Decomposition of quantum cohomology under blowups*

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Abstract

Quantum cohomology is a deformation of the cohomology ring defined by counting rational curves. A close relationship between quantum cohomology and birational geometry has been expected. For example, when the quantum parameter Q approaches an "extremal ray", the spectrum of the quantum cohomology ring clusters in a certain way (predicted by the corresponding extremal contraction), inducing a decomposition of the quantum cohomology. In this talk, I will discuss such a decomposition for blowups: quantum cohomology of the blowup of X along a smooth center Z will decompose into $\mathrm{QH}(X)$ and $(\operatorname{codim} Z\text{-}1)$ copies of $\mathrm{QH}(Z)$. The proof relies on Fourier analysis and shift operators for equivariant quantum cohomology. We can describe blowups as a variation of GIT of a certain space W with $\mathbb{C}^*$ action. The equivariant quantum cohomology of W acts as a "global" mirror family connecting X and its blowup.

1 Introduction

First we explain how a decomposition of quantum cohomology naturally arises from birational geometry.

Let X be a smooth projective variety. The quantum cohomology $\mathrm{QH}(X)$ is a supercommutative algebra over the *Novikov* ring

$$\mathbb{C}[\![Q]\!] = \mathbb{C}[\![\mathrm{NE}_{\mathbb{N}}(X)]\!]$$

which is the completed group ring of the monoid $\mathrm{NE}_{\mathbb{N}}(X) \subset H_2(X,\mathbb{Z})$ generated by effective curve classes. It consists of formal sums $\sum_{d\in \mathrm{NE}_{\mathbb{N}}(X)} c_d Q^d$ with $c_d \in \mathbb{C}$.

Consider the Mori cone $\overline{\mathrm{NE}}(X) \subset H_2(X,\mathbb{R})$ spanned by $\mathrm{NE}_{\mathbb{N}}(X)$. An *extremal ray* $R = \mathbb{R}_{\geq 0} d_0 \subset \overline{\mathrm{NE}}(X)$ is a 1-dimensional face of the Mori cone generated by a class d_0 satisfying $c_1(X)\cdot d_0 > 0$. By the Contraction Theorem, we have an *extremal contraction* $f\colon X \to Y$ to a normal projective variety Y such that a curve $C \subset X$ contracts to a point if and only if the homology class $[C]$ lies in the ray R.

Associated to the extremal contraction f, we have the *f-exceptional quantum cohomology* defined by counting only f-exceptional curves[1]. The f-exceptional (small)

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[1] A similar quantum ring has been introduced by Yongbin Ruan in his study of crepant resolution conjecture.

quantum product is defined by

$$(\alpha \star_f \beta, \gamma) = \sum_{n\geq 0} \langle \alpha, \beta, \gamma \rangle_{0,3,nd_0} Q^{nd_0}$$

where $(\cdot,\cdot)$ is the Poincaré pairing and $\langle \alpha,\beta,\gamma\rangle_{0,3,d}$ is the 3-point Gromov-Witten invariant. An important thing is that the left-hand side is a *finite* sum since $c_1(X)\cdot d_0 > 0$. If we write $\mathrm{QH}_{\mathrm{exc}}(X)$ for the exceptional quantum cohomology, it is therefore defined over the polynomial ring $\mathbb{C}[q]$ with $q := Q^{d_0}$.

The exceptional quantum cohomology $\mathrm{QH}_{\mathrm{exc}}(X)$ arises as a limit of $\mathrm{QH}(X)$. Conversely we may regard $\mathrm{QH}(X)$ as a deformation[2] of $\mathrm{QH}_{\mathrm{exc}}(X)$. Therefore, we have a natural ring decomposition

$$\mathrm{QH}(X) \cong \bigoplus_{\alpha} A_\alpha$$

indexed by points α of the zero-dimensional scheme $\mathrm{Spec}(\mathrm{QH}^{\mathrm{ev}}_{\mathrm{exc}}(X)|_{q=1})$. A natural question here is:

Does each summand A_α arise from (big) quantum cohomology of a space?

There are several examples where the answer is affirmative.

Example 1.1 ([I19]). Let X be a smooth toric DM stack. Let $X \dashrightarrow X^+$ be a toric flip (or more generally a discrepant transformation arising from a toric VGIT). Then $\mathrm{QH}(X)$ contains (big) quantum cohomology of X^+ as a direct summand.

Example 1.2 ([Koto-I]). Let $V \to Y$ be a vector bundle of rank $r \geq 2$. Let $X = \mathbb{P}(V) \to Y$ be the projective bundle over Y. In this case the exceptional quantum cohomology associated with the extremal contraction $f\colon X \to Y$ is

$$\begin{aligned} \mathrm{QH}_{\mathrm{exc}}(X) &\cong H^*(Y)[p,q]/(p^r + c_1(V)p^{r-1} + \cdots + c_r(V) - q) \\ &\cong \bigoplus_{i=1}^{r} H^*(Y) \quad \text{as a ring at } q \neq 0. \end{aligned}$$

We have the corresponding decomposition ("quantum Leray-Hirsch")

$$\mathrm{QH}(X)_\tau \cong \bigoplus_{i=1}^{r} \mathrm{QH}(Y)_{\sigma_i(\tau)}$$

where the subscripts $\tau \in H^*(X)$, $\sigma_i(\tau) \in H^*(Y)$ are big quantum cohomology parameters (bulk deformation parameters): the map $\tau \mapsto (\sigma_1(\tau),\dots,\sigma_r(\tau))$ defines a formal isomorphism. [Draw a picture of the deformation.]

[2] More precisely, we take the graded completion when defining the Novikov ring (with respect to the grading $\deg Q^d = 2c_1(X)\cdot d$). Therefore, the Novikov ring $\mathbb{C}[\![Q]\!]$ can be presented as a ring of the form $\mathbb{C}[q][\![Q']\!]$ consisting of formal power series in the other Novikov variables Q' with coefficients in the polynomial ring in $q = Q^{d_0}$.

Example 1.3 ([I23])**.** This is the main theme of this talk. Let $\widetilde{X} = \mathrm{Bl}_Z X$ be the blowup of a smooth projective variety X along a smooth subvariety $Z \subset X$. Let r be the codimension of Z. The blowdown morphism $\widetilde{X} \to X$ is an extremal contraction and the associated decomposition is

$$\mathrm{QH}(\widetilde{X})_{\tilde{\tau}} \cong \mathrm{QH}(X)_\tau \oplus \bigoplus_{i=1}^{r-1} \mathrm{QH}(Z)_{\sigma_i}$$

where $\tilde{\tau} \in H^*(\widetilde{X})$, $\tau = \tau(\tilde{\tau}) \in H^*(X)$, $\sigma_i = \sigma_i(\tilde{\tau}) \in H^*(Z)$ are big quantum cohomology parameters. Again $\tilde{\tau} \mapsto (\tau(\tilde{\tau}), \sigma_1(\tilde{\tau}), \dots, \sigma_{r-1}(\tilde{\tau}))$ is a formal isomorphism. [Draw a picture of the deformation.]

For example, if $X = \mathbb{C}^n$, $Z = \mathrm{pt}$, then $\widetilde{X} = \mathcal{O}_{\mathbb{P}^{r-1}}(-1)$ and

$$\operatorname{Spec} \mathrm{QH}(\widetilde{X}) = \{(p, q) : p(p^{r-1} + q) = 0\}.$$

The configuration of the eigenvalues of the Euler vector field (or the quantum multiplication by $c_1(X)$ over the small quantum cohomology locus) is similar to $\{0\} \cup \{(-1)^{\frac{1}{r-1}}\}$ near the "exceptional quantum cohomology limit".

2 More precise statement

We give a slightly more precise statement of the result of [I23]. We discuss the big quantum cohomology and a D-module structure on it. The big quantum cohomology is a supercommutative algebra

$$(H^*(X) \otimes \mathbb{C}[\![Q, \tau]\!], \star_\tau)$$

parametrized (formally) by $\tau \in H^*(X)$ (sometimes called bulk deformation parameter). Let $\{\phi_i\}$ be a basis of $H^*(X)$ and we expand $\tau = \sum_i \tau^i \phi_i$. The ring $\mathbb{C}[\![Q, \tau]\!]$ is defined[3] to be $\mathbb{C}[\![Q]\!][\![\tau^0, \dots, \tau^s]\!]$. The quantum product $\star_\tau$ is given by

$$(\alpha \star_\tau \beta, \gamma) = \sum_{n \geq 0} \langle \alpha, \beta, \gamma, \tau, \dots, \tau \rangle_{0,n+3,d} \frac{Q^d}{n!}.$$

We introduce a new variable z and write

$$\mathrm{QDM}(X) = H^*(X) \otimes \mathbb{C}[z][\![Q, \tau]\!]$$

for the *quantum D-module*. It is equipped with the following *quantum connection* (also known as *Dubrovin connection*):

$$\begin{aligned}
\nabla_{\tau^i} &= \frac{\partial}{\partial \tau^i} + \frac{1}{z}(\phi_i \star_\tau) \\
\nabla_{z\partial_z} &= z\frac{\partial}{\partial z} - \frac{1}{z}(E \star_\tau) + \mu \\
\nabla_{\xi Q \partial_Q} &= \xi Q \frac{\partial}{\partial Q} + \frac{1}{z}(\xi \star_\tau) \quad \text{with } \xi \in H^2(X)
\end{aligned}$$

[3] We treat τ^i as supercommuting variables.

where $E = c_1(X)+\sum_i(1-\frac{1}{2}\deg\phi_i)\tau^i\phi_i$ is the Euler vector field and $\mu \in \mathrm{End}(H^*(X))$ is the grading operator defined by $\mu(\phi_i) = (\frac{1}{2}\deg\phi_i - \frac{n}{2})\phi_i$ (with $n = \dim_{\mathbb{C}} X$). They are supercommuting operators

$$\nabla_{\tau^i}, \nabla_{z\partial_z}, \nabla_{\xi Q\partial_Q} \colon \mathrm{QDM}(X) \to z^{-1}\,\mathrm{QDM}(X).$$

The quantum D-module is also equipped with the pairing

$$P(f,g) = \int_X f(-z) \cup g(z), \qquad f, g \in \mathrm{QDM}(X)$$

given by the Poincaré pairing. It is flat with respect to the connection.

Theorem 2.1 ([I23])**.** *There exists a formal invertible map* $(\tau, \sigma_1, \ldots, \sigma_{r-1}) \colon H^*(\widetilde{X}) \to H^*(X) \oplus H^*(Z)^{\oplus(r-1)}$ *and an isomorphism*

$$\mathrm{QDM}(\widetilde{X}) \cong \tau^*\,\mathrm{QDM}(X) \oplus \bigoplus_{i=1}^{r-1} \sigma_i^*\,\mathrm{QDM}(Z)$$

preserving the connection and the pairing.

Remark 2.2. (1) The quantum D-modules $\mathrm{QDM}(\widetilde{X})$, $\mathrm{QDM}(X)$, $\mathrm{QDM}(Z)$ are originally defined over the different Novikov rings $\mathbb{C}[\![\mathrm{NE}_{\mathbb{N}}(\widetilde{X})]\!]$, $\mathbb{C}[\![\mathrm{NE}_{\mathbb{N}}(X)]\!]$, $\mathbb{C}[\![\mathrm{NE}_{\mathbb{N}}(Z)]\!]$. We need an extension of the Novikov rings to a common one. The common ring arises from the *equivariant second homology group* of a space W with $\mathbb{C}^\times$-action (see below).

(2) We need to invert the variable q associated with the extremal ray. It corresponds to the class of a line in the exceptional divisor $\mathbb{P}(\mathcal{N}_{Z/X})$ that contracts to a point in Z under $\widetilde{X} \to X$. The isomorphism is defined over the ring of the form

$$\mathbb{C}[z](\!(q^{-1/s})\!)[\![\mathcal{Q}, \tilde{\tau}]\!]$$

where $s = r-1$ when r is even and $s = 2(r-1)$ when r is odd.

The above result has the following corollary:

Corollary 2.3. *The derivative of* $\tilde{\tau} \mapsto (\tau, \sigma_1, \ldots, \sigma_{r-1})$ *defines a ring isomorphism*

$$\mathrm{QH}(\widetilde{X})_{\tilde{\tau}} \cong \mathrm{QH}^*(X)_\tau \oplus \bigoplus_{i=1}^{r-1} \mathrm{QH}(Z)_{\sigma_i}$$

Remark 2.4. Katzarkov-Kontsevich-Pantev-Yu have announced application of this result to birational geometry, e.g. irrationality of generic cubic 4-fold. They also give a reconstruction algorithm of $\mathrm{QH}(\widetilde{X})$ from $\mathrm{QH}(X)$ and $\mathrm{QH}(Z)$ together with certain classical topological data (Chern classes of normal bundles $\mathcal{N}_{Z/X}$ etc).

3 Idea of Proof

The basic idea is to describe blowups as a variation of GIT quotient. We also use Teleman's conjecture for quantum cohomology of reduction, adapted to quantum D-modules. We prove Teleman's conjecture in a special case, and the blowup result follows from it.

Let $W = \mathrm{Bl}_{Z\times\{0\}}(X \times \mathbb{P}^1)$ be the blowup of $X \times \mathbb{P}^1$ along $Z \times \{0\}$. Consider the $\mathbb{C}^\times$ action on W induced by the $\mathbb{C}^\times$-action on $\mathbb{P}^1$. We have two smooth GIT quotients:

$$W/\!/\mathbb{C}^\times = X \text{ or } \widetilde{X}$$

depending on the stability conditions. [Draw a picture of W]. The $\mathbb{C}^\times$-fixed locus of W is

$$W^{\mathbb{C}^\times} = X \sqcup \widetilde{X} \sqcup Z.$$

Roughly speaking, a quantum D-module version of Teleman's conjecture says that we have an isomorphism:

$$\mathrm{QDM}_{\mathbb{C}^\times}(W) \cong \mathrm{QDM}(W/\!/\mathbb{C}^\times). \tag{3.1}$$

This isomorphism should intertwine

$$\begin{cases} \lambda\text{: equivariant parameter} \\ S\text{: shift operator} \end{cases} \quad \text{with} \quad \begin{cases} z\nabla_{q\partial_q}\text{: quantum connection in the direction } \kappa(\lambda) \\ q \end{cases}$$

where $\kappa(\lambda) \in H^2(W/\!/\mathbb{C}^\times)$ is the image of λ under the Kirwan map. The isomorphism (3.1) should be viewed as a Fourier transformation between *difference* modules and D-modules.

Remark 3.1. The $\mathbb{C}^\times$-equivariant quantum D-module is equipped with the action of a shift operator $S\colon \mathrm{QDM}_{\mathbb{C}^\times}(W) \to \mathrm{QDM}_{\mathbb{C}^\times}(W)$ of equivariant parameters, and we have the commutation relation $[\lambda, S] = zS$, i.e. $S \circ \lambda = (\lambda - z) \circ S$.

The isomorphism (3.1) yields an obvious contradiction $\mathrm{QDM}(X) \cong \mathrm{QDM}_{\mathbb{C}^\times}(W) \cong \mathrm{QDM}(\widetilde{X})$, so it should be interpreted with care.

We explain how the equivariant quantum D-module $\mathrm{QDM}_{\mathbb{C}^\times}(W)$, after Fourier transformation, plays a role of a 'global mirror'.

The equivariant second homology $H_2^{\mathbb{C}^\times}(W,\mathbb{Z}) \cong H_2(W,\mathbb{Z}) \oplus \mathbb{Z}$ acts on $\mathrm{QDM}_{\mathbb{C}^\times}(W)[Q^{-1}]$ with Novikov variables inverted: the $H_2(W,\mathbb{Z})$ part acts as ordinary Novikov variables and the $\mathbb{Z} = H_2^{\mathbb{C}^\times}(\mathrm{pt})$ part acts as shift operators. We can introduce an *equivariant Mori monoid*

$$\mathrm{NE}_{\mathbb{N}}^{\mathbb{C}^\times}(W) \subset H_2^{\mathbb{C}^\times}(W,\mathbb{Z})$$

that preserves the lattice $\mathrm{QDM}_{\mathbb{C}^\times}(W) \subset \mathrm{QDM}_{\mathbb{C}^\times}(W)[Q^{-1}]$. The equivariant Mori monoid is an extension of $\mathrm{NE}_{\mathbb{N}}(W) \subset H_2(W,\mathbb{Z})$.

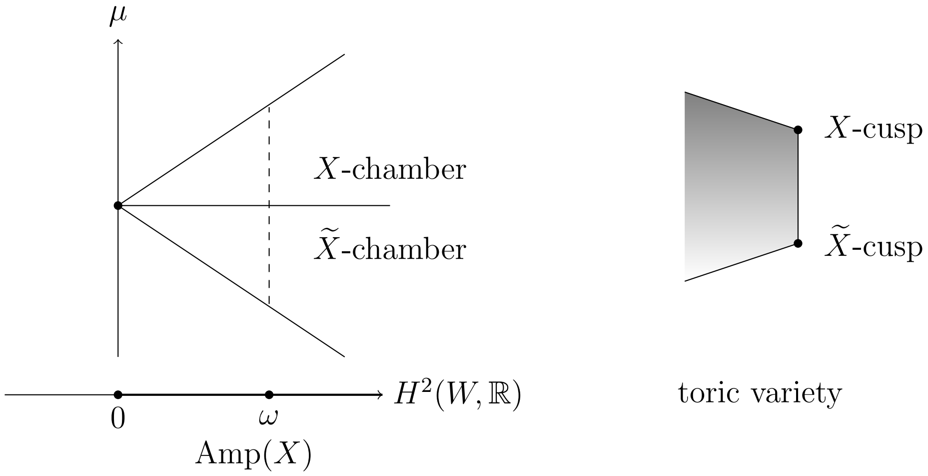
The dual cone of the equivariant Mori cone $\mathrm{NE}^{\mathbb{C}^\times}(W)$ is the $\mathbb{C}^\times$-ample cone[4] $C_{\mathbb{C}^\times}(W) \subset H^2_{\mathbb{C}^\times}(W,\mathbb{R})$ consisting of equivariant ample classes $[L]$ such that the

[4] Introduced by Dolgachev-Hu and Thaddeus

stable locus $W_{\mathrm{st}}(L)$ is nonempty. In the case at hand, there are two GIT chambers of this cone:

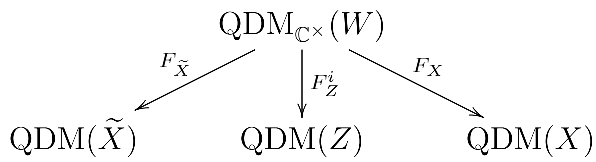
$$\overline{C_{\mathbb{C}^\times}(W)} = \overline{C}_X \cup \overline{C}_{\widetilde{X}}$$

corresponding to the GIT quotients X and $\widetilde{X}$. This gives a “fan” as in the following picture.



By the Fourier transformation, we can regard $\mathrm{QDM}_{\mathbb{C}^\times}(W)$ as a sheaf over the toric variety associated with this fan. The quantum D-modules $\mathrm{QDM}(W/\!/\mathbb{C}^\times)$ of the GIT quotients arise as certain completions of $\mathrm{QDM}_{\mathbb{C}^\times}(W)$ at the corresponding cusps.

We construct Fourier transformations:



and show that

- $F_{\widetilde{X}}$ is an isomorphism after completion at the $\widetilde{X}$-cups;
- $F_X \oplus F_Z^1 \oplus \cdots \oplus F_Z^{r-1}$ extends to the completion and is also an isomorphism.