Decomposition of quantum cohomology under blowups

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https://www.math.kyoto-u.ac.jp/~iritani/talk_Kansas.pdf ←

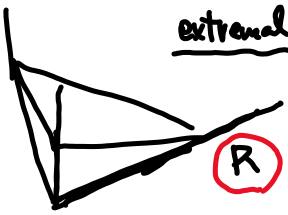
§ Introduction

X : sm proj variety

$\rightsquigarrow$ $QH(X)$ defined over $\mathbb{C}[\![Q]\!] = \mathbb{C}[\![NE_{\mathbb{N}}(X)]\!] \ni \sum_{d \in NE} a_d Q^d$

$NE_{\mathbb{N}}(X) \subset H_2(X, \mathbb{Z})$ effective curves

Mori cone $\overline{NE}(X) \subset H_2(X, \mathbb{R})$



extremal ray $R = \mathbb{R}_{\geq 0} \underline{\underline{d_0}} \subset \overline{NE}(X)$ 1-dim'al face

$c_1(X) \cdot d_0 > 0$

$\rightsquigarrow$ extremal contraction $f : X \longrightarrow Y$

st $\underset{\text{curve}}{C} \subset X$ contracts to a pt $\iff$ $[C] \in R$

f-exc quantum cohomology :

$$(\alpha *_f \beta, \gamma) = \sum_{n \geq 0} \langle \alpha, \beta, \gamma \rangle_{0,3,nd_0} Q^{nd_0} \qquad \alpha, \beta, \gamma \in H^*(X)$$

↑ finite sum (☺ $c_1(X) \cdot d_0 > 0$)

$QH_{exc}(X)$: defined over $\mathbb{C}[q]$ $\qquad q = Q^{d_0}$

$QH(X)$ is a deformation of $QH_{exc}(X)\big|_{q=1}$ $\rightsquigarrow$ $\mathrm{Spec}(QH_{exc}(X)|_{q=1})$

0-dim'al scheme $\ni d$

$\rightsquigarrow$ ring decomp $QH(X) \cong \bigoplus_{d} A_d$

Question $A_d \cong$ (big) QH of a space ?

Example 1 X : smooth toric DM stack $\quad X \dashrightarrow X^+$ toric flip (VGIT)

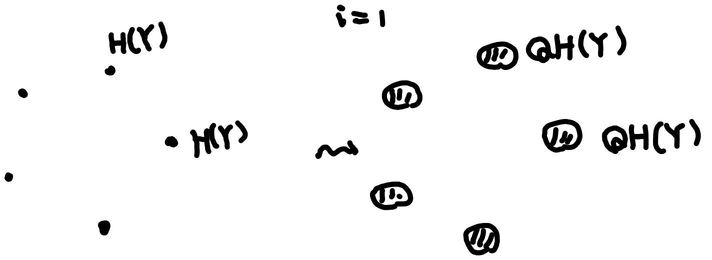
$QH(X)$ contains $QH(X^+)$ as a direct summand

Example 2 (I-Koto) $V \to Y$ vector bundle of rk r

$$f : X = \mathbb{P}(V) \longrightarrow Y \qquad \text{proj bdl}$$

$$QH_{\text{exc}}(X) \cong H^*(Y)[p, q] \Big/ \left(p^r + c_1(V)p^{r-1} + \cdots + c_r(V) - q\right)$$

$$\cong \bigoplus_{i=1}^{r} H^*(Y) \qquad \text{for } q = 1$$



Spec $(QH_{\text{exc}}(X))$

Thm

$$QH(X)_\tau \cong \bigoplus_{i=1}^{r} QH(Y)_{\sigma_i}$$

$\tau \in H^*(X)$, $\sigma_i \in H^*(Y)$ } big QC parameters

Example 3 $\tilde{X} = Bl_Z X \longrightarrow X$ blowup along a smooth subvariety Z, $r = \operatorname{codim} Z$

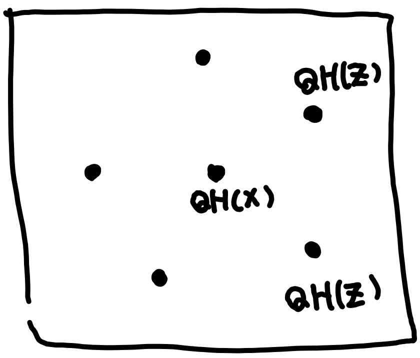
$$\rightsquigarrow \quad QH(\tilde{X})_{\hat{\tau}} \cong QH(X)_\tau \oplus \bigoplus_{i=1}^{r-1} QH(Z)_{\sigma_i}$$

$\sigma_i = \sigma_i(\hat{\tau})$

$\tau = \tau(\hat{\tau})$

$$Z = pt \subset X = \mathbb{C}^r \qquad \tilde{X} = \mathcal{O}_{\mathbb{P}^{r-1}}(-1)$$

$$\operatorname{Spec} QH(\tilde{X}) = \{ p(p^{r-1} + q) = 0 \}$$



More precise statement

$$\text{big } QH \quad \left(H^*(X) \otimes \mathbb{C}[\![Q]\!][\![\tau]\!], \ *_\tau\right)$$

$$\tau \in H^*(X) \qquad \tau = \sum \tau^i \phi_i$$

$\{\phi_i\}$: basis of $H^*(X)$

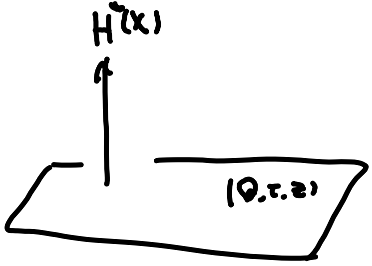
quantum D-module

$$QDM(X) := \underbrace{H^*(X)}_{\text{fiber}} \otimes \underbrace{\mathbb{C}[z][\![Q, \tau]\!]}_{\text{base}}$$

new variable

- quantum conn (flat)

$$\begin{cases} \boxed{\nabla_{\tau^i} = \dfrac{\partial}{\partial \tau^i} + \dfrac{1}{z}(\phi_i *_\tau)} \\ \nabla_{z\partial_z} = z\dfrac{\partial}{\partial z} - \dfrac{1}{z}(E *_\tau) + \mu \end{cases} : QDM(X) \to z^{-1} QDM(X)$$



$\nabla_{Q_i \frac{\partial}{\partial Q_i}} = \cdots$ $c_1(X) + \cdots$ grading operator

$z\nabla_{\tau^i} \xrightarrow{z\to 0} \phi_i^*$

• Pairing $P(f,g) = \int_X f(\underset{\sim}{-z}) \cup g(z)$

Thm $\exists$ formal invertible map $(\tau, \sigma_1, \cdots, \sigma_{r-1}) : H^*(\tilde{X}) \to H^*(X) \oplus H(Z)^{\oplus(r-1)}$

& $\exists$ isom $QDM(\tilde{X}) \cong \boxed{\tau^* QDM(X) \oplus \bigoplus_{i=1}^{r-1} \sigma_i^* QDM(Z)}$

preserving conn & pairing $\quad$ $q=0$ $\quad$ $q=\infty$ $\quad$ $\boxed{Q=\tilde{\tau}=0}$ $\quad$ $\mathbb{C}[q]$, $\hat{\mathbb{C}}((q^{-1}))$

Rem • need an extension of Novikov ring to a $\underline{\text{common ring}}$
$\uparrow$ equiv Mori cone of W

$K_{top}(\tilde{X}) \hookrightarrow \{\text{flat sections}\}$

• need to invert $\underline{q}$: extremal ray
line in the exceptional divisor $\mathbb{P}(N_{Z/X}) \to Z$

$K_{top}(\tilde{X}) \cong K_{top}(X) \oplus K_{top}(Z)^{\oplus(r-1)}$

isom is defined over $\mathbb{C}[z]((q^{-1/s}))\boxed{[[Q, \tilde{\tau}]]}$ $\quad$ $s = r-1$ or $2(r-1)$

rest of the Novikov

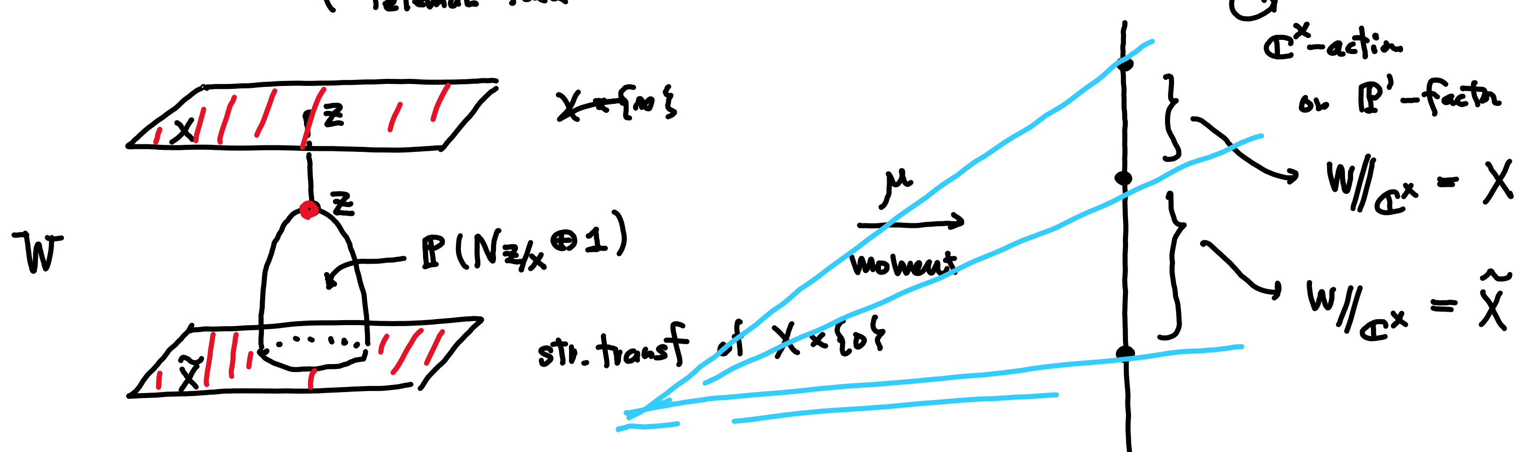
Rem Katzarkov-Kontsevich-Pantev-Yu

: application to rationality

: reconstruct $QH(\tilde{X})$ from $\underline{QH(X)}$ and $\underline{QH(Z)}$

Idea of Pf (blowup as a VGIT
Teleman's idea

$W := \underline{Bl_{Z\times\{0\}}} (X \times \underline{\mathbb{P}^1})$ ↺ $\mathbb{C}^\times$-action on $\mathbb{P}^1$-factor



$W^{\mathbb{C}^\times} = X \sqcup Z \sqcup \tilde{X}$

Fourier transf

Thm $QDM_{\mathbb{C}^\times}(W) \cong QDM(W/\!/\mathbb{C}^\times)$

$\kappa : H^2_{\mathbb{C}^\times}(W) \to H^2(W/\!/\mathbb{C}^\times)$

Teleman's conj

• $QDM_{\mathbb{C}^\times}(W) \cong QDM(W /\!/ \mathbb{C}^\times) \quad H^*(W/\!/\mathbb{C}^\times)$ Kirwan map

$$QDM_{\mathbb{C}^\times}(W) \circlearrowleft$$

$$\begin{cases} \lambda : \text{equiv para} & \longleftrightarrow \; z\nabla_{q\partial_q} : \text{quantum conn w.r.t } \underline{\kappa(\lambda)} \\ S : \underline{\text{shift operator}} & \longleftrightarrow \; q \end{cases}$$

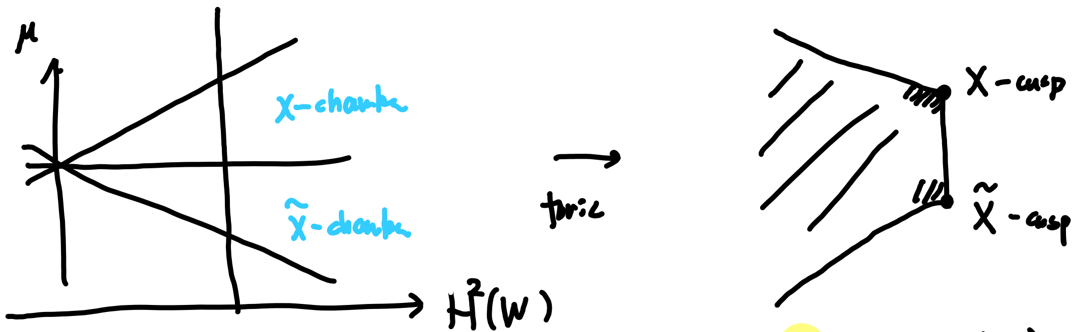
$$S \circ \lambda = (\lambda - z) \circ S$$

• $\exists$ Equiv Mori cone $NE_N^{\mathbb{C}^\times}(W) \subset H_2^{\mathbb{C}^\times}(W, \mathbb{Z}) \cong H_2(W,\mathbb{Z}) \oplus \mathbb{Z}$

$$\underset{Q^d}{H_2(W,\mathbb{Z})} \qquad \underset{S}{\mathbb{Z}}$$

s.t $QDM_{\mathbb{C}^\times}(W)$ is a module over $\mathbb{C}[\![NE_N^{\mathbb{C}^\times}(W)]\!]$

• $NE_N^{\mathbb{C}^\times}(W)$ is dual to the $\underline{\mathbb{C}^\times\text{-ample cone}}$ $C_{\mathbb{C}^\times}(W) \subset H^2_{\mathbb{C}^\times}(W, \mathbb{R})$



$\underline{QDM_{\mathbb{C}^\times}(W)}$ is a sheaf on this toric var