. Roughly speaking, a flat section near $z = \infty$ corresponds one-to-one with a cohomology class in $H$

$$H \to \{\text{flat sections near } z = \infty\}, \quad \alpha \mapsto s_{\alpha}(z) \sim z^{\mu} z^{c_1(X)} \alpha$$

\footnote{This is called mutation system by Sanda-Shamoto.}
which is then related to a $K$-class by the map

$$K_{\text{top}}(X) \to H, \quad \mathcal{E} \mapsto \frac{1}{(2\pi)^{\dim X/2}} \Gamma_X \cup (2\pi i)^{\deg/2} \text{ch}(\mathcal{E})$$

This map intertwines the Euler pairing on the $K$-group with the above pairing $\langle \cdot, \cdot \rangle$ on the space of flat sections.

**Conclusion:** if we know the formal decomposition of $\overline{QC}(X)$ and the corresponding SOD of the $K$-group, we can recover the quantum connections (by gluing the flat connections over $S$, $-S$ and around $z = \infty$). This amounts to solving a *Riemann-Hilbert problem* (which Dubrovin studied in the semisimple case in details).

**Example** (Sanda-Shamoto): Let $X \subset \mathbb{P}^n$ be a degree $k < n$ Fano hypersurface. The eigenvalues of the small quantum multiplication $E_\tau = c_1(X) \star_\tau$ at $\tau = 0$ are

$$\{0\} \cup \{T\zeta : \zeta^{n+1-k} = 1\}$$

with $T = (n+1-k)k^{-\frac{b}{n+1-k}}$. After certain mutation, the corresponding SOD of the derived category is given by

$$D^b(X) = (\mathcal{A}, \mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n-k)),$$

where $\mathcal{O}, \ldots, \mathcal{O}(n-k)$ is an exceptional collection and $\mathcal{A}$ is the right orthogonal of $(\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n-k))$. The category $\mathcal{A}$ corresponds to the eigenvalue 0.

3. **Discrepant transformation**

**Conjecture:** Suppose that we have a birational map $\phi: X_1 \to X_2$ such that $K_{X_1} > K_{X_2}$. Then

1. orthogonal decomposition $\overline{QC}(X_1) = \mathcal{L} \oplus \overline{QC}(X_2)_{\tau} \oplus \mathcal{R}$ for some connections $\mathcal{L}, \mathcal{R}$.
2. the analytic lift

$$\text{QC}(X_1)_{\tau}|_S \cong (\mathcal{L} \oplus \overline{QC}(X_2)_{\tau} \oplus \mathcal{R})|_S$$

over a sector $S$ corresponds to an SOD

$$K(X_1) \cong K_L \oplus K(X_2) \oplus K_R$$

**Example:** Let $\tilde{X}$ be the blowup of $X$ along a smooth subvariety $Z \subset X$ of codimension $c$. We have Orlov’s SOD:

$$D^b(\tilde{X}) = \left\langle D^b(Z)_{-(c-1)}, \ldots, D^b(Z)_{-1}, D^b(X) \right\rangle$$

where $D^b(Z)_k$ is the image of the fully faithful functor $D^b(Z) \to D^b(\tilde{X})$ given by $\alpha \mapsto j_*(\mathcal{O}(k) \otimes \pi^*(\alpha))$:

$$\begin{array}{ccc}
E & \xrightarrow{j} & \tilde{X} \\
\pi \downarrow & & \downarrow \\
Z & \longrightarrow & X
\end{array}$$

where $E$ is the exceptional divisor. The quantum connection $\text{QC}(\tilde{X})$ should be recovered from those for $Z$ and $X$ as follows:
(1) Put $\mathcal{Q}C(\tilde{X}) := \mathcal{Q}C(Z)_{\tau_1} \oplus \cdots \oplus \mathcal{Q}C(Z)_{\tau_{c-1}} \oplus \mathcal{Q}C(X)_{\sigma}$. Here we assume that $\tau_i \in H^*(Z)$ and $\sigma \in H^*(X)$ are chosen so that the Euler eigenvalues align as in the following picture:

$$
\begin{align*}
\text{Spec}(E^{Z \ast \tau_1}) & \quad : \quad e^{i\theta} \\
\vdots & \\
\text{Spec}(E^{Z \ast \tau_{c-1}}) & \\
\text{Spec}(E^{X \ast \sigma})
\end{align*}
$$

(2) Using Orlov’s SOD, we can reconstruct the Stokes data of $\mathcal{Q}C(\tilde{X})$ from the Stokes data of each piece. This recovers the analytic germ of $\mathcal{Q}C(\tilde{X})$ at $z = 0$.

(3) we can glue it with the germ of the flat connection near $z = \infty$ given by

$$
\nabla_{\partial_z} \sim \frac{\partial}{\partial z} - \frac{c_1(\tilde{X})}{z^2} + \frac{\mu}{z}
$$

via the $\widehat{\Gamma}$-integral structure. The parameter $f(\tau_1, \ldots, \tau_{c-1}, \sigma)$ for $\mathcal{Q}C(\tilde{X})$ can be determined. (Locally the parameter space should be the product of $(c - 1)$ copies of $H^*(Z)$ and $H^*(X)$.)

**Rem:** This picture has been partially verified in [arXiv:1906.00801] when $X$ is a weak Fano toric orbifold, $Z$ is a toric suborbifold, and $\tilde{X}$ is a weighted blowup of $X$ along $Z$ (we assume that $\tilde{X}$ is also weak Fano). See Example 7.34 (blowup of $\mathbb{P}^4$ along $\mathbb{P}^1$) where this picture is fully verified.