QUANTUM COHOMOLOGY OF BLOWUPS: A CONJECTURE

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Abstract. In this talk, I discuss a conjecture that a semiorthogonal decomposition of topological K-groups (or derived categories) due to Orlov should induce a relationship between quantum cohomology under blowups. The relationship between quantum cohomology can be described in terms of solutions to a Riemann-Hilbert problem.

References: [1906.00801] (particularly the last section 8), discussion with Sergey Galkin

1. Introduction

Let $X$ be a smooth projective variety and let $\text{QH}(X) = (H^*(X), \star_\tau)_{\tau \in H^*(X)}$ be the quantum cohomology. It is a family of (super)commutative product structures parametrized by $\tau \in H^*(X)$. It is defined in terms of genus-zero Gromov-Witten invariants.

$$(\alpha \star_\tau \beta, \gamma) = \sum_{n \geq 0, d \in H_2(X, Z)} (\alpha, \beta, \gamma, \tau_0, \ldots, \tau_n)_{0, n+3, d} \frac{1}{n!}$$

where $(\cdot, \cdot)$ is the Poincaré pairing. We don’t know the convergence in general, but we will assume it.

Rem: We have $\star_\tau \to \cup$ in the following large radius limit

$$\tau \in H^2(X), \quad \Re \left( \int \frac{d}{d\tau} \right) \to -\infty \quad \text{for all effective curve classes } d \in H_2(X, Z) \setminus \{0\}$$

e.g. $\tau = -r\omega$ with $\omega$ ample and $r \to \infty$.

Crepant transformation conjecture (Y. Ruan)

A birational map $\phi: X_1 \dasharrow X_2$ is crepant (or $K$-equivalent) if there exist a smooth projective variety $Z$ and a commutative diagram

$$\begin{array}{ccc}
Z & \xrightarrow{f} & X_1 \\
\downarrow & & \downarrow \phi \\
X_2 & \xrightarrow{g} & X_2
\end{array}$$

with $f, g$ birational morphisms, such that $f^*K_{X_1} = g^*K_{X_2}$.

Conjecture: Then, $\text{QH}(X_1) \cong \text{QH}(X_2)$ after analytic continuation in $\tau$.

Rem: The isomorphism would depend on the choice of a path connecting the large radius limit points.

Rem: $\exists$ an isomorphism as graded vector spaces: $H^*(X_1) \cong H^*(X_2)$. (Kontsevich, Batyrev, Yasuda, ...)

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Discrepant transformation conjecture (?) Suppose that $f^*K_{X_1} \leq g^*K_{X_2}$, i.e. $g^*K_{X_2} - f^*K_{X_1}$ is an effective divisor.

$\implies QH(X_1)$ would be a direct summand of $QH(X_2)$ (after analytic continuation).

**Goal:** We want to understand a precise relationship in terms of quantum differential equations and certain Betti data (coming from the topological $K$-group).

### 2. Structure of quantum connection

Fix $\tau \in H \coloneqq H^\ast(X)$. Consider the meromorphic flat connection $\nabla^{(\tau)}$ on the trivial bundle $H \times \mathbb{C}_z \to \mathbb{C}_z$:

$$\nabla^{(\tau)} = \frac{\partial}{\partial z} - \frac{1}{z^2} E \ast \tau + \frac{1}{z} \mu$$

where

$$E = c_1(X) + \sum_i (1 - \frac{1}{2} \deg \phi_i) \tau^i \phi_i$$

is the Euler vector field and

$$\mu \in \text{End}(H), \quad \mu(\phi_i) = \frac{1}{2}(\deg \phi_i - \dim_{\mathbb{C}} X) \phi_i$$

is the grading operator. Here $\{\phi_i\}$ is a homogeneous basis of $H = H^\ast(X)$ and $\tau = \sum_i \tau^i \phi_i$.

It is called quantum connection or Dubrovin connection.

**Fact (Dubrovin):** the family of flat connections $\{\nabla^{(\tau)}\}_{\tau \in H}$ is isomonodromic, i.e. can be extended to a flat connection on the bundle $H \times (H \times \mathbb{C}) \to H \times \mathbb{C} \ni (\tau, z)$.

A formula for the extended connection:

$$\nabla_{\frac{\partial}{\partial \tau^i}} = \frac{\partial}{\partial \tau^i} + \frac{1}{z} \phi_i \ast \tau^i$$

**Rem:** $\nabla^{(\tau)}$ has regular singularity (or better, logarithmic singularity) at $z = \infty$;

but has irregular singularities at $z = 0$ (in general) since it has order two poles at $z = 0$.

**Rem:** the quantum connection is self-dual with respect to the Poincaré pairing between the fibers at $z$ and $-z$.

**Conjecture/Expectation:** Write $QC(X) \coloneqq (H \times \mathbb{C}_z \to \mathbb{C}_z, \nabla^{(\tau)}_{\partial_\tau})$.

1. (formal decomposition); this is expected from mirror symmetry and was introduced by Hertling-Sevenheck under the name require no ramifications; Katzarkov-Kontsevich-Pantev called it of exponential type.

Consider the restriction of $QC(X)_{\tau}$ to the formal neighbourhood of $z = 0$

$$QC(X)_{\tau} \coloneqq QC(X) \otimes_{\mathbb{C}[z]} \mathbb{C}[z]$$

Then it should admit the following orthogonal decomposition:

$$QC(X)_{\tau} \cong \bigoplus_{u \in \text{Spec}(E \ast \tau)} (e^{u/z} \otimes \mathcal{F}_u) \otimes_{\mathbb{C}[z]} \mathbb{C}[z]$$

where $\text{Spec}(E \ast \tau)$ is the set of eigenvalues of $E \ast \tau$ and

- $e^{u/z}$ denotes the rank one connection $(\mathbb{C}[z], d + d(u/z))$;
- $\mathcal{F}_u$ is a free $\mathbb{C}[z]$-module with regular singular connection (and a pairing)

**Rem:** In general, the Hukuhara-Turrittin theorem says that we have a formal decomposition after pulling back by a ramified covering $w \mapsto z = w^r$, $r \in \mathbb{Z}_{>0}$. But here we do ‘not require ramifications’.
(2) (analytic lift): this is a fact (the Hukuhara-Turrittin theorem) when (1) holds. The above formal decomposition lifts uniquely to an analytic decomposition over a sector $S = S_\phi$ centered at the angle $\phi$ and of angle $> \pi$

$$\text{QC}(X)_\tau|_S \cong \bigoplus_{u \in \text{Spec}(E^*)} e^{u/z} \otimes F_u|_S$$

Here

$$S = S_\phi = \{ z \in \mathbb{C}^\times : |\arg z - \phi| < \frac{\pi}{2} + \epsilon \}$$

and we assume that the direction $e^{i\phi}$ is admissible (in the sense that $u_1 - u_2 \notin \mathbb{R}_{>0} e^{i\phi}$ for all $u_1, u_2 \in \text{Spec}(E^*)$).

(3) (SOD and Stokes data): this is also a general fact provided (1) holds. The above analytic decomposition induces a semiorthogonal decomposition (SOD) of the space $V_S$ of flat sections of the quantum connection over $S$:

$$V_S = \bigoplus_{u \in \text{Spec}(E^*)} V_u$$

If we equip $V_S$ with the pairing

$$[s_1, s_2] = (s_1(e^{-\pi i}z), s_2(z))_{\text{Poincaré}}$$

the decomposition is semiorthogonal in the sense that

$$[V_{u_1}, V_{u_2}] = 0 \quad \text{if} \quad \Im(u_1/e^{i\phi}) < \Im(u_2/e^{i\phi}).$$

The analytic decomposition associated with the opposite sector $-S$ is dual to the above decomposition

$$V_{-S} = \bigoplus_{u \in \text{Spec}(E^*)} V'_u$$

with respect to the natural pairing $V_{-S} \times V_S \to \mathbb{C}$. Since the sectors $S$ and $-S$ overlap in two connected components (see the figure below), we have two analytic continuation maps $t_\pm$:

These maps satisfy $\langle t_-(\alpha), \beta \rangle = [\alpha, \beta]$ and $\langle t_+(\alpha), \beta \rangle = [\beta, \alpha]$. The maps $t_\pm$ constitute the Stokes data: the formal decomposition together with the Stokes data reconstructs the analytic germ of $\text{QC}(X)_\tau$ at $z = 0$.

(4) (Dubrovin/Gamma conjecture): See Galkin-Golyshev-Iritani (in the semisimple case) and Sanda-Shamoto (in general case). Under analytic continuation from $z = 0$ to $z = \infty$, the above SOD of the space of flat sections induces an SOD of the topological $K$-group (or derived category) via the $\tilde{T}$-integral structure. Roughly speaking, a flat section near $z = \infty$ corresponds one-to-one with a cohomology class in $H$

$$H \to \{ \text{flat sections near } z = \infty \}, \quad \alpha \mapsto s_\alpha(z) = (1 + O(1/z)) z^{-\mu} e^{\text{cl}(X)} \alpha$$

This is called mutation system by Sanda-Shamoto.
which is then related to a $K$-class by the map
\[ K_{\text{top}}(X) \to H, \quad \mathcal{E} \mapsto \frac{1}{(2\pi)^{\dim X/2}} \tilde{\Gamma}_X \cup (2\pi i)^{\deg /2} \text{ch}(\mathcal{E}) \]
This map intertwines the Euler pairing on the $K$-group with the above pairing $[\cdot, \cdot]$ on the space of flat sections.

Conclusion: if we know the formal decomposition of $\overline{QC}(X)$ and the corresponding SOD of the $K$-group, we can recover the quantum connections (by gluing the at connections over $S$, $-S$ and around $z = \infty$). This amounts to solving a Riemann-Hilbert problem (recall the work of Dubrovin in the semisimple case).

**Example** (Sanda-Shamoto): Let $X \subset \mathbb{P}^n$ be a degree $k < n$ Fano hypersurface. The eigenvalues of the small quantum multiplication $E^\ast = c_1(X)^\ast$ at $\tau = 0$ are
\[ \{0\} \cup \{ T \zeta : \zeta^{n+1-k} = 1 \} \]
with $T = (n+1-k) k^{\frac{k}{n+1-k}}$. After certain mutation, the corresponding SOD of the derived category is given by
\[ D^b(X) = \langle A, \mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n-k) \rangle. \]
where $\mathcal{O}, \ldots, \mathcal{O}(n-k)$ is an exceptional collection and $A$ is the right orthogonal of $\langle \mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n-k) \rangle$ (Kuznetsov). The category $A$ corresponds to the eigenvalue 0.

3. Discrepant transformation/Blowups

**Conjecture**: Suppose that we have a birational map $\phi : X_1 \dashrightarrow X_2$ such that $K_{X_1} \geq K_{X_2}$. Then we have

1. an orthogonal decomposition $\overline{QC}(X_1) = \mathcal{L} \oplus \overline{QC}(X_2)_{f(\tau)} \oplus \mathcal{R}$ for some connections $\mathcal{L}, \mathcal{R}$,
2. the analytic lift $QC(X_1)_{\tau}|_S \cong (\mathcal{L} \oplus QC(X_2)_{f(\tau)} \oplus \mathcal{R})|_S$

over a sector $S$ corresponds to an SOD
\[ K(X_1) \cong K_L \oplus K(X_2) \oplus K_R \]

**The case of blowups**: Let $\tilde{X}$ be the blowup of $X$ along a smooth subvariety $Z \subset X$ of codimension $c$. We have Orlov’s SOD:
\[ D^b(\tilde{X}) = \langle D^b(Z)_{-(c-1)}, \ldots, D^b(Z)_{-1}, D^b(X) \rangle \]
where $D^b(Z)_k$ is the image of the fully faithful functor $D^b(Z) \to D^b(\tilde{X})$ given by $\alpha \mapsto j_\ast (\mathcal{O}(k) \otimes \pi^\ast (\alpha))$:

\[
\begin{array}{ccc}
E & \xrightarrow{j} & \tilde{X} \\
\downarrow \pi & & \downarrow \\
Z & \longrightarrow & X
\end{array}
\]
where $E$ is the exceptional divisor. The quantum connection $QC(\tilde{X})$ should be recovered from those for $Z$ and $X$ as follows:
(1) Put $\text{QC}(\tilde{X}) := \text{QC}(Z_{\tau_1}) \oplus \cdots \oplus \text{QC}(Z_{\tau_{c-1}}) \oplus \text{QC}(X)_{\sigma}$. Here we assume that $\tau_j \in H^*(Z)$ and $\sigma \in H^*(X)$ are chosen so that the eigenvalues of the Euler vector fields align as in the following picture:

$$
\begin{align*}
\text{Spec}(E^Z_{\tau_1}) \\
\vdots \\
\text{Spec}(E^Z_{\tau_{c-1}}) \\
\text{Spec}(E^X_{\sigma})
\end{align*} \xrightarrow{e^{i\phi}}
$$

i.e. the collections $\text{Spec}(E^Z_{\tau_1}), \ldots, \text{Spec}(E^Z_{\tau_{c-1}}), \text{Spec}(E^X_{\sigma})$ of eigenvalues are separated by lines parallel to an admissible $e^{i\phi}$ and line up in this order.

(2) Using Orlov’s SOD, we reconstruct the Stokes data of $\text{QC}(\tilde{X})$ from those of $\text{QC}(Z_{\tau_1})$ and $\text{QC}(X)_{\sigma}$. Namely, the Stokes data for $\text{QC}(Z_{\tau_1})$ and $\text{QC}(X)_{\sigma}$ determine SODs of $K_{\text{top}}(Z)$ and $K_{\text{top}}(X)$ (for a given phase $e^{i\phi}$); they are glued together to give an SOD of $K_{\text{top}}(\tilde{X})$ via the Orlov decomposition

$$
K_{\text{top}}(\tilde{X}) \cong K_{\text{top}}(Z)_{-c-1} \oplus \cdots \oplus K_{\text{top}}(Z)_{-1} \oplus K_{\text{top}}(X)
$$

and give rise to the Stokes data for $\text{QC}(\tilde{X})$. This recovers the analytic germ of $\text{QC}(\tilde{X})$ at $z = 0$.

(3) we can glue it (via the $\hat{\tau}$-integral structure) with the germ $\nabla^{(\tau,z=\infty)}_{\partial_z}$ of the flat connection near $z = \infty$ given by

$$
\nabla^{(\tau)}_{\partial_z} \xrightarrow{\text{gauge equivalent}} \nabla^{(\tau,z=\infty)}_{\partial_z} := \frac{\partial}{\partial z} - \frac{c_1(\tilde{X})}{z^2} + \frac{\mu}{z}.
$$

Then we obtain a global vector bundle over $\mathbb{P}^1$ which should be trivial. The gauge transformation between $\nabla^{(\tau)}$ and $\nabla^{(\tau,z=\infty)}$ gives a fundamental solution (the so-called calibration). The parameter $\tau = f(\tau_1, \ldots, \tau_{c-1}, \sigma)$ for $\text{QC}(\tilde{X})$ can also be determined. (Locally the parameter space should be isomorphic to the product of $(c-1)$ copies of $H^*(Z)$ and $H^*(X)$ as an $F$-manifold.)

**Rem:** This picture has been partially verified in [1306.1080] when $X$ is a weak Fano toric orbifold, $Z$ is a toric suborbifold, and $\tilde{X}$ is a weighted blowup of $X$ along $Z$ (we assume that $X$ is also weak Fano).

**Example** (Example 7.34 in [1306.1080]) Let $X = \mathbb{P}^4$ and let $\varphi: \tilde{X} \to X$ be the blowup along a line $\mathbb{P}^1$. We have a projection $\pi: \tilde{X} \to \mathbb{P}^2$ and $\tilde{X}$ is a $\mathbb{P}^2$-bundle over $\mathbb{P}^2$. Set $p_1 = \pi^* H$ and $p_2 = \varphi^* H$ and consider the parameter

$$
\tau = p_1 \log q_1 + p_2 \log q_2
$$

There are two limiting pictures:

- **fibration picture:** $|q_1| \ll |q_2|$, $-\tau$ approaches the ample class on $\mathbb{P}^2$
- **blow-down picture:** $|q_2| \ll |q_1|$, $-\tau$ approaches the ample class on $X = \mathbb{P}^4$

In these limits, the quantum cohomology decomposes differently (see the figure in Example 7.34).

$$
\begin{align*}
\text{QH}^*(\tilde{X})_\tau &\cong \text{QH}(\text{base } \mathbb{P}^2)_t_1 \oplus \text{QH}(\text{base } \mathbb{P}^3)_t_2 \oplus \text{QH}(\text{base } \mathbb{P}^2)_t_3 \\
\text{QH}^*(\tilde{X})_\tau &\cong \text{QH}(\mathbb{P}^1)_{\tau_1} \oplus \text{QH}(\mathbb{P}^4)_{\sigma} \oplus \text{QH}(\mathbb{P}^1)_{\tau_2}
\end{align*}
$$