The S^1 Equivariant Cohomology of the Universal Covering of Free Loop Spaces

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Abstract

We compute S^1 equivariant cohomology $H^*_{S^1}(\widetilde{L_0M},\mathbb{R})$ of the universal covering $\widetilde{L_0M}$ of a contracting free loop space for a connected manifold M with respect to the natural S^1 action rotating loops. In particular, its localization $H^*_{S^1}(\widetilde{L_0M})\otimes_{\mathbb{R}[\hbar]}\mathbb{R}[\hbar,\hbar^{-1}]$ turns out to be freely generated by a basis of two dimensional cohomology of $\widetilde{M}$, where $\hbar$ is a generator of $H^*_{S^1}(\mathrm{pt})$. We use Chen's iterated integration for free loop spaces and its equivariant extension by Getzler, Jones and Petrack.

1 Introduction

Our motivation to compute the (equivariant) cohomology of $\widetilde{L_0M}$ lies in Floer theory for symplectic manifold M. Floer theory is known as the semi-infinite cohomology theory for a space $\widetilde{L_0M}$ and its product structure is identified with that of quantum cohomology of M. It conjecturally deals with differential forms of infinite degree. Therefore, we can naively expect that finite dimensional cohomology of $\widetilde{L_0M}$ would act on Floer cohomology. On the other hand, Givental proposed the equivariant version of Floer theory, and using a model for $\widetilde{L_0M}$, showed that it computes quantum cohomology of symplectic toric manifolds [Giv]. In this case, we can also expect that finite dimensional equivaiant cohomology would act on equivariant Floer cohomology. In fact, Givental's theory naturally has an action of two dimensional equivariant classes. We prove that the equivariant cohomology ring of $\widetilde{L_0M}$ is, after the localization, freely generated by two dimensional equivariant classes which Givental considered in his model. Therefore, it is posible that it acts on the equivariant Floer cohomology by a quantum multiplication. On the other hand, the ordinary cohomology of $\widetilde{L_0M}$ does not necessarily act on the Floer cohomology by a quantum multiplication. Because, non-equivariant two dimensional classes are sometimes nilpotent. In fact, we show in section 4 that a two dimensional class of $\widetilde{L\mathbb{P}^n}$ is nilpotent.

We introduce some notation in order to state the main theorem. In this paper, the free loop space LM denotes the set of all C^∞ maps from S^1 to M. It has a structure of a Frechet manifold and we can think differential forms on it. De Rham theorem holds in this case, therefore one can use de Rham complex to compute the singular cohomology of LM, see e.g. [Bry]. We assume that M is a compact connected manifold whose second homotopy group $\pi_2(M)$ is torsion free. Let L_0M denote the set of all contracting loops in M. Then its universal cover $\widetilde{L_0M}$ is equal to $\widetilde{L\widetilde{M}}$, so consists of all pairs $(\gamma,[g])$ of C^∞ maps $\gamma\colon S^1\to\widetilde{M}$ and homotopy types $[g]$ of disks $g\colon D^2\to\widetilde{M}$ contracting the loop γ (i.e. $g|_{\partial D^2}=\gamma$). $\widetilde{L_0M}$ has the natural S^1 action rotating loops, i.e.

$$(\gamma(\cdot),[g])\mapsto(\gamma(\cdot+t),[g]),\qquad \text{for } t\in S^1.$$

Let $H^*(\widetilde{L_0M},\mathbb{R})$ and $H^*_{S^1}(\widetilde{L_0M},\mathbb{R})$ denote the ordinary and equivariant cohomology rings of $\widetilde{L_0M}$. We set $\hbar$ to be a generator of $H^*_{S^1}(\mathrm{pt})$. Let ev_t denote the evaluation map at $t\in S^1$, i.e. $\mathrm{ev}_t\colon\widetilde{L_0M}\to\widetilde{M}$, $(\gamma,[g])\mapsto\gamma(t)$. Let $(\bigwedge V,\delta_0)$ be the Sullivan's minimal model for $\widetilde{M}$. V is a graded vector space over $\mathbb{R}$ and is decomposed as

$$V=H^2(\widetilde{M},\mathbb{R})\oplus V',\qquad V' \text{ is a higher degree part.}$$

We take a basis $\{p_1, \ldots, p_r\}$ of $H^2(\widetilde{M}, \mathbb{Z})$, then we have $\bigwedge V = \bigwedge V' \otimes \mathbb{R}[p_1, \ldots, p_r]$. Introduce variables $t_1, \ldots, t_r$ corresponding to $p_1, \ldots, p_r$. By the isomorphism $\bigwedge V \cong \bigwedge V' \otimes \mathbb{R}[t_1, \ldots, t_r]$, $p_i \mapsto -t_i$, δ_0 on $\bigwedge V$ induces a differential δ' on $\bigwedge V' \otimes \mathbb{R}[t_1, \ldots, t_r]$. Let $\Omega_{\bigwedge V'}$ denote the differential ring of $\bigwedge V'$, and $d\colon \Omega_{\bigwedge V'} \to \Omega_{\bigwedge V'}$ denote the universal derivation. δ' is uniquely extended to the $\mathbb{R}[t_1, \ldots, t_r]$-linear differential of $\Omega_{\bigwedge V'}[t_1, \ldots, t_r]$ satisfying $d\delta' + \delta' d = 0$. We set

$$Z^i = \ker(d\colon \Omega^i_{\bigwedge V'}[t_1, \ldots, t_r] \to \Omega^{i+1}_{\bigwedge V'}[t_1, \ldots, t_r]).$$

Let $\hat{p}_i$ be a differential form on $\widetilde{M}$ representing the class p_i. Then we have the following equivariant differential forms on $\widetilde{L_0 M}$.

$$\int_{S^1} \mathrm{ev}_t^*(\hat{p}_i) dt + \hbar \int_{D^2} g^*(\hat{p}_i) \tag{1}$$

Our main theorem is stated as follows.

Theorem 1.1 *The ordinary and equivariant cohomology rings of $\widetilde{L_0 M}$ are of the forms,*

$$\begin{aligned} H^*(\widetilde{L_0 M}, \mathbb{R}) &\cong H^*(\Omega_{\bigwedge V'}[t_1, \ldots, t_r], \delta'), \\ H^*_{S^1}(\widetilde{L_0 M}, \mathbb{R}) &\cong \mathbb{R}[t_1, \ldots, t_r, \hbar] \oplus \bigoplus_{i>0} H^*(Z^i, \delta'). \end{aligned}$$

Here, the variables t_i on the right hand side of the latter isomorphism correspond to the classes in $H_{S^1}(\widetilde{L_0 M}, \mathbb{R})$ represented by the forms in the equation (1). ∎

We have the following corollary because $\hbar$ acts on $H^*_{S^1}(\widetilde{L_0 M})$ as $\hbar(H^*(Z^i)) \subset H^*(Z^{i-1})$ $(i > 1)$, and $\hbar(H^*(Z^1)) \subset \mathbb{R}[t_1, \ldots, t_r, \hbar]$.

Corollary 1.2 *The localization of $H^*_{S^1}(\widetilde{L_0 M})$ with respect to $\hbar$ becomes*

$$H^*_{S^1}(\widetilde{L_0 M}, \mathbb{R}) \otimes_{\mathbb{R}[\hbar]} \mathbb{R}[\hbar, \hbar^{-1}] \cong \mathbb{R}[t_1, \ldots, t_r, \hbar, \hbar^{-1}].$$

∎

Let us explain the main idea of the proof of Theorem 1.1. First, we take a principal T^r bundle $\pi : U \to \widetilde{M}$ such that $\pi_1(U) = \pi_2(U) = 0$. Then, one finds a homotopy equivalence

$$\widetilde{L_0 M} \sim LU/T^r.$$

Therefore, there are isomorphisms

$$H^*(\widetilde{L_0 M}) \cong H^*_{T^r}(LU), \qquad H^*_{S^1}(\widetilde{L_0 M}) \cong H^*_{S^1 \times T^r}(LU).$$

Both right hand sides are computed by the method of [GJP].

The above method — take a 2-connected T^r bundle U and think LU/T^r — enables us to generalize our theorem to the case of orbifold loop spaces. Orbifold loop space is the set of morphisms from S^1 to orbifolds in the sense of orbifolds which is introduced by Chen and Ruan [CR2, ChW]. When an orbifold M can be obtained by the quotient of a $2-$connected manifold U by a T^r action, we prove that our method also computes the (equivariant) cohomology of $\widetilde{LM}_{\mathrm{orb}}$, where $\widetilde{LM}_{\mathrm{orb}}$ denotes the universal covering of the orbifold loop space. For example, we can apply this method when M is a toric orbifold. We obtain the following theorem.

Theorem 1.3 *Let U be a 2-connected manifold with T^r action. We assume that each point in U has a finite stabilizer. Let $M = U/T^r$ be a quotient orbifold. Then the universal cover of the orbifold loop space $\widetilde{LM}_{\mathrm{orb}}$ is homeomorphic to LU/L_0T^r, where L_0T^r denotes the set of all contracting loops in T^r. Moreover the same conclusion in Theorem 1.1 holds for the (equivariant) cohomology of the space $\widetilde{LM}_{\mathrm{orb}}$.* ∎

As is well known, S^1 equivariant cohomology for a compact finite dimensional manifold satisfies the following localization property: An equivariant cohomology ring becomes isomorphic to the sum of cohomology rings of S^1 fixed components after the localization with respect to a generator of $H^*_{S^1}(\mathrm{pt})$, see e.g. [AB, Aud]. This property is very useful when computing the cohomology of symplectic manifolds with hamiltonian S^1 action [Kir]. In an infinite dimensional case, this property does not hold in general. Jones and Petrack proposed a different version of equivariant cohomology theory which satisfies the above localization property also in an infinite dimensional setting [JP1, JP2]. This cohomology theory only gives the information of fixed components. Therefore, if we apply it to the space $\widetilde{L_0M}$, it only gives the infinite copy of the ring $H^*(\widetilde{M}) \otimes \mathbb{R}[\hbar, \hbar^{-1}]]$, so it is not very interesting. In this paper, we use the original S^1 equivariant cohomology theory for infinite dimensional spaces. What is surprising about our theorem is that the S^1 equivariant cohomology of $\widetilde{L_0M}$ becomes, after the localization, a rather trivial algebra generated only by a basis of the second cohomology group of $\widetilde{M}$.

The paper is organized as follows. In section 2, we explain Chen's iterated integrals [ChK] and its equivariant extension by Getzler, Jones and Petrack [GJP]. Chen's iterated integration theory replaces de Rham complex of LM with Hochschild complex of de Rham complex $\Omega(M)$. Getzler, Jones and Petrack used a variant of the Cartan model of the equivariant cohomology of LM and replaced it with the cyclic complex of $\Omega(M)$. In section 3, we prove the main theorem. We use some techniques in the cyclic homology theory and reduce the complex to more computable one. In section 4, we explicitly determine the ordinary and equivariant cohomology of $\widetilde{L_0M}$ when a manifold M is simply connected and has a cohomology ring of the special type. For example, we can compute the (equivariant) cohomology of $\widetilde{L\mathbb{P}^n}$. In section 5, we remark that our method can be applied to study the topology of orbifold loop space. We first explain their definitions and basic facts briefly and then prove Theorem 1.3.

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2 Iterated Integration Theory

In this section, we explain iterated integration theory following [GJP]. Let M be a simply connected manifold, and $\Omega(M)$ be its de Rham complex. Define **cyclic bar complex** of $\Omega(M)$ as

$$\mathbb{C}(\Omega(M)) = \bigoplus_{p=0}^{\infty} \Omega(M) \otimes \Omega(M)^{\otimes p}.$$

All tensor products are over $\mathbb{R}$. We write an element of $\mathbb{C}(\Omega(M))$ in the form $(\omega_0, \ldots, \omega_p)$, $\omega_i \in \Omega(M)$. Degree of elements in $\mathbb{C}(\Omega(M))$ is defined to be

$$\deg(\omega_0, \ldots, \omega_p) = |\omega_0| + |\omega_1| + \cdots + |\omega_p| - p.$$

Cyclic bar complex $\mathbb{C}(\Omega(M))$ has two differentials b_0, b_1.

$$\begin{aligned} b_0(\omega_0, \ldots, \omega_p) &= (d\omega_0, \omega_1, \ldots, \omega_p) - \sum_{i=1}^{p} (-1)^{\epsilon_{i-1}} (\omega_0, \ldots, d\omega_i, \ldots, \omega_p), && (2) \\ b_1(\omega_0, \ldots, \omega_p) &= -\sum_{i=0}^{p-1} (-1)^{\epsilon_i} (\omega_0, \ldots, \omega_i \wedge \omega_{i+1}, \ldots, \omega_p) && (3) \\ &\qquad + (-1)^{(|\omega_p|-1)\epsilon_{p-1}} (\omega_p \wedge \omega_0, \omega_1, \ldots, \omega_{p-1}), \\ &\text{where} \quad \epsilon_i = |\omega_0| + |\omega_1| + \cdots + |\omega_i| - i. \end{aligned}$$

These two differentials satisfy relations $b_0^2 = b_1^2 = b_0 b_1 + b_1 b_0 = 0$, and make $\mathbb{C}(\Omega(M))$ a bicomplex. We write the total differential as $b = b_0 + b_1$.

This complex have negative degree elements. To remove this negative part, we perform a normalization of the cyclic bar complex, and call resulting complex **Chen's normalized bar complex**. Later, we introduce a slightly different normalization, so we put a word "Chen's" on the head to distinguish two normalizations. We define operators $S_i(f), R_i(f)$ acting on $\mathbb{C}(\Omega(M))$ for $f \in C^\infty(M)$ to be

$$S_i(f)(\omega_0, \omega_1, \dots, \omega_p) = (\omega_0, \dots, \omega_{i-1}, f, \omega_i, \dots, \omega_p) \qquad (i \geqq 1),$$

and $R_i(f) = bS_i(f) + S_i(f)b$. Then we define Chen's normalized bar complex as follows.

$$\mathbb{N}(\Omega(M)) = \mathbb{C}(\Omega(M)) / \{\text{ subcomplex generated by images of } R_i(f), S_i(f)(f \in C^\infty(M))\}.$$

Chen's normalization does not affect its cohomology.

Cyclic bar complex (and Chen's normalized bar complex also) has a product structure called **shuffle product**. For two elements $(\alpha_0, \dots, \alpha_p), (\beta_0, \dots, \beta_q) \in \mathbb{C}(\Omega(M))$ we define

$$(\alpha_0, \dots, \alpha_p) \cdot (\beta_0, \dots, \beta_q) = (-1)^{|\beta_0|(|\alpha_0| + \cdots + |\alpha_p| - p)} \sum_\chi (\pm) \alpha_0 \wedge \beta_0 \otimes S_\chi(s\alpha_1, \dots, s\alpha_p, s\beta_1, \dots, s\beta_q),$$

where χ moves all (p, q) shuffles, i.e. permutations χ of $p + q$ letters satisfying $\chi(1) < \chi(2) < \cdots < \chi(p)$, $\chi(p+1) < \cdots < \chi(p+q)$, and the sign $(\pm)$ is Koszul sign which arises from the permutation of $s\alpha_1, \dots, s\alpha_p$, $s\beta_1, \dots, s\beta_q$, and $s\alpha$ signifies the shift of degree by one, i.e. $\deg(s\alpha) = \deg(\alpha) - 1$. This product is also well-defined on $\mathbb{N}(\Omega(M))$. The triple $(\mathbb{C}(\Omega(M)), b, \text{shuffle product})$ forms a differential graded algebra (DGA for short).

Next, we define **iterated integral map** $\sigma : \mathbb{C}(\Omega(M)) \to \Omega(LM)$. For $\omega \in \Omega(M)$, $t \in S^1$, we write $\omega(t) = \mathrm{ev}_t^*(\omega) \in \Omega(LM)$, and define

$$\sigma(\omega_0, \omega_1, \dots, \omega_p) = \omega_0(0) \wedge \int_{0 \leqq t_1 \leqq t_2 \leqq \cdots \leqq t_p \leqq 1} \imath\omega_1(t_1) \wedge \cdots \wedge \imath\omega_p(t_p) dt_1 \dots dt_p,$$

where $\imath$ denotes the contraction with fundamental vector field of the natural S^1 action. Iterated integral map σ defines a morphism of DGAs,

$$\sigma \colon (\mathbb{C}(\Omega(M)), b, \text{shuffle product}) \to (\Omega(LM), d, \wedge).$$

σ clearly induces a map $\sigma : \mathbb{N}(\Omega(M)) \to \Omega(LM)$ because $\imath f(t)$ vanishes for $f \in C^\infty(M)$. Chen's theorem is stated as follows.

Theorem 2.1 (Chen) *For a simply connected manifold M, iterated integral map $\sigma : \mathbb{C}(\Omega(M)) \to \Omega(LM)$ induces an isomorphism on cohomologies. In other words, σ is a quasi-isomorphism.*

$$\sigma \colon H^*(\mathbb{C}(\Omega(M)), b) \cong H^*(\mathbb{N}(\Omega(M)), b) \cong H^*(\Omega(LM), d)$$

∎

Getzler, Jones, Petrack extended Chen's theory to the equivariant case. To deal with the equivariant cohomology via differential forms, one can use Cartan model. In case of S^1 action, it consists of the equivariant de Rham complex

$$\Omega_{S^1}(LM) = \Omega(LM)^{S^1} \otimes S^*(\mathrm{Lie}(S^1)^*) = \Omega(LM)^{S^1} \otimes \mathbb{R}[\hbar],$$

and the differential $d_{S^1} = d + \hbar\imath$, where $\Omega(LM)^{S^1}$ denotes the S^1 invariant part of $\Omega(LM)$ and $\imath$ denotes the contraction with fundamental vector field of S^1 action. The triple $(\Omega_{S^1}(M), d_{S^1}, \wedge)$ forms a DGA and is called Cartan model. The cohomology of $(\Omega_{S^1}(M), d_{S^1}, \wedge)$ is identical with the equivariant cohomology $H^*_{S^1}(LM)$. However, images of Chen's iterated integral map σ are not invariant under the natural S^1 action. Therefore, they needed to introduce a variant of Cartan model which is chain homotopy equivalent to the original one. Take a complex $\Omega(M)[\hbar] = \Omega(LM) \otimes \mathbb{R}[\hbar]$, and introduce a slightly different differential $d + \hbar P_1$ as follows.

$$P_1(\omega) = \int_{S^1} \phi_t^*(\imath\omega) dt \quad \text{for } \omega \in \Omega(LM),$$

where $\phi_t(t \in S^1)$ denotes a diffeomorphism of LM induced by S^1 action. $d + \hbar P_1$ satisfies $(d + \hbar P_1)^2 = 0$, so that $\Omega(LM)[\hbar]$ becomes a complex.

For the product structure, usual wedge product $\wedge$ does not satisfy Leibnitz rule, but it can be deformed to satisfy Leibnitz rule. The deformed product, however, does not satisfy associativity relation. Getzler, Jones and Petrack discovered the sequence of higher products $m_3, m_4, \dots$ and that $\Omega(LM)[\hbar]$ has a structure of A_∞ algebra. In this paper, we do not use this A_∞ structure, so do not mention more about this.

We extend the iterated integral map linearly over $\mathbb{R}[\hbar]$ to the map $\sigma\colon \mathbb{N}(\Omega(M))[\hbar] \to \Omega(LM)[\hbar]$. Corresponding to the operator P_1 defined on $\Omega(LM)[\hbar]$, we define **Connes' operator** B on $\mathbb{N}(\Omega(M))$ as follows.

$$B(\omega_0, \dots, \omega_p) = \sum_{i=0}^{p} (-1)^{(\epsilon_{i-1}-1)(\epsilon_p - \epsilon_{i-1})} (1, \omega_i, \dots, \omega_p, \omega_0, \dots, \omega_{i-1}).$$

Then, $\sigma : (\mathbb{N}(\Omega(LM))[\hbar], b + \hbar B) \to (\Omega(LM)[\hbar], d + \hbar P_1)$ is a chain map and the following theorem holds.

Theorem 2.2 (Getzler, Jones, Petrack) *For a simply connected manifold M, $\sigma\colon (\mathbb{N}(\Omega(LM))[\hbar], b + \hbar B) \to (\Omega(LM)[\hbar], d + \hbar P_1)$ is a quasi isomorphism. Therefore, σ induces an isomorphism*

$$\sigma : H^*(\mathbb{N}(\Omega(LM))[\hbar], b + \hbar B) \cong H^*(\Omega(LM)[\hbar], d + \hbar P_1) = H^*_{S^1}(LM).$$

∎

[GJP] also considered the case when T^r acts on the original manifold M. In this case, one can consider $S^1 \times T^r$ equivariant cohomology of LM, where T^r action on LM comes from the T^r action on M. Here, we use the original Cartan model for the T^r action, and use its variant for the S^1 action. Let Ω_{T^r} be the T^r equivariant cohomology of a point, i.e. $\Omega_{T^r} = S^*(\mathrm{Lie}(T^r)^*) = \mathbb{R}[t_1, \dots, t_r]$, and $(\Omega_{T^r}(M), d + \sum_{i=1}^r t_i \imath_i)$ be the T^r equivariant Cartan model, where

$$\Omega_{T^r}(M) = \Omega(M)^{T^r} \otimes \Omega_{T^r},$$

and $\imath_i$ is the contraction with i-th fundamental vector field of the T^r action. Define a new complex $\mathbb{C}_{\Omega_{T^r}}(\Omega(M))$ as

$$\mathbb{C}_{\Omega_{T^r}}(\Omega(M)) = \bigoplus_{p=0}^{\infty} \Omega_{T^r}(M) \otimes_{\Omega_{T^r}} \overbrace{\Omega_{T^r}(M) \otimes_{\Omega_{T^r}} \cdots \otimes_{\Omega_{T^r}} \Omega_{T^r}(M)}^{p \text{ times}},$$

and its derivation b_0, b_1 as before. When defining b_0, however, we must replace exterior derivation d with $d_{T^r} = d + \sum_{i=1}^r t_i \imath_i$ in the equation (2). Denote the total differential of the complex $\mathbb{C}_{\Omega_{T^r}}(\Omega(M))$ by b_{T^r}. Chen's normalized complex $\mathbb{N}_{\Omega_{T^r}}(\Omega(M))$ is defined as

$$\mathbb{N}_{\Omega_{T^r}}(\Omega(M)) = \mathbb{C}_{\Omega_{T^r}}(\Omega(M)) / \{\text{subcomplex generated by images of } S_i(f), R_i(f), (f \in C^\infty(M)^{T^r})\}.$$

Connes' operator B can also be defined on $\mathbb{N}_{\Omega_{T^r}}(\Omega(M))$ similarly. The iterated integral map $\sigma : \mathbb{N}_{\Omega_{T^r}}(\Omega(M)) \to \Omega_{T^r}(LM)$ is also defined linearly over Ω_{T^r}. We have the following theorem.

Theorem 2.3 (Getzler, Jones, Petrack) *For a simply connected manifold M which has a smooth T^r action, iterated integral maps*

$$\sigma\colon (\mathbb{N}_{\Omega_{T^r}}(\Omega(M)), b_{T^r}) \to (\Omega_{T^r}(LM), d_{T^r}),$$
$$\sigma\colon (\mathbb{N}_{\Omega_{T^r}}(\Omega(M))[\hbar], b_{T^r} + \hbar B) \to (\Omega_{T^r}(LM)[\hbar], d_{T^r} + \hbar P_1),$$

are quasi-isomorphisms. Therefore, there exist isomorphisms

$$\sigma\colon H^*(\mathbb{N}_{\Omega_{T^r}}(\Omega(M)), b_{T^r}) \cong H^*(\Omega_{T^r}(LM), d_{T^r}) = H^*_{T^r}(LM),$$
$$\sigma\colon H^*(\mathbb{N}_{\Omega_{T^r}}(\Omega(M))[\hbar], b_{T^r} + \hbar B) \cong H^*(\Omega_{T^r}(LM)[\hbar], d_{T^r} + \hbar P_1) = H^*_{T^r \times S^1}(LM).$$

∎

With these ingredients, we proceed to the proof of our main theorem.

3 Proof of the Main Theorem

The proof of the theorem consists of two parts. The first part uses a geometrical construction and reduce the problem to the equivariant cohomology of a free loop space. The second part uses homological algebra. We reduce Chen's normalized bar complex to more computable one using some techniques in the cyclic homology theory and the comparison theorem of spectral sequences. The main technical difficulty comes from the fact that we deal with a cohomological complex (i.e. graded in non-negative negative degree and the differential increases the degree by one), so that some standard results in cyclic homology do not hold in general because the relevant spectral sequence may not converge.

Let M be a connected manifold, and $\widetilde{M}$ be its universal cover. We assume that $\pi_2(\widetilde{M}) = \pi_2(M)$ is a free abelian group. Let $p_1, \ldots, p_r$ be a basis of $H^2(\widetilde{M}, \mathbb{Z})$. Then, there exist C^∞ complex line bundles L_i over $\widetilde{M}$ such that $c_1(L_i) = p_i$. We put

$$\pi\colon U = \prod_{i=1}^{r} S(L_i) \to \widetilde{M},$$

where $S(L_i)$ is a circle bundle of L_i. U is a principal T^r bundle over $\widetilde{M}$. The following lemma is clear from the homotopy exact sequence of the fibration $\pi\colon U \to \widetilde{M}$ and the assumption of the freeness of $\pi_2(\widetilde{M})$.

Lemma 3.1 *U is 2-connected, i.e.*

$$\pi_1(U) = \pi_2(U) = 0.$$

∎

The key proposition is the following.

Proposition 3.2 *There exists a homeomorphism $\widetilde{L_0 M} \cong LU/L_0T^r$, where L_0T^r denotes the set of all contracting loops in T^r, and L_0T^r acts on LU by pointwise multiplication, i.e. $LU \ni \gamma(t) \mapsto c(t)\cdot\gamma(t) \in LU$ for $c(\cdot) \in L_0T^r$.* ∎

(proof) As stated in the introduction, an element of $\widetilde{L_0M} = \widetilde{L\widetilde{M}}$ is a pair $(\gamma, [g])$ where $\gamma\colon S^1 \to \widetilde{M}$, and $g\colon D^2 \to \widetilde{M}$ such that $g|_{\partial D^2} = \gamma$. If $(\gamma, [g]) \in \widetilde{L_0M}$ is given, we can lift g to $\tilde{g}\colon D^2 \to U$. Then, $\tilde{g}|_{\partial D^2}\colon S^1 \to U$ gives an element of LU. If we take another lifting $\tilde{g}'$, two maps $\tilde{g}, \tilde{g}'$ are related to each other as

$$\tilde{g}' = c \cdot \tilde{g} \qquad \text{for some } c\colon D^2 \to T^r,$$

so that $\tilde{g}|_{\partial D^2}$ and $\tilde{g}'|_{\partial D^2}$ defines the same element in LU/L_0T^r. Independence of the choice of a representative g in the homotopy class $[g]$ follows from the covering homotopy property. Hence we obtain a map $\widetilde{L_0M} \to LU/L_0T^r$. Next, we define the inverse map. Because U is 2-connected, for a given element $\tilde{\gamma} \in LU$ there exits a unique disk $\tilde{g}\colon D^2 \to U$ contracting $\tilde{\gamma}$ ($\tilde{g}|_{\partial D^2} = \tilde{\gamma}$) up to homotopy. Therefore, it defines an element $(\pi \circ \tilde{\gamma}, [\pi \circ \tilde{g}]) \in \widetilde{L_0M}$. Independence of the choice of $\tilde{\gamma}$ can be easily seen. ∎

Corollary 3.3 *There exists an S^1 equivariant homotopy equivalence between $\widetilde{L_0M}$ and LU/T^r. Therefore,*

$$H^*(\widetilde{L_0M}) \cong H^*_{T^r}(LU), \qquad H^*_{S^1}(\widetilde{L_0M}) \cong H^*_{S^1\times T^r}(LU).$$

∎

(proof) Consider the fibration $LU/T^r \to LU/L_0T^r = \widetilde{LM}$. This fibration is principal Ω_0T^r bundle, where Ω_0T^r is the set of all contracting based loops in T^r. So this fibration is homotopy equivalent because Ω_0T^r is contractible. The homotopy can be taken S^1 equivariantly. (For example, we can construct a section $s\colon LU/L_0T^r = \widetilde{L_0M} \to LU/T^r$ by a parallel transport by a connection of $\pi\colon U \to \widetilde{M}$ and twisting its phase factor.) ∎

According to the above corollary and Theorem 2.3, we can compute the (equivariant) cohomology of $\widetilde{L_0M}$ by the complex $\mathbb{N}_{\Omega_{T^r}}(\Omega_{T^r}(U))$ (or $\mathbb{N}_{\Omega_{T^r}}(\Omega_{T^r}(U))[\hbar]$).

Let k denote a commutative ring and $(A, \delta, \cdot)$ be a unital DGA over k. Define cyclic bar complex for A by

$$\mathbb{C}_k(A) = \bigoplus_{p=0}^{\infty} A \otimes_k \overbrace{A \otimes_k \cdots \otimes_k A}^{p \text{ times}}.$$

Its derivations $b_0, b_1, b = b_0 + b_1$ are defined in the same way as in (2), (3). Three kinds of degree for an element $(a_0, \ldots, a_p) \in \mathbb{C}_k(A)$ are defined as follows.

$$\begin{cases} \text{tensor degree} = p. \\ \text{weight} = |a_0| + \cdots + |a_p|. \\ \text{total degree} = |a_0| + \cdots + |a_p| - p. \end{cases}$$

For a general A, we can define normalized bar complex $\overline{\mathbb{C}}_k(A)$ different from Chen's normalization as follows.

$$\overline{\mathbb{C}}_k(A) = \bigoplus_{p=0}^{\infty} A \otimes_k \overbrace{\overline{A} \otimes_k \cdots \otimes_k \overline{A}}^{p \text{ times}} \qquad \text{, where } \overline{A} \text{ denotes } A/k.$$

Derivation b is well-defined on this normalization, and Connes' operator B is defined on $\overline{\mathbb{C}}_k(A)$ similarly. Shuffle product is defined on $\mathbb{C}_k(A), \overline{\mathbb{C}}_k(A)$ similarly and makes them DGAs. The following lemma shows the relation of two complexes $(\mathbb{C}_k(A), b)$ and $(\overline{\mathbb{C}}_k(A), b)$ when k is a field.

Lemma 3.4 *Let $(A, \delta, \cdot)$ be a unital DGA over a field k such that the differential raises the degree, $\delta \colon A^i \to A^{i+1}$, and $A^i = 0$ if $i < 0$. Then, the natural projection*

$$(\mathbb{C}_k(A), b) \to (\overline{\mathbb{C}}_k(A), b)$$

is a quasi-isomorphism. ∎

(proof) We think the spectral sequence about the tensor degree, more precisely, about the filtrations $F^{-p} = \oplus_{i \leqq p} A \otimes A^{\otimes i}$, $\overline{F}^{-p} p = \oplus_{i \leqq p} A \otimes \overline{A}^{\otimes i}$. Then the morphism between E_1 terms is

$$\bar{f} \colon (\mathbb{C}(H(A)), b_1) \to (\overline{\mathbb{C}}(H(A)), b_1),$$

where b_1 is defined in (3). As known in the theory of simplicial modules, the above morphism $\bar{f}$ is a quasi-isomorphism [Lod]. The filtrations $F, \overline{F}$ are bounded above (F^0=$\overline{F}^0$=0), therefore the spectral sequence converges. The lemma follows from the comparison theorem of spectral sequences [MacL], [McC]. ∎

Next lemma is useful in the proof.

Lemma 3.5 *Let $f \colon (A, \delta, \cdot) \to (A', \delta', \cdot)$ be a morphism of unital DGAs. If f is a quasi-isomorphism, then the induced morphism $f \colon (\mathbb{C}_k(A), b) \to (\mathbb{C}_k(A'), b')$ is also a quasi-isomorphism.*

(proof) See [GJP]. ∎

We need three lemmata about the comparison of complexes. Let $(C, \delta_0, \cdot)$ be a unital DGA over $\mathbb{R}$. Assume $\delta_0 \colon C^i \to C^{i+1}$, and $C^i = 0$ if $i < 0$. Introduce r variables $t_1, \ldots, t_r$ of degree 2 and the decreasing filtration F of $C[t_1, \ldots, t_r]$ such that $F^i = \sum_{j_1 + \cdots + j_r \geqq i} C t_1^{j_1} t_2^{j_2} \ldots t_r^{j_r}$. We call this kind of filtration $t-$filtration. Furthermore we assume that $C[t_1, \ldots, t_r]$ has a $\mathbb{R}[t_1, \ldots, t_r]$ linear differential δ which induces the original differential δ_0 on $C \cong F^0/F^1$, and that $(C[t_1, \ldots, t_r], F, \delta)$ is a filtered DGA (i.e. $\delta(F^i) \subset F^i$). Clearly the complex $\Omega_{T^r}(M) = \Omega(M)^{T^r} \otimes \mathbb{R}[t_1, \ldots, t_r]$ has a $t-$filtration F, and so does $\mathbb{N}_{\Omega_{T^r}}(\Omega_{\Omega_{T^r}}(M))$.

Lemma 3.6 *Let $f \colon (C[t_1, \ldots, t_r], F, \delta) \to (\Omega_{\Omega_{T^r}}(U), F, d_{T^r})$ be a morphism of filtered DGAs where F's are $t-$filtrations. If the morphism induced by f*

$$\bar{f} \colon (C, \delta_0) \to (\Omega(U)^{T^r}, d)$$

is a quasi-isomorphism, then

$$f\colon (\overline{\mathbb{C}}_{\mathbb{R}[t]}(C[t_1,\dots,t_r]),b)\to(\mathbb{N}_{\Omega_{T^r}}(\Omega_{\Omega_{T^r}}(U)),b_{T^r})$$

is a quasi-isomorphism. ∎

(proof) Consider the spectral sequence from $t-$filtration F. This spectral sequence converges because the $t-$filtration is bounded above (i.e. for fixed degree d, there exists p such that $F^p\cap\{\text{degree } d \text{ part}\}=\{0\}$.) Then the morphism between E_0 term is

$$f_0\colon(\overline{\mathbb{C}}_{\mathbb{R}}(C),b)\to(\mathbb{N}(\Omega(U)^{T^r}),b).$$

Consider the commutative diagram,

$$\begin{array}{ccc}(\overline{\mathbb{C}}_{\mathbb{R}}(C),b) & \xrightarrow{f_0} & (\mathbb{N}(\Omega(U)^{T^r}),b)\\ \uparrow & & \uparrow\\ (\mathbb{C}_{\mathbb{R}}(C),b) & \longrightarrow & (\mathbb{C}(\Omega(U)^{T^r}),b).\end{array}$$

Two vertical arrows are quasi-isomorphisms by Lemma 3.4 and the fact that Chen's normalization does not change its cohomology. The bottom arrow is also a quasi-isomorphism by the assumption and Lemma 3.5. Therefore the top arrow f_0 is a quasi-isomorphism, and the Lemma follows from the comparison theorem of spectral sequences. ∎

Let $(C'[t_1,\dots,t_r],F',\delta')$ be a filtered DGA which satisfies the same conditions as $(C[t_1,\dots,t_r],F,\delta)$ does. Following lemma gives a sufficient condition for two normalized bar complexes over $\mathbb{R}[t_1,\dots,t_r]$ to define the same cohomology.

Lemma 3.7 *Let $f\colon(C[t_1,\dots,t_r],F,\delta)\to(C'[t_1,\dots,t_r],F',\delta')$ be a morphism of filtered DGAs. If the induced morphism*

$$\bar{f}\colon(C,\delta_0)\to(C',\delta'_0)$$

is a quasi-isomorphism, then

$$f\colon(\overline{\mathbb{C}}_{\mathbb{R}[t]}(C[t_1,\dots,t_r]),b)\to(\overline{\mathbb{C}}_{\mathbb{R}[t]}(C'[t_1,\dots,t_r]),b')$$

is a quasi-isomorphism. ∎

(proof) Take the spectral sequence with respect to $t-$filtration, and discuss in the same way as in Lemma 3.6 ∎

For a commutative algebra A over k, let $\Omega^1_{A|k}$ denote the module of relative differential form of A over k and $d\colon A\to\Omega^1_{A|k}$ denote the universal derivation. For $p>0$ we define

$$\Omega^p_{A|k}=\bigwedge^p\Omega^1_{A|k}.$$

When A is graded commutative, we define the commutation relation of two elements by

$$da\wedge db=(-1)^{(|a|-1)(|b|-1)}db\wedge da,\quad \deg(da)=\deg(a)-1,\qquad \text{for } a,b\in A.$$

In this case, we must interpret $\bigwedge^p\Omega^1_{A|k}$ to be equal to

$$\bigoplus_{i+j=p}S^i(\Omega^1_{A|k})_{\text{even}}\otimes\bigwedge^j(\Omega^1_{A|k})_{\text{odd}}.$$

We put $\Omega^0_{A|k} = A$ and $\Omega_{A|k} = \oplus_{p=0}^{\infty}\Omega^p_{A|k}$, then $\Omega_{A|k}$ becomes an associative algebra and d is extended to the map $d\colon \Omega_{A|k} \to \Omega_{A|k}$ satisfying (graded) Leibnitz rule, i.e. $d(ab) = da \cdot b + (-1)^{|a|}a \cdot db$. If A has a differential δ and is a DGA, we can define $\delta\colon \Omega_{A|k} \to \Omega_{A|k}$ as follows.

$$\begin{aligned}\delta(a_0 da_1 \wedge \cdots \wedge da_p) &= \delta a_0 da_1 \wedge \cdots \wedge da_p - \sum_{i=1}^{p}(-1)^{\epsilon_{i-1}} a_0 da_1 \wedge \cdots \wedge d(\delta a_i) \wedge \cdots \wedge da_p \\ &\quad \text{where } \epsilon_i = |a_0| + \cdots + |a_i| - i, \quad a_i \in A.\end{aligned}$$

One can easily check that $\delta d + d\delta = d^2 = \delta^2 = 0$. For this ring $\Omega_{A|k}$ we can also define three kinds of degree as follows,

$$\begin{cases}\text{tensor degree } = p \\ \text{weight } = |a_0| + |a_1| + \cdots + |a_p| \\ \text{total degree } = |a_0| + |a_1| + \cdots + |a_p| - p,\end{cases} \qquad \text{for } a_0 da_1 \wedge \cdots \wedge da_p.$$

When k contains $\mathbb{Q}$, we define the map $\varphi\colon \overline{\mathbb{C}}_k(A) \to \Omega_{A|k}$ as follows,

$$\varphi(a_0, a_1, \ldots, a_p) = \frac{1}{p!} a_0 da_1 \wedge \cdots \wedge da_p.$$

φ turns out to be a morphism between complexes with two anticommuting differentials and preserves the product structure, $\varphi\colon (\overline{\mathbb{C}}_k(A), b, B, \text{shuffle product}) \to (\Omega_{A|k}, \delta, d, \wedge)$.

Lemma 3.8 *Furthermore, we assume that C is a free algebra $\bigwedge V$, where V is a graded vector space over $\mathbb{R}$, and that $C^0 = \mathbb{R}$, $C^1 = 0$. Then, there is a quasi-isomorphism*

$$\varphi\colon (\overline{\mathbb{C}}_{\mathbb{R}[t]}(C[t_1, \ldots, t_r]), b) \to (\Omega_{C[t]|\mathbb{R}[t]}, \delta).$$

∎

(proof) First we consider $t-$filtration and the associated spectral sequence. The morphism between E_0 terms is

$$(\overline{\mathbb{C}}_{\mathbb{R}}(C), b) \to (\Omega_{C|\mathbb{R}}, \delta_0).$$

It suffices to show that this morphism is a quasi-isomorphism. Next we take weight filtrations, then the morphism between E_0 terms of the associated spectral sequence is

$$(\overline{\mathbb{C}}_{\mathbb{R}}(C), b_1) \to (\Omega_{C|\mathbb{R}}, 0).$$

This morphism is known to be a quasi-isomorphism when C is a free algebra in the Hochschild homology theory [Lod]. The conditions $C^0 = \mathbb{R}$, $C^1 = 0$ ensure that the weight filtration is bounded above, so that the spectral sequence converges. Therefore the lemma follows from the comparison theorem of spectral sequences. ∎

Using the above three lemmata, we can reduce the complex $\mathbb{N}_{\Omega_{Tk}}(\Omega_{\Omega_{Tk}}(U))$ to more computable one. Take a connection form $\hat{\theta}_i$ on the circle bundle $S(L_i)$, then its curvature form $d\hat{\theta}_i$ represents the cohomology class $p_i \in H^*(\widetilde{M}, \mathbb{Z})$. Let $(\bigwedge V, \delta_0)$ be the Sullivan's minimal model for $\widetilde{M}$, where V is a graded vector space over $\mathbb{R}$ and is of the form

$$V = \bigoplus_{i=1}^{r} \mathbb{R}p_i \oplus V', \qquad \text{elements of } V' \text{ have degree more than three.}$$

Put $C = \mathbb{R}[\theta_1, \ldots, \theta_r] \otimes \bigwedge V$ and $C[t] = C \otimes \mathbb{R}[t_1, \ldots, t_r]$, where $\deg \theta_i = 1$ and $\deg t_i = 2$. We define the differential δ on $C[t]$ as

$$\delta(\theta_i) = p_i + t_i, \quad \delta(p_i) = 0, \quad \delta(v) = \delta_0(v) \text{ for } v \in V'.$$

Take $t-$filtration F of $C[t]$, then $(C[t],F,\delta)$ is a filtered DGA. The differential δ induces the differential δ_0 on $C\cong F^0/F^1$. Take the following morphism over $\mathbb{R}[t]$,

$$\begin{array}{rcl} f\colon (C[t_1,\dots,t_r],\delta) & \to & (\Omega_{T^r}(U)=\Omega(U)^{T^r}\otimes\mathbb{R}[t_1,\dots,t_r],d_{T^r}) \\ p_i & \longmapsto & d\hat{\theta}_i(\text{curvature forms}), \\ \theta_i & \longmapsto & \hat{\theta}_i(\text{connection forms}), \\ V'\ni v & \longmapsto & \pi^*v, \end{array}$$

where we think $v\in V'$ represents a differential form on $\widetilde{M}$. This preserves the $t-$filtration and the induced morphism

$$(C=\mathbb{R}[\theta_1,\dots,\theta_r]\otimes\bigwedge V,\delta_0)\to(\Omega(U)^{T^r},d)$$

is a quasi-isomorphism. Because, E_2 term of the Serre spectral sequence for the fibration $\pi\colon U\to\widetilde{M}$ is $H^*(\widetilde{M})\otimes\mathbb{R}[\theta_1,\dots,\theta_r]$, and is isomorphic to the E_2 term of the spectral sequence of C associated with the filtration $F^pC=\mathbb{R}[\theta_1,\dots,\theta_r]\otimes(\bigwedge V)^{\geqq p}$. Therefore, by Lemma 3.6, we conclude that

$$(\overline{\mathbb{C}}_{\mathbb{R}[t]}(C[t]),b)\to(\mathbb{N}_{\Omega_{T^r}}(\Omega_{T^r}(U)),b_{T^r})\quad\text{is a quasi isomorphism.}\tag{4}$$

Put $C'=\bigwedge V'$ and $C'[t]=C'\otimes\mathbb{R}[t_1,\dots,t_r]$. Define a morphism h as follows,

$$\begin{array}{rcl} h\colon (C[t]=\mathbb{R}[\theta,t]\otimes\bigwedge V,\delta) & \to & (C'[t]=\mathbb{R}[t]\otimes\bigwedge V',\delta') \\ \theta_i & \longmapsto & 0, \\ p_i & \longmapsto & -t_i, \\ V'\ni v & \longmapsto & v, \end{array}$$

where the $\mathbb{R}[t]$ linear differential δ' is defined on $C'[t]$ in order for h to be a chain map, i.e. $\delta'(v)=h(\delta_0(v))$ for $v\in V'$. Then h becomes a chain map and preserves the $t-$filtration. We claim that the induced morphism $\bar{h}\colon(C,\delta_0)\to(C',\delta_0')$ is a quasi-isomorphism. Take filtrations (bounded above) as

$$F^t(C)=\operatorname{span}\{\theta_{a_1}\theta_{a_2}\cdots\theta_{a_j}\otimes b|a_1<\cdots<a_j,2j+\deg b\geqq t\},\quad F^t(C')=(\bigwedge V')^{\geqq t}.$$

Then the morphism between E_0 terms of the associated spectral sequence is

$$\bar{h}_0\colon(C=\mathbb{R}[\theta]\otimes\bigwedge V,\bar{\delta_0})\to(C'=\bigwedge V',0),\quad \theta_i,p_i\mapsto 0,\quad v\mapsto v\text{ for }v\in V',$$

where $\bar{\delta_0}(\theta_i)=p_i,\bar{\delta_0}(v)=0$ for $v\in V'$. It is easy to see that $\bar{h}_0$ is a quasi-isomorphism and our claim is proved. Therefore by the Lemma 3.7, we conclude that

$$(\overline{\mathbb{C}}_{\mathbb{R}[t]}(C[t]),b)\to(\overline{\mathbb{C}}_{\mathbb{R}[t]}(C'[t]),b')\quad\text{is a quasi-isomorphism.}\tag{5}$$

Finally, by the Lemma 3.8,

$$(\overline{\mathbb{C}}_{\mathbb{R}[t]}(C'[t]),b')\to(\Omega_{C'[t]|\mathbb{R}[t]},\delta')\quad\text{is a quasi-isomorphism.}\tag{6}$$

From equations (4), (5), (6), one obtains the following.

Theorem 3.9 *For a connected manifold M whose second homotopy group $\pi_2(M)$ has no torsion, there exist isomorphisms*

$$\begin{array}{rcl} H^*(\Omega_{\bigwedge V'|\mathbb{R}}[t_1,\dots,t_r],\delta') & \cong & H^*_{T^r}(LU)=H^*(\widetilde{L_0M}), \\ H^*(\Omega_{\bigwedge V'|\mathbb{R}}[t_1,\dots,t_r,\hbar],\delta'+\hbar d) & \cong & H^*_{T^r\times S^1}(LU)=H^*_{S^1}(\widetilde{L_0M}). \end{array}$$

∎

The latter isomorphism follows from the fact that three complexes $\overline{\mathbb{C}}_{\mathbb{R}[t]}(C'[t])$, $\overline{\mathbb{C}}_{\mathbb{R}[t]}(C[t])$, $\mathbb{N}_{\Omega_{T^r}}(\Omega_{T^r}(U))$ have no negative degree part, and one has convergent spectral sequences about $\hbar-$filtrations. Because $\bigwedge V'$ is a free algebra, the differential d in the complex $\Omega_{\bigwedge V'|\mathbb{R}}[t_1,\ldots,t_r,\hbar]$ is acyclic by the Poincaré lemma. We can decompose this complex as

$$\Omega_{\bigwedge V'|\mathbb{R}}[t_1,\ldots,t_r,\hbar] \cong \bigoplus_{i\geqq 0} B_i \oplus \bigoplus_{i>0} B'_i$$

$$\begin{cases} B_i = \bigoplus_{p\geqq 0} \Omega^p_{\bigwedge V'|\mathbb{R}}[t]\hbar^{p+i} & \text{for } i \geqq 0 \\ B'_i = \bigoplus_{p\geqq i} \Omega^p_{\bigwedge V'|\mathbb{R}}[t]\hbar^{p-i} & \text{for } i > 0 \end{cases}$$

B_i, B'_i are subcomplexes with respect to the differential $\delta' + \hbar d$, and are acyclic with d, therefore we obtain

$$\begin{cases} H^*(B_i) \cong \mathbb{R}[t_1,\ldots,t_r]\hbar^i, \\ H^*(B'_i) \cong H^*(Z^i,\delta'), \quad \text{where we define } Z^i = \ker(d\colon \Omega^i_{\bigwedge V'|\mathbb{R}}[t] \to \Omega^{i+1}_{\bigwedge V'|\mathbb{R}}[t]) \end{cases}$$

On the other hand, to multiply by $\hbar$ in the complex $\Omega_{\bigwedge V'|\mathbb{R}}[t_1,\ldots,t_r,\hbar]$ defines a chain map, and we get the following sequence,

$$\longrightarrow B'_n \xrightarrow{\times\hbar} B'_{n-1} \longrightarrow \ldots \xrightarrow{\times\hbar} B'_1 \xrightarrow{\times\hbar} B_0 \xrightarrow{\times\hbar} B_1 \longrightarrow \ldots$$

Therefore if we repeat the multiplication of any element in the complex B'_i by $\hbar$'s, it is eventually contained in the complex B_0. To summarize, we obtain the following corollary.

Corollary 3.10 *For a connected manifold M whose second homotopy group $\pi_2(M)$ has no torsion, the equivariant cohomology of $\widetilde{LM}$ has the following splitting.*

$$H^*_{S^1}(\widetilde{L_0M}) \cong \mathbb{R}[t_1,\ldots,t_r,\hbar] \oplus \bigoplus_{i=1}^{\infty} H^*(Z^i,\delta').$$

These components satisfy $\hbar H^*(Z^i) \subset H^*(Z^{i-1})$ for $i > 1$ and $\hbar H^*(Z^1) \subset \mathbb{R}[t_1,\ldots,t_r,\hbar]$, therefore the following holds.

$$H^*_{S^1}(\widetilde{L_0M}) \otimes_{\mathbb{R}[\hbar]} \mathbb{R}[\hbar,\hbar^{-1}] \cong \mathbb{R}[t_1,\ldots,t_r,\hbar,\hbar^{-1}].$$

Remark 3.11 The cohomology class of $H^*_{S^1}(\widetilde{L_0M})$ corresponding to the variables t_i can be represented by the chain $\hat{p}_i + \hbar\hat{\theta}_i$ in $\mathbb{N}_{\Omega_{T^r}}(\Omega_{T^r}(U))[\hbar]$, where $\hat{p}_i = d\hat{\theta}_i$ is a curvature form. It maps to the following element of $\Omega_{T^r}(LU)[\hbar]$ by the iterated integral map.

$$\mathrm{ev}_0^*(\hat{p}_i) + \hbar\int_{S^1}\tilde{\gamma}^*(\theta_i), \qquad \text{where } \tilde{\gamma} \in LU.$$

Furthermore, this corresponds to the following element of Cartan model $\Omega(\widetilde{L_0M})^{S^1}[\hbar]$.

$$\int_{S^1}\mathrm{ev}_t^*(\hat{p}_i)dt + \hbar\int_{D^2} g^*(\hat{p}_i), \qquad \text{where } (\gamma,[g]) \in \widetilde{L_0M}.$$

4 Examples

In this section we present the case where the computation of the (equivariant) cohomology of $\widetilde{L_0M}$ can be carried out explicitly.

Theorem 4.1 *Let M be a simply connected manifold. We assume $H_2(M,\mathbb{Z}) = \pi_2(M)$ is torsion free and M has a cohomology ring of the form,*

$$H^*(M,\mathbb{R}) \cong \mathbb{R}[p_1,\ldots,p_r]/\langle f_1,\ldots,f_l\rangle,$$

where $\{p_1,\dots,p_r\}$ *is a basis of* $H^2(M)$ *and* f_i*'s are homogeneous polynomials of* p_i*'s and* $f_1,\dots,f_l$ *is a regular sequence. Then the (equivariant) cohomology of* $\widetilde{LM}$ *is*

$$\begin{aligned} H^*(\widetilde{LM}) &\cong \mathbb{R}[p_1,\dots,p_r]/\langle f_1\dots f_l\rangle\otimes\mathbb{R}[\tilde z_1,\dots,\tilde z_l],\\ H^*_{S^1}(\widetilde{LM}) &\cong \mathbb{R}[p_1,\dots,p_r,f_1/\hbar,\dots,f_l/\hbar,\hbar], \end{aligned}$$

where $\deg\tilde z_i=\deg f_i(p)-2$. ∎

(proof) We first claim that the minimal model for M has the form,

$$(\bigwedge V_l,\delta_l),\quad V_l=\bigoplus_{i=1}^r\mathbb{R}p_i\oplus\bigoplus_{i=1}^l\mathbb{R}z_i,\quad \deg p_i=2,\deg z_i=\deg f_i(p)-1,$$
$$\text{where } \delta_l(z_i)=f_i(p),\ \delta_l(p_i)=0.$$

We proceed by the induction on l. When $l=1$, the claim is clear. We assume the case $l-1$ and prove the case l. Consider the following exact sequence of complexes.

$$0\longrightarrow((f_l(p)-w)\bigwedge V_l\otimes\mathbb{R}[w],\tilde\delta_l)\longrightarrow(\bigwedge V_l\otimes\mathbb{R}[w],\tilde\delta_l)\overset{w\mapsto f_l(p)}{\longrightarrow}(\bigwedge V_l,\delta_l)\longrightarrow 0,$$
where w is a variable of degree equal to $\deg f_l(p)$,
$$\tilde\delta_l(p_i)=0\ (1\leqq i\leqq r),\ \tilde\delta_l(z_j)=f_j(p)\ (1\leqq j\leqq l-1),\ \tilde\delta_l(z_l)=w.$$

The newly defined left and middle complexes are quasi-isomorphic to $(\bigwedge V_{l-1},\delta_{l-1})$. By the induction hypothesis, $H^*(\bigwedge V_{l-1},\delta_{l-1})\cong\mathbb{R}[p]/I_{l-1}$, so that we get the following long exact sequence,

$$\longrightarrow\mathbb{R}[p]/I_{l-1}\xrightarrow{\times f_l(p)}\mathbb{R}[p]/I_{l-1}\longrightarrow H^*(\bigwedge V_l,\delta_l)\xrightarrow{\delta^*}\mathbb{R}[p]/I_{l-1}\longrightarrow,$$

where $I_j=\langle f_1,\dots,f_j\rangle$, $\mathbb{R}[p]=\mathbb{R}[p_1,\dots,p_r]$. Because $f_1,\dots,f_l$ is a regular sequence, $\times f_l(p)$ is a monomorphism, so δ^* is a 0 map and the above sequence is in fact a short exact sequence. Therefore we conclude that $H^*(\bigwedge V_l,\delta_l)\cong\mathbb{R}[p]/I_l$ and the claim is proved.

Next, we use Theorem 3.9 and reduces the problem to compute the cohomology of the following complexes

$$(\mathbb{R}[t_1,\dots,t_r,dz_1,\dots,dz_l]\otimes\bigwedge\nolimits_{\mathbb{R}}[z_1,\dots,z_l],\delta_l'),\quad(\mathbb{R}[t_1,\dots,t_r,dz_1,\dots,dz_l,\hbar]\otimes\bigwedge\nolimits_{\mathbb{R}}[z_1,\dots,z_l],\delta_l'+\hbar d),$$

where $\deg dz_i=\deg z_i-1$ and $\delta_l'(z_i)=f_i(t)$, $\delta_l'(dz_i)=0$. The first one is easy to compute and

$$H^*(\mathbb{R}[t_1,\dots,t_r,dz_1,\dots,dz_l]\otimes\bigwedge\nolimits_{\mathbb{R}}[z_1,\dots,z_l],\delta_l')\cong\mathbb{R}[t_1,\dots,t_r]/\langle f_1(t),\dots,f_l(t)\rangle\otimes\mathbb{R}[dz_1,\dots,dz_l].$$

Then we take $\tilde z_i$ to be dz_i and obtain the theorem. For the second complex, we take the $\hbar-$filtration,

$$F^p=\bigoplus_{j\geqq p}\mathbb{R}[t_1,\dots,t_r,dz_1,\dots,dz_l]\otimes\bigwedge\nolimits_{\mathbb{R}}[z_1,\dots,z_l]\hbar^j,$$

then the associated spectral sequence degenerates at E_1, where

$$E_1^{p,q}=\big(\mathbb{R}[t_1,\dots,t_r]/\langle f_1(t),\dots,f_l(t)\rangle\otimes\mathbb{R}[dz_1,\dots,dz_l]\big)^{q-p}\hbar^p.$$

By the relation $(\delta_l'+\hbar d)(z_i)=f_i(t)+\hbar dz_i$, we obtain $\hbar dz_i\equiv -f_i(t)$, so that the theorem follows. ∎

Example 4.2 In the case $M=\mathbb{P}^n$, we can apply Theorem 4.1 and obtain,

$$\begin{aligned} H^*(\widetilde{L\mathbb{P}^n}) &\cong \mathbb{R}[p]/\langle p^{n+1}\rangle\otimes\mathbb{R}[\tilde z],\quad \deg\tilde z=2n,\\ H^*_{S^1}(\widetilde{L\mathbb{P}^n}) &\cong \mathbb{R}[p,p^{n+1}/\hbar,\hbar]. \end{aligned}$$

Example 4.3 More generally, a certain class of toric manifolds has cohomology algebras of the type as stated in Theorem 4.1. For toric manifolds, or more generally toric varieties, see e.g. [Oda]. Let Σ be a fan defining a toric manifold X_Σ and denote its 1−skeleton by $\Sigma^{(1)}$. We define $[\Sigma]$, the combinatorial type of Σ, to be a subset of the power set $\mathscr{P}(\Sigma^{(1)})$ satisfying,

$$I \in [\Sigma] \Longleftrightarrow I \text{ is a set of edges of some cone } \sigma_I \in \Sigma.$$

We impose the following condition on Σ, $\Sigma^{(1)}$ has a decomposition $\Sigma^{(1)} = I_1 \amalg I_2 \amalg \cdots \amalg I_r$, and Σ satisfies

$$I \in [\Sigma] \Longleftrightarrow I \not\supseteq I_i \text{ for } \forall i.$$

Then corresponding toric manifold X_Σ has a cohomology ring of the type, see e.g. [Aud],

$$\begin{aligned} H^*(X_\Sigma) &\cong \mathbb{R}[p_1, \dots, p_r]/\langle f_1, \dots, f_r \rangle, \qquad f_1, \dots, f_r \text{ is a regular sequence,} \\ p_i &= \sum_{\rho \in I_i} [D_\rho], \quad D_\rho \text{ denotes the toric divisor corresponding to } \rho \text{ in } \Sigma^{(1)}. \end{aligned}$$

so that the conclusion of Theorem 4.1 holds.

5 A Relation to Orbifold Loop Spaces

The method we used in the proof of the main theorem can also be exploited to study orbifold loop spaces. Orbifold is a generalization of the notion of a differentiable manifold. It admits a mild singularity and is locally written as the quotient of a Euclidean space by a finite group [Sat]. Almost all notions about differentiable manifolds can be generalized about orbifolds, for example, differentiable functions or forms can be defined on orbifolds. Until recently, however, the adequate notion for a "morphism" between orbifolds was not understood well. It arose from the study of string theories in physics. String theories consider loops in the spacetime in place of point particles. Dixon, Harvey, Vafa and Witten studied string theories on Calabi-Yau orbifolds [DHVW]. They took a contribution from singularities of orbifolds into consideration, which is called a "twisted sector". Later, Chen and Ruan defined the notion of good maps or morphisms between orbifolds mathematically. Their objectives were to formulate orbifold Gromov-Witten invariants which study the topology of symplectic orbifolds [CR2]. On the other hand, Chen studied a general theory for a more general category of spaces, called orbispaces, which is an appropriate category for spaces locally written by the quotient of G equivariant charts [ChW]. In this context, orbifold loop space can be defined as the set of all "morphisms" from S^1 to an orbifold.

5.1 Orbifolds and Morphisms

To fix the notation, in this subsection we explain the definition of orbifolds and morphisms between them, following [CR2, ChW]. For a connected Hausdorff space U, we define a **uniformizing system** of U as a triple (V, G, π), where V is a connected smooth manifold, and G is a finite group acting on V smoothly and effectively, and $\pi\colon V \to U$ is a quotient map by the action of G, so that we have $U \cong V/G$. Let (V, G, π), (V', G', π') be uniformizing systems of U and U', where $U' \subset U$. An **injection** $j\colon (V', G', \pi') \to (V, G, \pi)$ is a pair (ϕ, τ) of a C^∞ embedding $\phi\colon V' \to V$ and a homomorphism $\tau\colon G' \to G$ such that $\pi \circ \phi = \pi'$ and $\phi(g \cdot x) = \tau(g) \cdot \phi(x)$, $x \in V'$, $g \in G'$. We can define the composition of two injections. Two uniformizing systems (V_1, G_1, π_1), (V_2, G_2, π_2) of the same U are called isomorphic if there exist injections $j_1\colon (V_1, G_1, \pi_1) \to (V_2, G_2, \pi_2)$ and $j_2\colon (V_2, G_2, \pi_2) \to (V_1, G_1, \pi_1)$ such that $j_1 \circ j_2 = \mathrm{id}_{(V_2, G_2, \pi_2)}$, $j_2 \circ j_1 = \mathrm{id}_{(V_1, G_1, \pi_1)}$, where we call j_1, j_2 isomorphisms between uniformizing systems. Let (V, G, π) be a uniformizing system of U and $U' \subset U$ be a connected open subset. We define an **induced uniformizing system** (V', G', π') on U' by (V, G, π) as

$$V' = \text{a connected component of } \pi^{-1}(U'), \quad G' = \left\{ g \in G \,|\, g \cdot V' \subset V' \right\}, \quad \pi' = \pi|_{V'}.$$

This turns out to be a uniformizing system of U'. From this definition, there exists a natural injection from (V', G', π') to (V, G, π). Important properties about uniformizing systems are as follows.

(1) Any automorphism (ϕ, τ) of a uniformizing system (V, G, π) can be written as $\phi(x) = g \cdot x$ and $\tau(h) = ghg^{-1}$ for some $g \in G$. We denote this automorphism by $\psi_g = (\phi, \tau)$.

(2) Any given two injections j_1, j_2 between uniformizing systems (V', G', π'), (V, G, π) are connected by an automorphism of (V, G, π), i.e. there exists an automorphism ψ_g satisfying the following commutative diagram.

$$\begin{tikzcd}
(V', G', \pi') \arrow[r, "j_1"] \arrow[d, equal] & (V, G, \pi) \arrow[d, "\psi_g"] \\
(V', G', \pi') \arrow[r, "j_2"] & (V, G, \pi).
\end{tikzcd}$$

(3) Any two induced uniformizing systems $(V_1, G_1.\pi_1)$, (V_2, G_2, π_2) on $U'(\subset U)$ by a uniformizing system (V, G, π) of U are isomorphic to each other. More precisely, there is an element $g \in G$ such that $\psi_g \colon (V_1, G_1, \pi_1) \cong (V_2, G_2, \pi_2)$, where we think V_i to be subsets of V and G_i to be subgroups of G.

Let M be a Hausdorff paracompact topological space. A **compatible cover** of M is an open cover $\mathcal{U}$ of M such that,

(1) any $U \in \mathcal{U}$ is a connected open set,

(2) for $U \in \mathcal{U}$, there exists a uniformizing system (V, G, π) of U, where $\pi \colon V \to U \cong V/G$,

(3) for $U, U' \in \mathcal{U}$ such that $U' \subset U$, there exists an injection $j_U^{U'} \colon (V', G', \pi') \to (V, G, \pi)$ between corresponding uniformizing systems,

(4) for $U, U' \in \mathcal{U}$ and $p \in U \cap U'$, there exists a $U'' \in \mathcal{U}$ such that $p \in U'' \subset U \cap U'$.

An orbifold is a pair $(M, \mathcal{U})$ of M and a compatible cover $\mathcal{U}$ of M. Two compatible covers $\mathcal{U}$, $\mathcal{U}'$ are equivalent when $\mathcal{U} \cup \mathcal{U}'$ is a compatible cover. We remark that orbifolds in this paper are all reduced, i.e. for any uniformizing system (V, G, π), G action on V is effective. We can define Riemannian metric and geodesics on orbifolds [CR2]. For any compatible cover $\mathcal{U}$, we can replace it with an equivalent compatible cover $\widetilde{\mathcal{U}}$ which satisfies the following conditions: **(C1)** Any $U \in \widetilde{\mathcal{U}}$ is geodesically convex for some fixed Riemannian metric on M, **(C2)** For any $U_1, U_2 \in \widetilde{\mathcal{U}}$, an induced uniformizing system of $U_1 \cap U_2$ from that of U_1 and an induced uniformizing system of $U_1 \cap U_2$ from that of U_2 are isomorphic to each other. Remark that the condition (C1) guarantees the connectedness of $U_1 \cap U_2$ in (C2). For example, we can choose a suitable positive function $f \colon M \to \mathbb{R}_{>0}$ such that the cover $\{B_r(x) \mid 0 < r < f(x), x \in M\}$ becomes a compatible cover and satisfies (C1), (C2), where $B_r(x)$ denotes the geodesic neighborhood of radius r centered at x. Hereafter we assume the above conditions (C1), (C2) for a compatible cover $\mathcal{U}$ without loss of generality.

Next, we define a morphism between orbifolds. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ be a compatible cover on M, and $(V_\alpha, G_\alpha, \pi_\alpha)$ be a uniformizing system of U_α for $\alpha \in \Lambda$. We set for $U_\alpha, U_\beta \in \mathcal{U}$,

$$\begin{aligned}
(V_{\alpha\beta}, G_{\alpha\beta}, \pi_{\alpha\beta}) &= \text{an induced uniformizing system on } U_{\alpha\beta} = U_\alpha \cap U_\beta \\
&\quad\ \text{by the uniformizing system } (V_\alpha, G_\alpha, \pi_\alpha) \text{ on } U_\alpha, \\
\mathrm{Tran}(U_\alpha, U_\beta) &= \big\{\text{all isomorphisms } (\phi, \tau) \colon (V_{\alpha\beta}, G_{\alpha\beta}, \pi_{\alpha\beta}) \to (V_{\beta\alpha}, G_{\beta\alpha}, \pi_{\beta\alpha})\big\}/\sim, \\
&\quad \text{where } (\phi, \tau) \sim \psi_{\tau(g)}^{-1} \circ (\phi, \tau) \circ \psi_g, \quad g \in G_\alpha.
\end{aligned}$$

When defining $\mathrm{Tran}(U_\alpha, U_\beta)$, we do not fix the choice of induced uniformizing systems $(V_{\alpha\beta}, G_{\alpha\beta}, \pi_{\alpha\beta})$, $(V_{\beta\alpha}, G_{\beta\alpha}, \pi_{\beta\alpha})$, but instead take the equivalence class by the relation $\sim$. By the condition (C2), $\mathrm{Tran}(U_\alpha, U_\beta)$ is non-empty if $U_\alpha \cap U_\beta$ is non-empty. By abuse of notations, for $\imath \in \mathrm{Tran}(U_\alpha, U_\beta)$, we write its representative as $\imath \colon V_\alpha \to V_\beta$ which means that $\imath$ is a map from a connected component of $\pi_\alpha^{-1}(U_\alpha \cap U_\beta) \subset V_\alpha$ to a connected component of $\pi_\beta^{-1}(U_\alpha \cap U_\beta) \subset U_\beta$. Note that this notation is implicit about the domain of the map. When $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$, we can define the composition

$$\circ \colon \mathrm{Tran}(U_\alpha, U_\beta) \times \mathrm{Tran}(U_\beta, U_\gamma) \to \mathrm{Tran}(U_\alpha, U_\gamma).$$

Let $(M, \mathcal{U})$, $(M', \mathcal{U}')$ be orbifolds, and $f \colon M \to M'$ be a continuous map. A **morphism** between orbifolds is given by the data $(f, \{U_\alpha\}, \{U'_\alpha\}, \{f_\alpha\}, \{\rho_{\beta\alpha}\})$,

(1) a subcover $\{U_\alpha\}_{\alpha\in\Lambda} \subset \mathcal{U}$ of M,

(2) a cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of M' (where each $U'_\alpha \in \mathcal{U}'$) which has the same index set as (1), and satisfies $f(U_\alpha) \subset U'_\alpha$, (here, U_α may coincide with U_β with $\alpha \neq \beta$,)

(3) C^∞ lifts $f_\alpha\colon V_\alpha \to V'_\alpha$ of $f|_{U_\alpha}$,

(4) maps $\rho_{\beta\alpha}\colon \mathrm{Tran}(U_\alpha, U_\beta) \to \mathrm{Tran}(U'_\alpha, U'_\beta)$,

which satisfies the following conditions.

(1) For $\imath \in \mathrm{Tran}(U_\alpha, U_\beta)$,

$$\begin{tikzcd} V_\alpha \arrow[r, "f_\alpha"] \arrow[d, "\imath"] & V'_\alpha \arrow[d, "\rho_{\beta\alpha}(\imath)"] \\ V_\beta \arrow[r, "f_\beta"] & V'_\beta \end{tikzcd} \qquad \text{commutes.}$$

(2) If $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$, for $\imath \in \mathrm{Tran}(U_\alpha, U_\beta)$, $\jmath \in \mathrm{Tran}(U_\beta, U_\gamma)$,

$$\begin{tikzcd} V'_\alpha \arrow[r, "\rho_{\gamma\alpha}(\jmath\circ\imath)"] \arrow[d, "\rho_{\beta\alpha}(\imath)"] & V'_\gamma \arrow[d, equal] \\ V'_\beta \arrow[r, "\rho_{\gamma\beta}(\jmath)"] & V'_\gamma \end{tikzcd} \qquad \text{commutes.}$$

The precise meaning of the above diagrams is that the diagram commutes when all maps restricted to their suitably chosen domains. Note that $\mathrm{Tran}(U_\alpha, U_\alpha) \cong G_\alpha$ and $\rho_{\alpha\alpha}\colon \mathrm{Tran}(U_\alpha, U_\alpha) \to \mathrm{Tran}(U'_\alpha, U'_\alpha)$ is a group homomorphism $G_\alpha \to G'_\alpha$ by the condition (2). Two data $(f, \{U_\alpha\}, \{U'_\alpha\}, \{f_\alpha\}, \{\rho_{\beta\alpha}\})$, $(f, \{U_\alpha\}, \{U'_\alpha\}, \{g_\alpha\}, \{\lambda_{\beta\alpha}\})$ are defined to be isomorphic if there exist automorphisms $\delta_\alpha\colon (V'_\alpha, G'_\alpha, \pi'_\alpha) \to (V'_\alpha, G'_\alpha, \pi'_\alpha)$ for $\alpha \in \Lambda$ satisfying

$$g_\alpha = \delta_\alpha \circ f_\alpha, \quad \lambda_{\beta\alpha}(\imath) = \delta_\beta \circ \rho_{\beta\alpha}(\imath) \circ \delta_\alpha^{-1} \quad \text{for } \imath \in \mathrm{Tran}(U_\alpha, U_\beta).$$

When we take a different cover from $(\{U_\alpha\}, \{U'_\alpha\})$, the equivalence relation of two data is defined by using the refinement of covers. We omit the detailed descriptions, see e.g. [ChW]. Roughly speaking, when a cover $\{U_i\}_{i\in I}$ is a refinement of $\{U_\alpha\}_{\alpha\in\Lambda}$, and $\{U'_i\}_{i\in I}$ is a cover of M' such that $U_i \in \mathcal{U}'$ and $f(U_i) \subset U'_i$, we can induce another data $(f, \{U_i\}, \{U'_i\}, \{f_i\}, \{\rho_{ji}\})$ from $(f, \{U_\alpha\}, \{U'_\alpha\}, \{f_\alpha\}, \{\rho_{\beta\alpha}\})$. Two data $(f, \{U_\alpha\}, \{U'_\alpha\}, \{f_\alpha\}, \{\rho_{\beta\alpha}\})$, $(f, \{\tilde{U}_\alpha\}, \{\tilde{U}'_\alpha\}, \{\tilde{f}_\alpha\}, \{\tilde{\rho}_{\beta\alpha}\})$ are defined to be equivalent when there exists a refinement $(\{U_i\}, \{U'_i\})$, over which the two induced data are isomorphic to each other. We can define the composition of two morphisms, so that the set of all orbifolds forms a category.

5.2 Orbifold Loop Spaces

Orbifold loop space is defined as the set of all morphisms from S^1 to an orbifold, where S^1 has a trivial orbifold structure. We consider an orbifold M given by the quotient of a 2−connected smooth manifold U by the action of T^r, where we assume that each point in U has a stabilizer of at most finite order. Our objective is to show the following homeomorphisms,

$$LM_{\mathrm{orb}} \cong LU/LT^r, \quad \widetilde{LM}_{\mathrm{orb}} \cong LU/L_0T^r, \tag{7}$$

where LM_{orb} denotes the orbifold loop space. By the equation (7), we can use the method in section 3 to compute the cohomology of the universal cover of orbifold loop spaces.

In this subsection, we let f denote the projection from U to M instead of π. We first show the next proposition.

Proposition 5.1 *The projection $f\colon U \to M$ is a morphism of orbifolds in the sense defined in the previous subsection.* ∎

(proof) We take a connected, geodesically convex, open covering $\{U_\alpha\}_{\alpha\in\Lambda}$ of U. We assume that each U_α is contained in a tubular neighborhood W_α of some T^r orbit $T^r\cdot x$ ($x\in U$). Furthermore we assume that W_α is of the form

$$W_\alpha \cong T^r \times_{G'_\alpha} V'_\alpha = T^r\times V'_\alpha/(t,v)\sim(tg,g^{-1}v),\ g\in G'_\alpha,$$
$$V'_\alpha = T_xW/T_x(T^k\cdot x),\quad G'_\alpha \text{ is a stabilizer of } x,$$

and that there exists an open convex set $F_\alpha\subset T^r$ satisfying $g\cdot F_\alpha\cap F_\alpha=\emptyset$ for $\forall g\in G_\alpha$ and $U_\alpha\subset F_\alpha\times V'_\alpha$, where $F_\alpha\times V'_\alpha$ is embedded in U as follows,

$$F_\alpha\times V'_\alpha \hookrightarrow T^r\times_{G'_\alpha} V'_\alpha = W_\alpha \hookrightarrow U.$$

Here, the convexity of F_α is defined by some fixed flat, translation invariant metric on T^r. We set $U'_\alpha = f(W_\alpha)$, then $\{U'_\alpha\}_{\alpha\in\Lambda}$ is a compatible cover of M with uniformizing systems $(V'_\alpha, G'_\alpha, \pi'_\alpha)$, where we define π'_α as the composite of the inclusion at the fiber of x, $V'_\alpha\hookrightarrow W_\alpha$ and the projection $f\colon W_\alpha\to M$.

Define a lift f_α of $f|_{U_\alpha}$ as a second projection $f_\alpha: U_\alpha\subset F_\alpha\times V'_\alpha\to V'_\alpha$. Next, we define a map $\rho_{\beta\alpha}\colon \mathrm{Tran}(U_\alpha,U_\beta)\to\mathrm{Tran}(U'_\alpha,U'_\beta)$. Now, we have $\mathrm{Tran}(U_\alpha,U_\beta)=\{1\}$ because U is a manifold, therefore we need only to define one element $\rho_{\beta\alpha}=\rho_{\beta\alpha}(1)\in\mathrm{Tran}(U'_\alpha,U'_\beta)$. Suppose $U_\alpha\cap U_\beta\neq\emptyset$. We put $V'_{\alpha\beta}$ to be a connected component of ${\pi'}_\alpha^{-1}(U'_\alpha\cap U'_\beta)$ which contains $f_\alpha(U_\alpha\cap U_\beta)$, then a uniformizing system $(V'_{\alpha\beta}, G'_{\alpha\beta}, \pi'_{\alpha\beta})$ over $U'_\alpha\cap U'_\beta$ is induced by $(V'_\alpha, G'_\alpha, \pi'_\alpha)$. We would like to define an isomorphism,

$$\rho_{\beta\alpha}\colon (V'_{\alpha\beta}, G'_{\alpha\beta}, \pi'_{\alpha\beta}) \to (V'_{\beta\alpha}, G'_{\beta\alpha}, \pi'_{\beta\alpha})$$

We use the next lemma to construct $\rho_{\beta\alpha}$.

Lemma 5.2 *Let (V,G,π), (V',G',π') be two isomorphic uniformizing systems on a connected open set U. Let $D\subset V$ be a connected open set and suppose we have a smooth map $\phi\colon D\to V'$ such that $\pi=\pi'\circ\phi$. Then ϕ is uniquely extended to an isomorphism $(\tilde\phi,\lambda)\colon (V,G,\pi)\to(V',G',\pi')$. ($\tilde\phi|_D=\phi$.)* ∎

The proof of the above lemma is postponed until the end of the subsection. f_α is an open map (essentially a projection), so that $f_\alpha(U_\alpha\cap U_\beta)$ is a connected open subset of $V'_{\alpha\beta}$. Moreover f_α is a submersion, so that we can take a local section s for f_α defined on a connected set $D\subset f_\alpha(U_\alpha\cap U_\beta)\subset V'_{\alpha\beta}$,

$$s\colon D\to U_\alpha\cap U_\beta,\quad f_\alpha\circ s=\mathrm{id}_D\,.$$

We apply the above lemma for $f_\beta\circ s\colon D\to V'_{\beta\alpha}$, then obtain an isomorphism $\rho_{\beta\alpha}$.

Finally we check that $\rho_{\beta\alpha}$ are independent of the choices made, and that the conditions required for $\rho_{\beta\alpha}$ are satisfied.

- The choice of s. Let $s'\colon D\to U_\alpha\cap U_\beta$ be another section. Take $t(x)\in T^r$ for $x\in D$ such that $t(x)s(x)=s'(x)$. Let $\gamma_x(t)\colon[0,1]\to T^r$ be the shortest path connecting $1\in T^r$ and $t(x)\in T^r$ i.e. $\gamma_x(0)=1,\gamma_x(1)=t(x)$. Noting that $s(x), s'(x)\in(F_\alpha\times V'_\alpha)\cap(F_\beta\times V'_\beta)$, we have

$$\gamma_x(t)s(x)\in(F_\alpha\times V'_\alpha)\cap(F_\beta\times V'_\beta)\quad\text{for any } t\in[0,1]$$

because F_α, F_β is convex. From this it easily follows that $f_\beta\circ s=f_\beta\circ s'$.
- The choice of D. Let D' be another choice. If $D\cap D'\neq\emptyset$, we set D'' to be a connected component of $D\cap D'$. Then two $\rho_{\beta\alpha}$'s determined by D and D'' coincide, and two $\rho_{\beta\alpha}$'s determined by D' and D'' also coincide, therefore the independency follows. Because $V'_{\alpha\beta}$ is connected, it is sufficient to consider the above case.
- The commutativity of the following diagram is clear from the construction.

$$\begin{array}{ccc} U_\alpha\cap U_\beta & \xrightarrow{f_\alpha} & V'_{\alpha\beta} \\ \| & & \downarrow{\scriptstyle\rho_{\beta\alpha}} \\ U_\alpha\cap U_\beta & \xrightarrow{f_\beta} & V'_{\beta\alpha} \end{array}$$

- The commutativity of the following diagram when $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$.

$$\begin{array}{ccc} V'_\alpha & \xrightarrow{\rho_{\gamma\alpha}} & V'_\gamma \\ {\scriptstyle \rho_{\beta\alpha}}\downarrow & & \| \\ V'_\beta & \xrightarrow[\rho_{\gamma\beta}]{} & V'_\gamma \end{array}$$

By the above lemma, it suffices to show that there is a connected open subset $D \subset V'_{\alpha\beta} \cap V'_{\alpha\gamma}$ over which the composite $\rho_{\gamma\beta} \circ \rho_{\beta\alpha}$ is defined and is equal to $\rho_{\gamma\alpha}$. Note that $V'_{\alpha\beta} \cap V'_{\alpha\gamma}$ is non-empty because it contains $f_\alpha(U_\alpha \cap U_\beta \cap U_\gamma)$. We take a local section s for f_α defined on a connected open set $D \subset f_\alpha(U_\alpha \cap U_\beta \cap U_\gamma)$,

$$s\colon D \to U_\alpha \cap U_\beta \cap U_\gamma, \quad f_\alpha \circ s = \mathrm{id}_D .$$

Then $f_\beta \circ s$ extends to an isomorphism $\rho_{\beta\alpha}\colon V'_{\alpha\beta} \to V'_{\beta\alpha}$ by the lemma. We put $D' = f_\beta \circ s(D) \subset f_\beta(U_\alpha \cap U_\beta \cap U_\gamma)$ and take a section s' for f_β defined on D' as

$$s'(f_\beta \circ s(x)) = s(x) \quad \text{for } x \in D, \quad s'\colon D' \to U_\alpha \cap U_\beta \cap U_\gamma, \quad f_\beta \circ s' = \mathrm{id}_{D'} .$$

It is possible because $f_\beta \circ s$ is an embedding. Similarly, $f_\gamma \circ s'$ extends to $\rho_{\gamma\beta}\colon V'_{\beta\gamma} \to V'_{\gamma\beta}$. So we obtain

$$\rho_{\gamma\beta} \circ \rho_{\beta\alpha}|_D = (f_\gamma \circ s') \circ (f_\beta \circ s) = f_\gamma \circ s = \rho_{\gamma\alpha}|_D.$$

∎

By this proposition, given a loop γ in U, the composite $f \circ \gamma$ gives an orbifold loop in M, i.e. there is a map $LU \to LM_{\mathrm{orb}}$. Next proposition shows that this map induces a map $LU/LT^r \to LM_{\mathrm{orb}}$.

Proposition 5.3 *Let X be a smooth manifold and $h\colon X \to U$, $c\colon X \to T^r$ be C^∞ maps. Then two morphisms $f \circ h$ and $f \circ (c \cdot h)$ are identical, where $f\colon U \to M$ is the projection and $c \cdot h$ denotes the pointwise multiplication.* ∎

(proof) We use the notations used in the proof of Proposition 5.1. We can decompose two morphisms as $f \circ h = f \circ \mathrm{id}_U \circ h$, $f \circ (c \cdot h) = f \circ (c \cdot \mathrm{id}_U) \circ h$, therefore it suffices to prove the case $X = U$ and $h = \mathrm{id}_U$. By taking the refinement if necessary, we can take a covering $\{U_\alpha\}_{\alpha\in\Lambda}$ of U such that $U_\alpha \subset F_\alpha \times V'_\alpha$ and $(c \cdot \mathrm{id}_U)(U_\alpha) \subset (t_\alpha \cdot F_\alpha) \times V'_\alpha$ for some $t_\alpha \in T^r$. We can easily argue that by this choice and the same procedure as in the proof of previous proposition, we obtain the same data for two morphisms f and $f \circ (c \cdot \mathrm{id}_U)$. ∎

Proposition 5.4 *There exist homeomorphisms $LU/LT^r \cong LM_{\mathrm{orb}}$, $LU/L_0T^r \cong \widetilde{LM}_{\mathrm{orb}}$* ∎

(proof) So far, we showed that there exists a map $LU/LT^r \to LM_{\mathrm{orb}}$. Now we construct the inverse map. Let $\gamma\colon S^1 \to M$ be an orbifold loop given by the data $(\gamma, \{U_i\}, \{U'_i\}, \{\gamma_i\}, \{\rho_{ji}\})_{i=1}^n$. We assume that each U_i is an interval in $S^1 \cong \mathbb{R}/\mathbb{Z}$ i.e. $U_i = (a_i, b_i)$, $U_i \cap U_{i+1} \neq \emptyset$, $U_i \cap U_j = \emptyset$ for $|i-j| \geqq 2$ except for $U_1 \cap U_n \neq \emptyset$, and that each U'_i is an image of a tubular neighborhood W_i of a T^r orbit in U of the form $W_i = T^r \times_{G'_i} V'_i$. Therefore U'_i is uniformized by (V'_i, G'_i, π'_i). Because S^1 has a trivial orbifold structure, each ρ_{ji} can be thought to be an element of $\mathrm{Tran}(U'_i, U'_j)$. We remark that $\rho_{ii} = \mathrm{id}$ because ρ_{ii} is a group homomorphism.

The first step is to lift the map $\gamma|_{U_1}$ to U. We take an arbitrary curve $c_1\colon U_1 \to T^r$, and then put $\tilde{\gamma}(\cdot) = [c_1(\cdot), \gamma_1(\cdot)]\colon U_1 \to T^r \times_{G'_1} V'_1 = W_1 \subset U$.

The second step is to extend the lift $\tilde{\gamma}$ to the domain $U_1 \cup U_2$. On the interval $U_1 \cap U_2 = (a_2, b_1)$, $\tilde{\gamma}$ is already determined as a map with values in W_2. We lift the map $\tilde{\gamma}\colon U_1 \cap U_2 \to T^r \times_{G'_2} V'_2$ to $T^r \times V'_2$ as

$$\tilde{\tilde{\gamma}}(x) = (\hat{c}_2(x), \hat{\gamma}_2(x)) \in T^r \times V'_2, \qquad x \in (a_2, b_1).$$

Let $V'_{12} \subset {\pi'_1}^{-1}(U'_1 \cap U'_2)$ and $\hat{V}'_{21} \subset {\pi'_2}^{-1}(U'_1 \cap U'_2)$ be connected components which contain $\gamma_1(U_1 \cap U_2)$ and $\hat{\gamma}_2(U_1 \cap U_2)$ respectively, and let (V'_1, G'_1, π'_1) and $(\hat{V}'_{21}, \hat{G}'_{21}, \hat{\pi}'_{21})$ be corresponding induced uniformizing systems over $U'_1 \cap U'_2$.

Lemma 5.5 *There exists an isomorphism* $\hat{\rho}_{21}\colon (V'_{12}, G'_{12}, \pi'_{12}) \to (\hat{V}'_{21}, \hat{G}'_{21}, \hat{\pi}'_{21})$ *such that* $\hat{\rho}_{21}(\gamma_1(x)) = \hat{\gamma}_2(x)$ *for* $x \in U_1 \cap U_2$. ∎

(proof of Lemma) The construction of $\hat{\rho}_{21}$ is similar to that of $\rho_{\beta\alpha}$ in Proposition 5.1. For any $x_0 \in U_1 \cap U_2$, we take the following section s for the second projection $T^r \times V'_1 \to V'_1$ as,

$$s\colon V'_{12} \to T^r \times V'_1, \quad v \mapsto (c_1(x_0), v).$$

Let $pr\colon T^r \times V'_1 \to T^r \times_{G'_1} V'_1$ be a projection. Note that $pr \circ s$ has values in $(T^r \times_{G'_1} V'_1) \cap (T^r \times_{G'_2} V'_2)$. Then we lift $pr \circ s\colon V'_{12} \to T^r \times_{G'_2} V'_2$ to the map $\tilde{s}\colon V'_{12} \to T^r \times \hat{V}'_2$ such that $\tilde{s}(\gamma_1(x_0)) = \tilde{\tilde{\gamma}}(x_0)$. The composition of $\tilde{s}$ and the second projection $T^r \times V'_2 \to V'_2$ gives a map $\phi\colon V'_{12} \to V'_{21}$. By Lemma 5.2, ϕ determines an isomorphism $\hat{\rho}_{21}\colon (V'_{12}, G'_{12}, \pi'_{12}) \to (\hat{V}'_{21}, \hat{G}'_{21}, \hat{\pi}'_{21})$ such that $\hat{\rho}_{21}(\gamma_1(x_0)) = \hat{\gamma}_2(x_0)$. This isomorphism does not depend on the choice of x_0 in $U_1 \cap U_2$ because the above ϕ depends continuously on x_0 but the set of isomorphisms is discrete. So it follows that $\hat{\rho}_{21}(\gamma_1(x)) = \hat{\gamma}_2(x)$ for any $x \in U_1 \cap U_2$, and the lemma is proved. ∎

On the other hand, ρ_{21} is an isomorphism from $(V'_{12}, G'_{12}, \pi'_{12})$ to $(V'_{21}, G'_{21}, \pi'_{21})$. Two isomorphisms are related as $\rho_{21} = \psi_{g_0} \circ \hat{\rho}_{21}$ for some $g_0 \in G'_2$, so that we have

$$\gamma_2(x) = \rho_{21} \circ \gamma_1(x) = g_0 \cdot (\hat{\rho}_{21} \circ \gamma_1(x)) = g_0 \cdot \hat{\gamma}_2(x) \quad \text{for } x \in U_1 \cap U_2.$$

We replace the lift $\tilde{\tilde{\gamma}} = (\hat{c}_2, \hat{\gamma}_2)$ with $\tilde{\tilde{\gamma}}' \overset{\text{def}}{=} (c_2, \gamma_2) \overset{\text{def}}{=} (\hat{c}_2 \cdot g_0^{-1}, g_0 \cdot \hat{\gamma}_2)$ which is also the lift of $\tilde{\gamma}$ and defined on $U_1 \cap U_2$. Then we extend c_2 to the domain U_2 arbitrarily and define

$$\tilde{\gamma} = [c_2, \gamma_2]\colon U_2 \to T^r \times_{G'_2} V'_2 = W_2 \subset U.$$

We can continue this process and obtain the lift $\tilde{\gamma}\colon U_1 \cup \cdots \cup U_n \to U$, where it is easy to make the extension to U_n coincide with the map $\tilde{\gamma}|_{U_1}$ over $U_1 \cap U_n$.

From this construction, we can obtain a lift $\tilde{\gamma}$ whose indeterminacy comes only from the multiplication by an element in LT^r. Thus we get a map $LM_{\text{orb}} \to LU/LT^r$ which is the inverse to $LU/LT^r \to LM_{\text{orb}}$. To show the isomorphism $LU/L_0T^r \cong \widetilde{LM}_{\text{orb}}$, it suffices to show that LU/L_0T^r is simply connected. Stabilizers of the action of L_0T^r on LU at each point in LU are of finite order, so that the projection $LU \to LU/L_0T^r$ has a covering homotopy property for a point. Therefore we obtain the following exact sequence,

$$\pi_1(LU) \to \pi_1(LU/L_0T^r) \to \pi_0(L_0T^r).$$

Therefore LU/L_0T^r is simply connected and the proof is completed. ∎

Note that each stabilizer of the action of L_0T^r on LU is a finite group contained in the set of constant loops $T^r \subset L_0T^r$. Thus the set of contracting based loops $\Omega_0 T^r$ acts on LU/T^r freely and we obtain a principal $\Omega_0 T^r$ bundle $LU/T^r \to LU/L_0T^r$. This bundle is trivial because $\Omega_0 T^r$ is contractible, so that we have a S^1 equivariant homotopy equivalence,

$$LU/L_0T^r \sim LU/T^r.$$

From this equivalence, we can deduce the following isomorphisms as stabilizers of T^r action are of finite order.

$$H^*(\widetilde{LM}_{\text{orb}}) \cong H^*_{T^r}(LU), \quad H^*_{S^1}(\widetilde{LM}_{\text{orb}}) \cong H^*_{T^r \times S^1}(LU).$$

These isomorphisms admit us to compute the (equivariant) cohomology of $\widetilde{LM}_{\text{orb}}$. In this orbifold case, we can also think connection forms $\hat{\theta}_i$ on U and its curvature forms $\hat{p}_i = d\hat{\theta}_i$. The conditions imposed on connection forms $\hat{\theta}_i$ are

$$\imath_i \hat{\theta}_j = \delta_{ij}, \quad \mathcal{L}_i \hat{\theta}_j = 0,$$

where $\imath_i$, $\mathcal{L}_i$ denote the contraction and the Lie derivation by $i-$th fundamental vector field of T^r. These curvature forms are pulled back from forms on orbifold M which form a basis of the second cohomology group of M over $\mathbb{R}$.

Let p_i be a cohomology class represented by $\hat{p}_i$ and $(\bigwedge V, \delta_0)$ be the Sullivan's minimal model for M. V can be decomposed as $V = H^2(M,\mathbb{R}) \oplus V'$, where V' denotes the higher degree part. As before, δ_0 induces a differential δ' on $\bigwedge V' \otimes \mathbb{R}[t_1, \dots, t_r]$ and we set

$$Z^i = \ker(d\colon \Omega^i_{\bigwedge V'}[t_1, \dots, t_r] \to \Omega^{i+1}_{\bigwedge V'}[t_1, \dots, t_r]).$$

In the same way when we proved Theorem 3.9, we obtain the following.

Theorem 5.6 *Let U be a $2-$connected manifold with T^r action. We assume that each point in U has a finite stabilizer. Let $M = U/T^r$ be a quotient orbifold. Then the following hold.*

(1) *The universal covering of the orbifold loop space $\widetilde{LM}_{\mathrm{orb}}$ is homeomorphic to LU/L_0T^r, where L_0T^r denotes the set of contracting loops in T^r.*

(2) *The ordinary and equivariant cohomology rings of $\widetilde{LM}_{\mathrm{orb}}$ are of the forms*

$$\begin{aligned} H^*(\widetilde{LM}_{\mathrm{orb}}, \mathbb{R}) &\cong H^*(\Omega_{\bigwedge V'|\mathbb{R}}[t_1, \dots, t_r], \delta'), \\ H^*_{S^1}(\widetilde{LM}_{\mathrm{orb}}, \mathbb{R}) &\cong \mathbb{R}[t_1, \dots, t_r, \hbar] \oplus \bigoplus_{i>0} H^*(Z^i, \delta'). \end{aligned}$$

(3) *The variables t_i on the right hand side of the latter isomorphism correspond to the classes represented by the following forms*

$$t_i = \int_{S^1} \mathrm{ev}_t^*(\hat{p}_i) dt + \hbar \int_{D^2} g^*(\hat{p}_i),$$

where $g\colon D^2 \to U$ is a disk contracting the loop $\gamma \in LU/L_0T^r = \widetilde{LM}_{\mathrm{orb}}$.

(4) *The localization of $H^*_{S^1}(\widetilde{LM_{\mathrm{orb}}})$ with respect to $\hbar$ becomes*

$$H^*_{S^1}(\widetilde{LM}_{\mathrm{orb}}, \mathbb{R}) \otimes_{\mathbb{R}[\hbar]} \mathbb{R}[\hbar, \hbar^{-1}] \cong \mathbb{R}[t_1, \dots, t_r, \hbar, \hbar^{-1}].$$

∎

Remark 5.7 Chen and Ruan proposed an "orbifold cohomology theory" defined for orbifolds [CR1]. However, cohomology rings of M and $\widetilde{LM_{\mathrm{orb}}}$ (this is an infinite dimensional orbifold) in the above theorem are not Chen-Ruan's orbifold cohomology rings but ordinary ones.

Example 5.8 Any symplectic toric orbifold can be written as a quotient of $2-$connected manifold by a T^r action, see e.g. [Aud]. Thus the above theorem can be applied to toric orbifolds. ∎

We close this subsection with the proof of Lemma 5.2.

(proof of Lemma 5.2) Because (V, G, π) and (V', G', π') are isomorphic to each other, we can take an isomorphism $(\psi, \tau)\colon (V, G, \pi) \to (V', G', \pi')$. Put $D' = \psi(D)$ and consider a map $\alpha \stackrel{\mathrm{def}}{=} \phi \circ \psi^{-1}\colon D' \to V'$. There exists at least one $g_x \in G'$ for $x \in D'$ such that $\alpha(x) = g_x \cdot x$, because $\pi' \circ \phi = \pi$. We would like to take a common g_x for all $x \in D'$. We define $H(x)$ for $x \in D'$ to be

$$H(x) = \{g \in G' \mid \alpha(x) = g \cdot x\}.$$

We claim that the set $\{x \in D' \mid \sharp(H(x)) = 1\}$ is dense in D'. Let $Stab(x)$ denotes the stabilizer at x in D', then

$$\begin{aligned} \{x \in D' \mid \sharp(H(x)) = 1\} &= \{x \in D' \mid Stab(x) = \{e_{G'}\}\} \\ &= D' - \bigcup_{g \in G' - \{e_{G'}\}} \{x \in D' \mid g \cdot x = x\}. \end{aligned}$$

By the effectivity of the action of G' and the connectedness of D', we deduce that $\{x \in D' \mid g \cdot x = x\}$ is a submanifold of D' whose dimension is strictly less than $\dim D'$. The claim is proved because G' is finite. Next put $B_g = \{x \in D' \mid H(x) = \{g\}\}$ for g in G'. Then,

$$\alpha(x) = g \cdot x \qquad \text{for } x \in \overline{B}_g,$$

where $\overline{(\cdots)}$ denotes the closure in D'. Let $S_g \colon V' \to V'$ denote the map $x \mapsto g \cdot x$. If $\overline{B}_{g_1} \cap \overline{B}_{g_2}$ is non-empty for $g_1 \neq g_2$, by the above equation, the next relation holds for $p \in \overline{B}_{g_1} \cap \overline{B}_{g_2}$,

$$d_p S_{g_1} = d_p \alpha = d_p S_{g_2}.$$

This contradicts that $g_1 \cdot g_2^{-1}$ acts on $T_p V'$ non-trivially, consequently we have $\overline{B}_{g_1} \cap \overline{B}_{g_2} = \emptyset$. Finally,

$$D' = \overline{\{x \in D' \mid \sharp(H(x)) = 1\}} = \bigcup_{g \in G'} \overline{B_g},$$

and D' is connected, therefore there exists a unique $g_0 \in G'$ such that $D' = \overline{B_{g_0}}$. Thus $\phi \circ \psi^{-1} = \alpha = S_{g_0}$, and we can extend ϕ to a morphism $(S_{g_0} \circ \psi, g_0 \cdot \tau \cdot g_0^{-1})$. ∎

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