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Fourier analysis of equivariant quantum cohomology

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available at https://www.math.kyoto-u.ac.jp/~iritani/talk_Colorado2025.pdf

References:

[I-Koto] *Quantum cohomology of projective bundles*, arXiv:2307.03696

[I] *Quantum cohomology of blowups*, arXiv:2307.13555

[I] *Fourier analysis of equivariant quantum cohomology*, arXiv:2501.18849

[I-Sanda] in preparation

Talk Plan:

1. Shift operators on equivariant quantum cohomology
2. A D-module version of Teleman's conjecture
3. Reduction conjecture (with Sanda)
4. Decomposition of quantum cohomology
   — projective bundles
   — blowups

Quantum Cohomology

X: smooth (semi-)projective variety

$$\mathrm{QH}(X) = (H^*(X) \otimes \mathbb{C}[\![Q]\!], \star)$$

- $\mathbb{C}[\![Q]\!] := \mathbb{C}[\![\mathrm{NE}_{\mathbb{N}}(X)]\!]$: Novikov ring
- $\star$ is defined by counting rational curves in X, s.t. $\lim_{Q \to 0} \star = \cup$.
- equipped with a D-module structure

$$z\nabla_{Q\partial_Q} = zQ\partial_Q + p\star$$

with $p \in H^2(X)$ dual to Q (assuming $\dim H_2 = 1$).

Equivariant Quantum Cohomology

X: smooth (semi-)projective T-variety

$$\mathrm{QH}_T(X) = \left(H_T^*(X)\hat{\otimes}\mathbb{C}[\![Q]\!], \star\right)$$

- equipped with a D-module structure

$$z\nabla_{Q\partial_Q} = zQ\partial_Q + \hat{p}\star$$

- and a difference module structure (shift operator)

$$\mathbb{S}^k\colon\ \mathrm{QH}_T(X) \to \mathrm{QH}_T(X)[Q^{-1}]$$

  for $k \in \mathrm{Hom}(\mathbb{C}^\times, T)$, that shifts the equivariant parameter $\lambda \in H_T^2(\mathrm{pt})$ by $-z$ (assuming $\mathrm{rank}\, T = 1$)

$$\mathbb{S} \circ \lambda = (\lambda - z) \circ \mathbb{S} \iff [\lambda, \mathbb{S}] = z\mathbb{S}$$

Four operators on $\mathrm{QH}_T(X)$

$z\nabla_{Q\partial_Q}$	Q
λ	$\mathbb{S}$

satisfying the "canonical commutation relations"

$$[z\nabla_{Q\partial_Q}, Q] = zQ$$
$$[\lambda, \mathbb{S}] = z\mathbb{S}$$

All other commutators are zero.

Four operators on $\mathrm{QH}_T(X)$

$z\nabla_{Q\partial_Q}$	Q
λ	$\mathbb{S}$

(usually Q and λ are treated as variables)

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Four operators on $\mathrm{QH}_T(X)$

$z\nabla_{Q\partial_Q}$	Q
λ	$\mathbb{S}$

(could use Q and $\mathbb{S}$ as variables)

satisfying the "canonical commutation relations"

$$[z\nabla_{Q\partial_Q}, Q] = zQ$$
$$[\lambda, \mathbb{S}] = z\mathbb{S}$$

All other commutators are zero.

Teleman's conjecture (D-module version)

$$\mathrm{QH}_T(X) \quad \underset{\text{Fourier dual}}{\Longleftrightarrow} \quad \mathrm{QH}(X /\!\!/_t T)$$

equiv param λ	quantum conn $z\nabla_{q\partial_q}$
shift op $\mathbb{S}$	Kähler param $q = e^{-t}$
$[\lambda, \mathbb{S}] = z\mathbb{S}$	$[z\nabla_{q\partial_q}, q] = zq$

- $\mathrm{QH}_T(X)$ connects different GIT quotients continuously (cf. Woodward's quantum Kirwan map)
- modulo bulk deformation and completion
- [Pomerleano-Teleman]: algebra version, Fano case

What are shift operators? [Seidel, Okounkov-Pandharipande]

Symplectic Floer theory: we interpret $\mathrm{QH}(X)$ as a (semi-infinite) Morse homology of $\widetilde{LX}$.

$$\mathrm{QH}^*(X) = \text{“}H^{\frac{\infty}{2}}(\widetilde{LX})\text{”}$$

A cocharacter $k \in \mathrm{Hom}(\mathbb{C}^\times, T)$ gives rise to a map

$$LX \to LX, \quad \gamma(e^{\mathtt{i}\theta}) \mapsto k(e^{\mathtt{i}\theta}) \cdot \gamma(e^{\mathtt{i}\theta}) \qquad (\star)$$

and thus a map $\mathbb{S}^k \colon \mathrm{QH}(X) \to \mathrm{QH}(X)$.

Point: the map ($\star$) is $(T \times S^1_{\text{loop}})$-equivariant w.r.t. the group auto $(e^\lambda, e^z) \mapsto (e^{\lambda+kz}, e^z)$ of $T \times S^1_{\text{loop}}$.

Definition: by counting holomorphic sections of an X-fibration $E_k \to \mathbb{P}^1$, $z = S^1_{\text{loop}}$-equivariant param.

Why Fourier transformations?

Classical Fourier transformations for symplectic volumes [Duistermaat-Heckman, Jeffrey-Kirwan]

$$\underbrace{\int_X^T e^{\omega-\lambda\mu}}_{\text{equivariant volume}} \overset{\text{Fubini}}{=} \int_{t\in\mathbb{R}} e^{-\lambda t}dt \int_{\mu^{-1}(t)} \frac{\omega^n/n!}{d\mu}$$

$$= \int_{t\in\mathbb{R}} e^{-\lambda t}dt \underbrace{\int_{X/\!\!/_t T} e^{\omega_{\text{red}}}}_{\text{volume of a reduction}}$$

- $\mu\colon X\to\mathbb{R}$ is a moment map
- GIT quotient $X/\!\!/_t T=\mu^{-1}(t)/T$ with a reduced symplectic form ω_{red}
- stability param t and equivariant param λ are Fourier dual.

Why Fourier transformations?

"Quantum volume" (Givental's path integral)

$$\Pi_X = \int_{\mathrm{Hol}(D^2,X)}^{S^1_{\text{loop}}} e^{(\Omega - z\mathcal{A})/z} \qquad S^1_{\text{loop}}\text{-equivariant volume of } \mathrm{Hol}(D^2, X)$$

$$\underset{\text{"localization"}}{=} \int_X J_X(-z) \cup z^{n-\frac{\deg}{2}} z^{c_1} \widehat{\Gamma}_X$$

- $\mathcal{A}(g) = \int_{D^2} g^* \omega$: action functional on $\mathrm{Hol}(D^2, X)$
- Ω is a symplectic form on the (positive) loop space $\mathrm{Hol}(D^2, X) \subset \widetilde{LX}$
- J_X: the J-function (a solution of the quantum connection)
- quantum cohomology central charge of $\mathcal{O}_X$
- [Cassia-Longhi-Zabzine]: quantum volume of a GLSM

Thus we (naively) hope:

$$\underbrace{\Pi_X^{\mathsf{equiv}}}_{\substack{T\text{-equivariant} \\ \text{quantum volume}}} = \int_{t\in\mathbb{R}} e^{-\lambda t/z} dt \underbrace{\Pi_{X/\!\!/_t T}}_{\substack{\text{quantum volume} \\ \text{of a reduction}}}$$

and its inverse transformation.

— can be checked in many examples (modulo bulk deformation and asymptotic expansion)

— we propose a more robust conjecture in terms of the J-function (or the Givental cone).

Givental cone and the shift operator $\mathcal{S}$

The Givental cone $\mathcal{L}_X^{\text{equiv}} \subset \mathcal{H}_{\text{rat}}^X$ is the union of the images of a standard fundamental solution M_τ

$$M_\tau \colon \mathrm{QH}_{T\times S^1_{\text{loop}}}(X)_\tau \longrightarrow \mathcal{H}_{\text{rat}}^X := H^*_{T\times S^1_{\text{loop}}}(X)_{\text{loc}}$$

(rational) Givental space

such that $M_\tau(1) = J_X(\tau, z)$. M_τ intertwines $\mathbb{S}^k$ with a purely topological operator $\mathcal{S}^k$ on $\mathcal{H}_{\text{rat}}^X$

$$\mathcal{S}^k \mathbf{g}\Big|_F = Q^{\hat{p}_F\cdot k} \frac{\prod_{c\le 0}(\rho_{F,\alpha,j}+\alpha+cz)}{\prod_{c\le -\alpha\cdot k}(\rho_{F,\alpha,j}+\alpha+cz)} e^{-zk\partial_\lambda}\mathbf{g}|_F$$

F: T-fixed component, $c_T(\mathcal{N}_F) = \prod_{\alpha,j}(1+\rho_{F,\alpha,j}+\alpha)$

Reduction Conjecture (with Fumihiko Sanda)

Let $Y = X/\!\!/_t T$ be a smooth GIT quotient. The discrete Fourier transformation

$$\mathcal{F}\colon \mathcal{H}^X_{\mathrm{rat}} \dashrightarrow \mathcal{H}^Y, \quad \mathbf{g} \mapsto \sum_{k \in \mathrm{Hom}(\mathbb{C}^\times, T)} \kappa(\mathcal{S}^{-k}\mathbf{g}) q^k$$

sends $\mathcal{L}_X^{\mathrm{equiv}}$ to $\mathcal{L}_Y$, where $\kappa\colon \mathcal{H}^X_{\mathrm{rat}} \dashrightarrow \mathcal{H}^Y$ is the Kirwan map.

— $I_Y = \mathcal{F}(J_X)$ plays a role of the I-function of Y.
— analogous to $g(\lambda) \mapsto \sum_{k\in\mathbb{Z}} g(kz) q^k$.
— intertwining property (for $\mathcal{S}$ and λ)

Example 1

$X = \mathbb{C}^n$ with diagonal T-action. $J_X = 1$ and $\mathcal{S} = \lambda^n e^{-z\partial_\lambda}$. We have:

$$\mathcal{S}^{-k} 1 = \frac{1}{\prod_{c=1}^{k} (\lambda + kz)^n}$$

Using $\kappa(\lambda) = p$ (the hyperplane class of $\mathbb{P}^{n-1}$),

$$\mathcal{F}(J_X) = \sum_k \kappa(\mathcal{S}^{-k} 1) q^k$$

$$= \sum_k \frac{q^k}{\prod_{c=1}^{k} (p + kz)^n} = I_{\mathbb{P}^{n-1}}$$

Example 2

$X = \mathbb{P}^1$ with a standard T-action. The Fourier transform of the equivariant J-function

$$J_X = \sum_d \frac{Q^d}{\prod_{c=1}^d ([0] + cz)([\infty] + cz)}$$

gives the exponentiated mirror LG model

$$\mathcal{F}(J_X) = \sum_{k \in \mathbb{Z}} \kappa(\mathcal{S}^{-k} J_X) x^k$$

$$= e^{\left(x + \frac{Q}{x}\right)/z} \in \mathcal{L}_{\mathrm{pt}}$$

$W = x + \frac{Q}{x}$: mirror LG model for $\mathbb{P}^1$.

Decomposition of Quantum Cohomology

Fourier analysis of $\mathrm{QH}_T(\mathbb{C}^n)$ leads to the decomposition

$$\mathrm{QH}(\mathbb{P}^{n-1}) \cong \mathrm{QH}(\mathrm{pt})^{\oplus n}.$$

More generally: for a rank n vector bundle $V \to B$ [I-Koto]

$$\mathrm{QH}(\mathbb{P}(V))_{\tau} \cong \bigoplus_{j=1}^{n} \mathrm{QH}(B)_{\sigma_j(\tau)}.$$

$\tau \in H^*(\mathbb{P}(V))$, $\sigma_j(\tau) \in H^*(B)$: bulk parameters

I will explain the blowup case

X: smooth projective variety,

$Z \subset X$: smooth subvariety of codimension r,

$\widetilde{X} = \mathrm{Bl}_Z X$.

Theorem [I]

$$\mathrm{QH}(\widetilde{X})_{\tilde{\tau}} \cong \mathrm{QH}(X)_{\tau(\tilde{\tau})} \oplus \bigoplus_{i=1}^{r-1} \mathrm{QH}(Z)_{\sigma_i(\tilde{\tau})}$$

Remark [Hinault-Yu-Zhang-Zhang] The decomposition is uniquely reconstructible from a topological initial condition $\rightsquigarrow$ reconstruction of the GW invariants of $\widetilde{X}$ from those of X and Z.

Problems/Applications

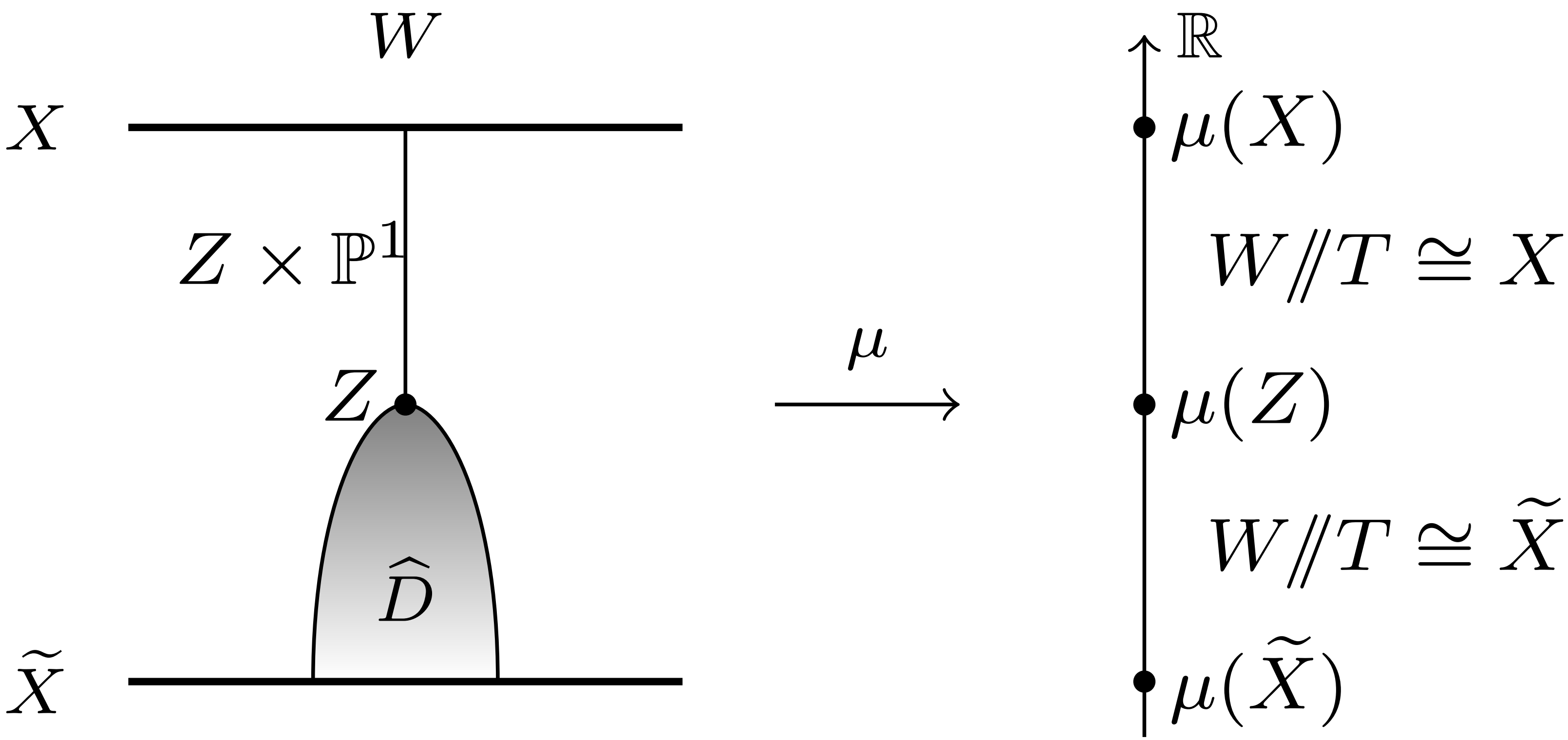
- Relative $\widehat{\Gamma}$-conjecture: relate this result to an SOD of derived categories such as:

$$D^b(\widetilde{X}) \cong \left\langle D^b(X), D^b(Z), \ldots, D^b(Z) \right\rangle$$

  A partial affirmative answer by [Shen-Shoemaker]
- announced by [Katzarkov-Kontsevich-Pantev-Yu] Application to rationality question: e.g. irrationality of generic cubic fourfolds

Strategy of Proof

Consider $W = \mathrm{Bl}_{Z\times\{0\}}(X \times \mathbb{P}^1)$ with T-action on the $\mathbb{P}^1$-factor. This has two GIT quotients X, $\widetilde{X}$.

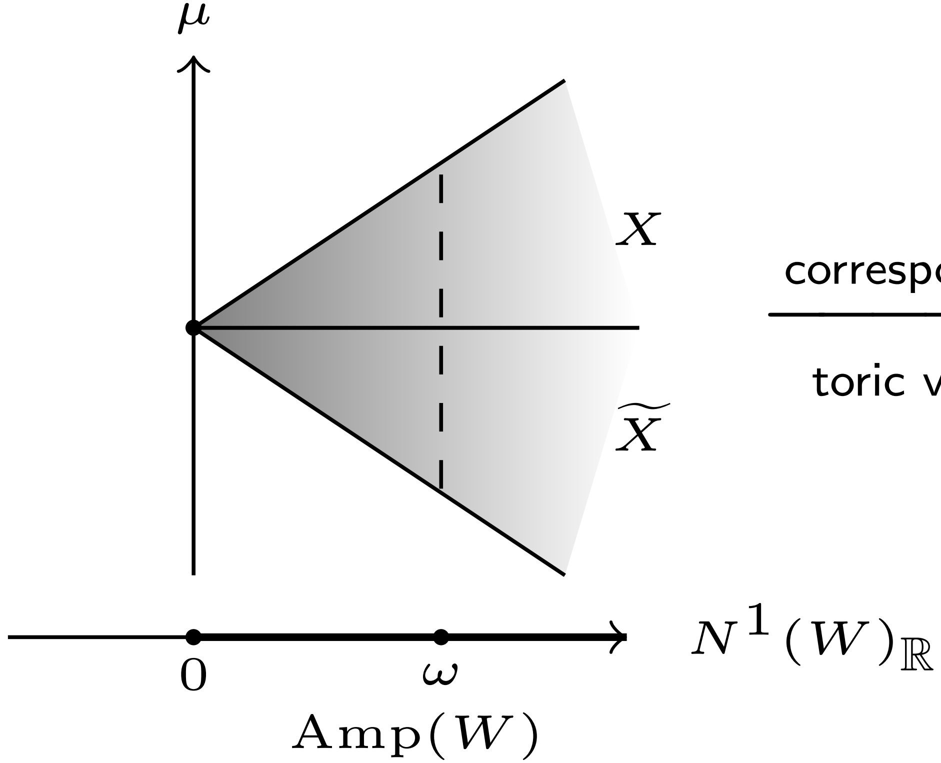


“Equivairant” Kähler moduli space

$\omega \in N^1_T(W)$ gives a stability condition
$\rightsquigarrow$ GIT fan in $N^1_T(W)$ [Dolgachev-Hu, Thaddeus]

GIT fan

equivariant Kähler moduli
$= (Q, \mathbb{S})$-space



$\mathrm{QH}_T(W)$ gives
a sheaf over $\mathfrak{M}$

Wall-crossing via Fourier transformations:

$$
\begin{array}{ccccc}
 & & \mathrm{QH}_T(W) & & \\
 & \swarrow^{\mathcal{F}_{\widetilde{X}}} & \Big\downarrow{\scriptstyle \mathscr{F}_Z} & \searrow^{\mathcal{F}_X} & \\
\mathrm{QH}(\widetilde{X}) & & & & \mathrm{QH}(X) \\
 & & \bigoplus_{j=1}^{r-1} \mathrm{QH}(Z) & &
\end{array}
$$

where $\mathcal{F}_X$, $\mathcal{F}_{\widetilde{X}}$ are discrete FTs, $\mathscr{F}_Z$ is a continuous FT.

- $\mathcal{F}_{\widetilde{X}}$ is an isomorphism (after completion of $\mathrm{QH}_T(W)$)
- $\mathcal{F}_X \oplus \mathscr{F}_Z$ is also an isomorphism

Continuous FT for a fixed component

Proposition (follows from [Coates-Givental]). Let $F \subset W$ be a T-fixed component and set

$$G_F := \prod_{\substack{\varrho \\ \text{Chern roots of } \mathcal{N}_F}} \frac{1}{\sqrt{-2\pi z}} (-z)^{-\varrho/z} \Gamma(-\varrho/z)$$

The formal stationary phase asymptotics $\mathscr{I}(z)$

$$\underbrace{\int J_W|_F \cdot G_F \, e^{\lambda \log q/z} d\lambda}_{\text{integrals of Mellin-Barnes type}} \underset{z \to 0}{\sim} \sqrt{2\pi z}\, e^{u/z} \mathscr{I}(z)$$

lies in the Givental cone of F.

It remains to show the
reduction conjecture for X and $\widetilde{X}$

Note: Both X and $\widetilde{X}$ arise as a GIT quotient and also as a fixed component of W.

The reduction conjecture for X and $\widetilde{X}$ follows from the fact that their discrete and continuous FTs coincide. This coincidence is established by applying the residue theorem to the Mellin-Barnes integrals (or the continuous FTs).