QUANTUM COHOMOLOGY OF BLOWUPS: A CONJECTURE

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Abstract. In this talk, I discuss a conjecture that a semiorthogonal decomposition of topological K-groups (or derived categories) due to Orlov should induce a relationship between quantum cohomology under blowups. The relationship between quantum cohomology can be described in terms of solutions to a Riemann-Hilbert problem.

References: 1906.00801 (particularly the last section 8), discussion with Sergey Galkin

1. Introduction

Let X be a smooth projective variety and let $\mathrm{QH}(X) = (H^*(X), \star_\tau)_{\tau \in H^*(X)}$ be the quantum cohomology. It is a family of (super)commutative product structures parametrized by $\tau \in H^*(X)$. It is defined in terms of genus-zero Gromov-Witten invariants.

$$(\alpha \star_\tau \beta, \gamma) = \sum_{n \geq 0, d \in H_2(X,\mathbb{Z})} \langle \alpha, \beta, \gamma, \overbrace{\tau, \dots, \tau}^{n} \rangle_{0,n+3,d} \frac{1}{n!}$$

where $(\cdot, \cdot)$ is the Poincaré pairing. We don't know the convergence in general, but we will assume it.

Rem: We have

$$\star_\tau \to \cup$$

in the following *large radius limit*

$$\tau \in H^2(X), \quad \Re\left(\int_d \tau\right) \to -\infty \quad \text{for all effective curve classes } d \in H_2(X,\mathbb{Z}) \setminus \{0\}$$

e.g. $\tau = -r\omega$ with ω ample and $r \to \infty$.

Crepant transformation conjecture (Y. Ruan)

A birational map $\phi\colon X_1 \dashrightarrow X_2$ is *crepant* (or K-equivalent) if there exist a smooth projective variety Z and a commutative diagram

$$\begin{array}{ccc} & Z & \\ {}^{f}\swarrow & & \searrow^{g} \\ X_1 & \overset{\phi}{\dashrightarrow} & X_2 \end{array}$$

with f, g birational morphisms, such that $f^*K_{X_1} = g^*K_{X_2}$.

Conjecture: Then, $QH(X_1) \cong QH(X_2)$ after analytic continuation in τ.

Rem: The isomorphism would depend on the choice of a path connecting the large radius limit points.

Rem: $\exists$ an isomorphism as graded vector spaces: $H^*(X_1) \cong H^*(X_2)$. (Kontsevich, Batyrev, Yasuda, ...)

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Discrepant transformation conjecture (?) Suppose that $f^*K_{X_1} \le g^*K_{X_2}$, i.e. $g^*K_{X_2} - f^*K_{X_1}$ is an effective divisor.

$\Longrightarrow QH(X_1)$ would be a *direct summand* of $QH(X_2)$ (after analytic continuation).

Goal: We want to understand a precise relationship in terms of quantum differential equations and certain Betti data (coming from the topological K-group).

2. Structure of quantum connection

Fix $\tau \in H = H^*(X)$. Consider the meromorphic flat connection $\nabla^{(\tau)}$ on the trivial bundle $H \times \mathbb{C}_z \to \mathbb{C}_z$:

$$\nabla^{(\tau)}_{\partial_z} = \frac{\partial}{\partial z} - \frac{1}{z^2} E \star_\tau + \frac{1}{z}\mu$$

where

$$E = c_1(X) + \sum_i (1 - \tfrac{1}{2}\deg \phi_i)\tau^i \phi_i$$

is the *Euler vector field* and

$$\mu \in \mathrm{End}(H), \qquad \mu(\phi_i) = \frac{1}{2}(\deg \phi_i - \dim_{\mathbb{C}} X)\phi_i$$

is the grading operator. Here $\{\phi_i\}$ is a homogeneous basis of $H = H^*(X)$ and $\tau = \sum_i \tau^i \phi_i$. It is called *quantum connection* or *Dubrovin connection.*

Fact (Dubrovin): the family of flat connections $\{\nabla^{(\tau)}_{\partial_z}\}_{\tau \in H}$ is *isomonodromic*, i.e. can be extended to a flat connection on the bundle $H \times (H \times \mathbb{C}) \to H \times \mathbb{C} \ni (\tau, z)$.

A formula for the extended connection:

$$\nabla_{\frac{\partial}{\partial \tau^i}} = \frac{\partial}{\partial \tau^i} + \frac{1}{z}\phi_i \star_\tau .$$

Rem: $\nabla^{(\tau)}_{\partial_z}$ has regular singularity (or better, logarithmic singularity) at $z = \infty$;
but has *irregular singularities* at $z = 0$ (in general) since it has order two poles at $z = 0$.

Rem: the quantum connection is self-dual with respect to the Poincaré pairing between the fibers at z and $-z$.

Conjecture/Expectation: Write $\mathrm{QC}(X)_\tau := (H \times \mathbb{C}_z \to \mathbb{C}_z, \nabla^{(\tau)}_{\partial_z})$.

(1) (formal decomposition); this is expected from mirror symmetry and was introduced by Hertling-Sevenheck under the name *require no ramifications*; Katzarkov-Kontsevich-Pantev called it *of exponential type.*

Consider the restriction of $\mathrm{QC}(X)_\tau$ to the formal neighbourhood of $z = 0$

$$\overline{\mathrm{QC}}(X)_\tau := \mathrm{QC}(X) \otimes_{\mathbb{C}[z]} \mathbb{C}[\![z]\!]$$

Then it should admit the following orthogonal decomposition:

$$\overline{\mathrm{QC}}(X)_\tau \cong \bigoplus_{u \in \mathrm{Spec}(E\star_\tau)} (e^{u/z} \otimes \mathcal{F}_u) \otimes_{\mathbb{C}\{z\}} \mathbb{C}[\![z]\!]$$

where $\mathrm{Spec}(E\star_\tau))$ is the set of eigenvalues of $E\star_\tau$ and

- $e^{u/z}$ denotes the rank one connection $(\mathbb{C}\{z\}, d + d(u/z))$;
- $\mathcal{F}_u$ is a free $\mathbb{C}\{z\}$-module with regular singular connection (and a pairing)

Rem: In general, the Hukuhara-Turrittin theorem says that we have a formal decomposition after pulling back by a ramified covering $w \mapsto z = w^r$, $r \in \mathbb{Z}_{>0}$. But here we do 'not require ramifications'.

(2) (analytic lift): this is a fact (the Hukuhara-Turrittin theorem) when (1) holds. The above formal decomposition lifts uniquely to an analytic decomposition over a sector $S = S_\phi$ centered at the angle ϕ and of angle $> \pi$

$$\mathrm{QC}(X)_\tau\Big|_S \cong \bigoplus_{u\in \mathrm{Spec}(E\star_\tau)} e^{u/z}\otimes \mathcal{F}_u\Big|_S$$

Here

$$S = S_\phi = \{z \in \mathbb{C}^\times : |\arg z - \phi| < \tfrac{\pi}{2} + \epsilon\}$$

and we assume that the direction $e^{i\phi}$ is admissible (in the sense that $u_1 - u_2 \notin \mathbb{R}_{>0}e^{i\phi}$ for all $u_1, u_2 \in \mathrm{Spec}(E\star_\tau)$).

(3) (SOD and Stokes data): this is also a general fact provided (1) holds. The above analytic decomposition induces a *semiorthogonal decomposition*[1] (SOD) of the space V_S of flat sections of the quantum connection over S:

$$V_S = \bigoplus_{u\in \mathrm{Spec}(E\star_\tau)} V_u$$

If we equip V_S with the pairing

$$[s_1, s_2) = \big(s_1(e^{-\pi i}z), s_2(z)\big)_{\text{Poincaré}}$$

the decomposition is semiorthogonal in the sense that

$$[V_{u_1}, V_{u_2}) = 0 \quad \text{if } \Im(u_1/e^{i\phi}) < \Im(u_2/e^{i\phi}).$$

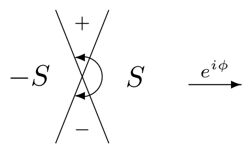
The analytic decomposition associated with the opposite sector $-S$ is *dual* to the above decomposition

$$V_{-S} = \bigoplus_{u\in \mathrm{Spec}(E\star_\tau)} V'_u$$

with respect to the natural pairing $V_{-S} \times V_S \to \mathbb{C}$. Since the sectors S and $-S$ overlap in two connected components (see the figure below), we have two analytic continuation maps $t_\pm$:

$$V_{-S} \underset{t_-}{\overset{t_+}{\leftleftarrows}} V_S$$

These maps satisfy $\langle t_-(\alpha), \beta\rangle = [\alpha, \beta)$ and $\langle t_+(\alpha), \beta\rangle = [\beta, \alpha)$. The maps $t_\pm$ constitute the *Stokes data*: *the formal decomposition together with the Stokes data reconstructs the analytic germ of* $\mathrm{QC}(X)_\tau$ *at* $z = 0$.



(4) (Dubrovin/Gamma conjecture): See Galkin-Golyshev-Iritani (in the semisimple case) and Sanda-Shamoto (in general case). Under analytic continuation from $z = 0$ to $z = \infty$, the above SOD of the space of flat sections induces an SOD of the topological K-group (or derived category) via the $\widehat{\Gamma}$-integral structure. Roughly speaking, a flat section near $z = \infty$ corresponds one-to-one with a cohomology class in H

$$H \to \{\text{flat sections near } z = \infty\}, \quad \alpha \mapsto s_\alpha(z) = (1 + O(1/z))z^{-\mu}z^{c_1(X)}\alpha$$

[1]This is called *mutation system* by Sanda-Shamoto.

which is then related to a K-class by the map

$$K_{\mathrm{top}}(X) \to H, \quad \mathcal{E} \mapsto \frac{1}{(2\pi)^{\dim X/2}} \widehat{\Gamma}_X \cup (2\pi i)^{\deg/2} \operatorname{ch}(\mathcal{E})$$

This map intertwines the Euler pairing on the K-group with the above pairing $[\cdot, \cdot)$ on the space of flat sections.

Conclusion: if we know the formal decomposition of $\overline{\mathrm{QC}}(X)_\tau$ and the corresponding SOD of the K-group, we can recover the quantum connections (by gluing the flat connections over S, $-S$ and around $z = \infty$). This amounts to solving a *Riemann-Hilbert problem* (recall the work of Dubrovin in the semisimple case).

Example (Sanda-Shamoto): Let $X \subset \mathbb{P}^n$ be a degree $k < n$ Fano hypersurface. The eigenvalues of the small quantum multiplication $E\star_\tau = c_1(X)\star_\tau$ at $\tau = 0$ are

$$\{0\} \cup \{T\zeta : \zeta^{n+1-k} = 1\}$$

with $T = (n+1-k)k^{\frac{k}{n+1-k}}$. After certain mutation, the corresponding SOD of the derived category is given by

$$D^b(X) = \langle \mathcal{A}, \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n-k) \rangle \,.$$

where $\mathcal{O}, \dots, \mathcal{O}(n-k)$ is an exceptional collection and $\mathcal{A}$ is the right orthogonal of $\langle \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n-k) \rangle$ (Kuznetsov). The category $\mathcal{A}$ corresponds to the eigenvalue 0.

3. Discrepant transformation/Blowups

Conjecture: Suppose that we have a birational map $\phi \colon X_1 \dashrightarrow X_2$ such that $K_{X_1} \geq K_{X_2}$. Then we have

(1) an orthogonal decomposition $\overline{\mathrm{QC}}(X_1)_\tau = \overline{L} \oplus \overline{\mathrm{QC}}(X_2)_{f(\tau)} \oplus \overline{R}$ for some connections $\overline{L}$, $\overline{R}$.
(2) the analytic lift

$$\mathrm{QC}(X_1)_\tau|_S \cong (L \oplus \mathrm{QC}(X_2)_{f(\tau)} \oplus R)|_S$$

over a sector S corresponds to an SOD

$$K(X_1) \cong K_L \oplus K(X_2) \oplus K_R$$

The case of blowups: Let $\widetilde{X}$ be the blowup of X along a smooth subvariety $Z \subset X$ of codimension c. We have Orlov's SOD:

$$D^b(\widetilde{X}) = \left\langle D^b(Z)_{-(c-1)}, \dots, D^b(Z)_{-1}, D^b(X) \right\rangle$$

where $D^b(Z)_k$ is the image of the fully faithful functor $D^b(Z) \to D^b(\widetilde{X})$ given by $\alpha \mapsto j_*(\mathcal{O}(k) \otimes \pi^*(\alpha))$:

$$\begin{array}{ccc} E & \xrightarrow{j} & \widetilde{X} \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

where E is the exceptional divisor. The quantum connection $\mathrm{QC}(\widetilde{X})$ should be recovered from those for Z and X as follows:

(1) Put $\overline{\mathrm{QC}}(\widetilde{X}) := \overline{\mathrm{QC}}(Z)_{\tau_1} \oplus \cdots \oplus \overline{\mathrm{QC}}(Z)_{\tau_{c-1}} \oplus \overline{\mathrm{QC}}(X)_\sigma$. Here we assume that $\tau_i \in H^*(Z)$ and $\sigma \in H^*(X)$ are chosen so that the eigenvalues of the Euler vector fields align as in the following picture:

$$\begin{array}{cc} \boxed{\mathrm{Spec}(E^Z\star_{\tau_1})} & \\ \vdots & \xrightarrow{e^{i\phi}} \\ \boxed{\mathrm{Spec}(E^Z\star_{\tau_{c-1}})} & \\ \boxed{\mathrm{Spec}(E^X\star_{\sigma})} & \end{array}$$

i.e. the collections $\mathrm{Spec}(E^Z\star_{\tau_1}), \dots, \mathrm{Spec}(E^Z\star_{\tau_{c-1}}), \mathrm{Spec}(E^X\star_\sigma)$ of eigenvalues are separated by lines parallel to an admissible $e^{i\phi}$ and line up in this order.

(2) Using Orlov's SOD, we reconstruct the Stokes data of $\mathrm{QC}(\widetilde{X})$ from those of $\mathrm{QC}(Z)_{\tau_i}$ and $\mathrm{QC}(X)_\sigma$. Namely, the Stokes data for $\mathrm{QC}(Z)_{\tau_i}$ and $\mathrm{QC}(X)_\sigma$ determine SODs of $K_{\mathrm{top}}(Z)$ and $K_{\mathrm{top}}(X)$ (for a given phase $e^{i\phi}$); they are glued together to give an SOD of $K_{\mathrm{top}}(\widetilde{X})$ via the Orlov decomposition

$$K_{\mathrm{top}}(\widetilde{X}) \cong K_{\mathrm{top}}(Z)_{-(c-1)} \oplus \cdots \oplus K_{\mathrm{top}}(Z)_{-1} \oplus K_{\mathrm{top}}(X)$$

and give rise to the Stokes data for $\mathrm{QC}(\widetilde{X})$. This recovers the analytic germ of $\mathrm{QC}(\widetilde{X})$ at $z = 0$.

(3) we can glue it (via the $\widehat{\Gamma}$-integral structure) with the germ $\nabla^{(\tau,z=\infty)}_{\partial_z}$ of the flat connection near $z = \infty$ given by

$$\nabla^{(\tau)}_{\partial_z} \underset{\substack{\text{gauge}\\ \text{equivalent}}}{\sim} \nabla^{(\tau,z=\infty)}_{\partial_z} := \frac{\partial}{\partial z} - \frac{c_1(\widetilde{X})}{z^2} + \frac{\tilde{\mu}}{z}.$$

Then we obtain a global vector bundle over $\mathbb{P}^1_z$ which should be trivial. The gauge transformation between $\nabla^{(\tau)}$ and $\nabla^{(\tau,z=\infty)}$ gives a fundamental solution (the so-called *calibration*). The parameter $\tau = f(\tau_1, \dots, \tau_{c-1}, \sigma)$ for $\mathrm{QC}(\widetilde{X})$ can also be determined. (Locally the parameter space should be isomorphic to the product of $(c-1)$ copies of $H^*(Z)$ and $H^*(X)$ as an F-manifold.)

Rem: This picture has been partially verified in 1906.00801 when X is a weak Fano toric orbifold, Z is a toric suborbifold, and $\widetilde{X}$ is a weighted blowup of X along Z (we assume that $\widetilde{X}$ is also weak Fano).

Example (Example 7.34 in 1906.00801) Let $X = \mathbb{P}^4$ and let $\varphi \colon \widetilde{X} \to X$ be the blowup along a line $\mathbb{P}^1$. We have a projection $\pi \colon \widetilde{X} \to \mathbb{P}^2$ and $\widetilde{X}$ is a $\mathbb{P}^2$-bundle over $\mathbb{P}^2$. Set $p_1 = \pi^* H$ and $p_2 = \varphi^* H$ and consider the parameter

$$\tau = p_1 \log q_1 + p_2 \log q_2$$

There are two limiting pictures:

fibration picture: $|q_1| \ll |q_2|$, $-\tau$ approaches the ample class on $\mathbb{P}^2$

blow-down picture: $|q_2| \ll |q_1|$, $-\tau$ approaches the ample class on $X = \mathbb{P}^4$

In these limits, the quantum cohomology decomposes differently (see the figure in Example 7.34).

$$\mathrm{QH}^*(\widetilde{X})_\tau \cong \mathrm{QH}(\text{base } \mathbb{P}^2)_{t_1} \oplus \mathrm{QH}(\text{base } \mathbb{P}^2)_{t_2} \oplus \mathrm{QH}(\text{base } \mathbb{P}^2)_{t_3}$$

$$\mathrm{QH}^*(\widetilde{X})_\tau \cong \mathrm{QH}(\mathbb{P}^1)_{\tau_1} \oplus \mathrm{QH}(\mathbb{P}^4)_\sigma \oplus \mathrm{QH}(\mathbb{P}^1)_{\tau_2}$$