QUANTUM RING AND QUANTUM LEFSCHETZ

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0.1. **Modification of quantum rings.** We study the quantum Lefschetz with respect to a nef line bundle $L \to X$ with $c_1(L) = v$. Let $Y \subset X$ denote a smooth hypersurface with respect to L. We choose a nef integral basis $\{p_1, \dots, p_r\}$ of $H^2(X)$ such that $v = \sum_{a=1}^r v^a p_a$ with $v^a \ge 0$. Let $Q = (Q_1, \dots, Q_r)$ denote the Novikov variables dual to $\{p_1, \dots, p_r\}$; we write

$$Q^d = Q_1^{p_1 \cdot d} \cdots Q_r^{p_r \cdot d}$$

with $d \in H_2(X)$. We also choose a basis $\{\phi_i\}_{i=0}^s$ of $H^*(X)$ such that $\phi_0 = 1$, $\phi_a = p_a$ for $1 \le a \le r$. Let $\{t^i\}_{i=0}^s$ denote the dual co-ordinate system on $H^*(X)$. We write $t = \sum_{i=0}^s t^i \phi_i$. Let $\mathrm{Eff}(X) \subset H_2(X, \mathbb{Z})$ denote the set of classes of effective curves.

Write the J-function in the form:

$$J = \sum_{d \in \mathrm{Eff}(X)} Q^{d+p/z} J_d(t,z) \tag{1}$$

Here we use a redundant co-ordinate system $(Q_1, \dots, Q_r, t^0, \dots, t^s)$ and adopt the convention that J contains the factor $Q^{p/z}$, so that J satisfies the *divisor equation*

$$Q_a \partial_{Q_a} J = \partial_{t^a} J, \qquad 1 \le a \le r.$$

For simplicity, we write $\partial_i = \partial/\partial t^i$ and $\boldsymbol{\partial}_a = Q_a \partial / \partial Q_a$, so that $\boldsymbol{\partial}_a J = \partial_a J$. The *hypergeometric modification* with respect to the line bundle L is given by:

$$\mathrm{modif}(J) = \sum_{d \in \mathrm{Eff}(X)} Q^{d+p/z} J_d(t,z) \prod_{i=1}^{v \cdot d} (v + iz).$$

Note that $J \mapsto \mathrm{modif}(J)$ is a well-defined operation for any cohomology-valued power series J of the form (1). The proof of the following lemma is straightforward:

Lemma 1. *We have*

$$\mathrm{modif}(\partial_i J) = \partial_i \, \mathrm{modif}(J), \quad \mathrm{modif}(\boldsymbol{\partial}_a J) = \boldsymbol{\partial}_a \, \mathrm{modif}(J),$$
$$\mathrm{modif}(Q_a J) = Q_a \left(\prod_{i=1}^{v^a} (z \boldsymbol{\partial}_v + iz) \right) \mathrm{modif}(J)$$

where $\boldsymbol{\partial}_v := \sum_{a=1}^r v^a \boldsymbol{\partial}_a$. *In particular, if J satisfies the divisor equation, so is* $\mathrm{modif}(J)$.

Remark 2. The differential operators $Q_a \prod_{i=1}^{v^a} (z \boldsymbol{\partial}_v + iz)$, $a = 1, \dots, r$ commute each other and the map

$$Q_a \mapsto Q_a \prod_{i=1}^{v^a} (z \boldsymbol{\partial}_v + iz), \quad z \boldsymbol{\partial}_a \mapsto z \boldsymbol{\partial}_a$$

defines an automorphism of the Weyl algebra $\mathbb{C}\langle Q_1, \dots, Q_r, z\boldsymbol{\partial}_1, \dots, z\boldsymbol{\partial}_r, z \rangle$.

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By the lemma, we see that if J satisfies a differential relation:

$$RJ = 0$$

for some differential operator $R = R(Q, t, z\boldsymbol{\partial}, z\partial, z)$, then then modif$(J)$ satisfies

$$\widehat{R}\,\mathrm{modif}(J) = 0, \quad \text{with } \widehat{R} = R|_{Q_a \to Q_a \prod_{i=1}^{v^a}(z\boldsymbol{\partial}_v + iz)}.$$

Recall the following basic fact:

Proposition 3. *If the J-function of X satisfies a differential equation $RJ = 0$ for $R = R(Q, t, z\boldsymbol{\partial}, z\partial, z)$, then we have a relation $R(Q, t, z, p\star, \phi\star, 0)1 = 0$ in the quantum cohomology algebra of X. Conversely, for any relation $f(Q, t, z, p\star, \phi\star)1 = 0$ in the quantum cohomology, there is a differential operator $R(Q, t, z\boldsymbol{\partial}, z\partial, z)$ such that $R(Q, t, p\star, \phi\star, 0) = f$ and that $RJ = 0$.*

Let $\tau = \tau(t)$ denote the mirror map. This is the canonical parameter for the position of modif(J) in the ruling of the L-twisted Givental cone[1], i.e. a cohomology class $\tau \in H^*(X)$ such that

$$\mathrm{modif}(J) \in zT_\tau$$

where $\mathcal{L}^{\mathrm{tw}} = \bigcup_\tau zT_\tau$ denotes the ruling of the twisted Givental cone such that $zT_\tau \cap (-z + z\mathcal{H}_-) = \{-z + \tau + O(z^{-1})\}$. Since we have added the prefactor $Q^{p/z}$ to the J-function, the mirror map is of the form $\tau(t) = p \log Q + t + O(Q)$. Then we have

Proposition 4. *If* modif(J) *satisfies a differential equation $\widehat{R}\,\mathrm{modif}(J) = 0$ for some $\widehat{R} = \widehat{R}(Q, t, z\boldsymbol{\partial}, z\partial, z)$, then we have the relation:*

$$\widehat{R}(Q, t, (\boldsymbol{\partial}_1\tau)\star_\tau, \dots, (\boldsymbol{\partial}_r\tau)\star_\tau, (\partial_0\tau)\star_\tau, \dots, (\partial_s\tau)\star_\tau, 0)1 = 0$$

in the twisted quantum cohomology of X (thus in the quantum cohomology of the hypersurface Y with respect to L) with $\tau = \tau(t)$.

Corollary 5. *If we have a relation $f(Q, t, p, \phi) = 0$ in the big quantum cohomology algebra of X (with parameter t), then we have a relation*

$$f(Q_1(\boldsymbol{\partial}_v\tau)^{v^1}, \dots, Q_r(\boldsymbol{\partial}_v\tau)^{v^r}, t, \boldsymbol{\partial}_1\tau, \dots, \boldsymbol{\partial}_r\tau, \partial_0\tau, \dots, \partial_s\tau) = 0 \tag{2}$$

in the L-twisted quantum cohomology of X with the parameter $\tau = \tau(t)$.

Remark 6. Under the current convention, the mirror map satisfies the divisor equation $\boldsymbol{\partial}_a\tau = \partial_a\tau$.

The corollary says that the twisted quantum cohomology can be obtained from the untwisted quantum cohomology just by a co-ordinate change: more precisely, if we regard the spectrum of the quantum ring as a Lagrangian subvariety in the cotangent bundle of $H^2(X, \mathbb{C}^\times) \times H^*(X)$, then the *Lagrangian subvariety is transformed by a certain symplectic transformation under the twisting.*

Example 7. Consider $X = \mathbb{P}^n$ and line bundle $L = \mathcal{O}(k)$ with $0 < k < n$. In this case the mirror map is trivial for small quantum cohomology ($t = 0$). Let p be a positive generator of $H^2(\mathbb{P}^n)$. The relation in small quantum cohomology

$$p^{n+1} - Q = 0$$

[1]More precisely, we consider the non-equivariant limit of the twisted cone. We do not need to restrict ourselves to the case where modif(J) has a good asymptotics: we may allow the hypersurface to be non-weak-Fano (the mirror map τ still makes sense).

is turned into the relation

$$p^{n+1} - Q(kp)^k = 0.$$

Factoring p out, we get a relation $p^n = Qk^k p^{k-1}$ in the small quantum cohomology ring of the degree k hypersurface.

Example 8. In the above example, consider the case where $k = n$, i.e. the hypersurface is of index 1. In this case the mirror map is nontrivial: since the mirror map is given by $\tau(t = 0) = n!Q + p \log Q$, we obtain the relation

$$(p + n!Q)^{n+1} - Qn^n(p + n!Q)^n = 0.$$

Again we remove one factor of $p + n!Q$ to get a relation for the hypersurface.

Question 9. The twisted quantum cohomology contains a "trivial" factor which is the kernel of the restriction map $H^*(X) \to H^*(Y)$. In terms of the Lagrangian subvariety, we have a closed embedding $\mathrm{Spec}(QH^*_{\mathrm{amb}}(Y)) \hookrightarrow \mathrm{Spec}(QH^*_{\mathrm{tw}}(X))$. Can we see this directly from the relations (2)?

A possible answer to this question is as follows. Suppose v is ample. If we have a relation $f(Q, v) = 0$ in the small quantum cohomology of X such that $f(0, 0) = 0$. Then the modified relation $f(Q_1(\boldsymbol{\partial}_v\tau)^{v^1}, \ldots, Q_r(\boldsymbol{\partial}_v\tau)^{v^r}, \boldsymbol{\partial}_v\tau)$ always contain a factor of $\boldsymbol{\partial}_v\tau$. Then it is likely that we can factor out $\boldsymbol{\partial}_v\tau$ from the relation on the hypersurface.

Let us give a more precise answer: suppose again that v is ample and that we have a differential operator $R = R(Q, t, z\boldsymbol{\partial}, z\partial, z)$ that annihilates J and belongs to the *right* ideal[2] generated by Q^d with $d \neq 0$ and $z\boldsymbol{\partial}_v$, i.e. R is of the form

$$R = z\boldsymbol{\partial}_v S + \sum_{d\neq 0} Q^d R_d.$$

Then it follows that

$$\left[z\boldsymbol{\partial}_v S' + \sum_{d\neq 0} \left(\prod_{i=0}^{v\cdot d-1} (z\boldsymbol{\partial}_v - iz) \right) Q^d R'_d \right] \mathrm{modif}(J) = 0$$

for some differential operators R'_d and S'. By factoring out $z\boldsymbol{\partial}_v$ from the left, we have

$$z\boldsymbol{\partial}_v T \,\mathrm{modif}(J) = 0, \quad \text{with } T := S' + \sum_{d\neq 0} \left(\prod_{i=1}^{v\cdot d-1} (z\boldsymbol{\partial}_v - iz) \right) Q^d R'_d.$$

Thus $T \,\mathrm{modif}(J)$ is a power series annihilated by $z\boldsymbol{\partial}_v$. Using the fact that v is ample, we find that $T \,\mathrm{modif}(J)$ is of the form $Q^{p/z}\phi(t, z)$ for some Q-independent cohomology class $\phi(t, z)$ such that $v \cup \phi(t, z) = 0$. Again using the fact that v is ample, we find $i^*\phi(t, z) = 0$ for the inclusion $i\colon Y \to X$ ($\phi(t, z)$ is "coprimitive"). This implies

$$Ti^* \,\mathrm{modif}(J) = 0$$

and thus T gives rise to a relation in the hypersurface:

$$\frac{R(Q_1(\boldsymbol{\partial}_v\tau)^{v^1}, \ldots, Q_r(\boldsymbol{\partial}_v\tau)^{v^r}, t, \boldsymbol{\partial}\tau, \partial\tau, 0)}{\boldsymbol{\partial}_v\tau} = 0$$

Question 10. We have observed that the quantum cohomology are related by a certain co-ordinate change under hyperplane sections. Can we find any (symplectic) "invariants" of quantum rings preserved by this operation?

[2] This is reminiscent of the method of Mann-Mignon.

0.2. **Subcanonical hyperplane sections.** Suppose that X is a Fano manifold and $Y \subset X$ is a Fano hypersurface with respect to a line bundle L whose first Chern class is a multiple of $c_1(X) = -K_X$. Let ℓ be the Fano index and we write $-K_X = \ell h$ and $c_1(L) = kh$ for some ample class h and $0 < k < \ell$. Suppose we have a relation in the small quantum cohomology of X:

$$f(Q,h) = 0.$$

We may assume that the relation is homogeneous

$$f(\lambda^{\ell w_1} Q_1, \dots, \lambda^{\ell w_r} Q_r, \lambda h) = \lambda^{|f|} f(Q,h),$$

where we set $c_1(X) = \sum_{a=1}^r \ell w_a p_a$ with $w_a \in \mathbb{Z}_{\ge 0}$. The mirror map is of the form:

$$\tau(t=0) = \begin{cases} p \log Q & \text{if } \ell - k > 1 \\ p \log Q + k! \sum_{h\cdot d = 1} \left\langle [\mathrm{pt}] \psi^{\ell-2} \right\rangle_{0,1,d}^X Q^d & \text{if } \ell - k = 1 \end{cases}$$

Thus we get a relation in the twisted quantum cohomology:

$$f(Q_1 (kh')^{k w_1}, \dots, Q_r (kh')^{k w_r}, h') = 0 \qquad \text{in } QH_{\mathrm{tw}}$$

where

$$h' = \boldsymbol{\partial}_h \tau = \begin{cases} h & \text{if } \ell - k > 1 \\ h + k! \sum_{h\cdot d = 1} \left\langle [\mathrm{pt}] \psi^{\ell-2} \right\rangle_{0,1,d}^X Q^d & \text{if } \ell - k = 1. \end{cases} \tag{3}$$

Using the above homogeneity, we get

$$(kh')^{k|f|/\ell} f(Q_1, \dots, Q_r, h'^{1-(k/\ell)} k^{-k/\ell}) = 0 \qquad \text{in } QH_{\mathrm{tw}}.$$

Note that this is a polynomial in h' (non-integral powers do not appear). Thus the eigenvalues of h' in the twisted quantum cohomology is either 0 or a solution x to the equation $f(Q, k^{-k/\ell} x^{1-k/\ell}) = 0$. This gives the following corollary:

Corollary 11. *For a non-zero eigenvalue u' of $(h' \star^Y)$, there is an eigenvalue u of $(h \star^X)$ such that*

$$(u')^{l-k} = k^k u^k. \tag{4}$$

Question 12. Can we show the converse statement: is every complex number u' satisfying (4) for some eigenvalue u of $h \star^X$ an eigenvalue of $h' \star^Y$?

0.3. **(Super?) characteristic polynomials.** Suppose again that X is a Fano manifold and that Y is a Fano hypersurface in X. Now $v = c_1(L)$ is not necessarily proportional to $c_1(X)$ but we assume that v is ample. We consider the characteristic polynomial $P(Q,\lambda)$ of $v\star$ in the small quantum cohomology of X, that is:

$$P(Q,\lambda) = \det(\lambda - v\star) = \lambda^N - \lambda^{N-1} \operatorname{tr}(v\star) + \cdots + (-1)^N \det(v\star)$$

where $N = \dim H^*(X)$. We can also write this as $P(Q,\lambda) = \det((\lambda - v)\star^X)$. We could also consider the super-characteristic polynomial:

$$P(Q,\lambda) = \operatorname{sdet}(\lambda - v\star) = \frac{\det_{H^{\mathrm{ev}}}(\lambda - v\star)}{\det_{H^{\mathrm{od}}}(\lambda - v\star)} = \lambda^{\chi(X)} - \lambda^{\chi(X)-1} \operatorname{str}(v\star) + \cdots.$$

I am not sure which is better. In the following, I only consider the usual characteristic polynomial. It gives the relation $P(Q,v) = 0$ in the quantum ring $QH^*(X)$. Thus we have a relation:

$$P'(Q,w) := P(Q_1 w^{v^1}, \dots, Q_r w^{v^r}, w) = 0$$

in the twisted quantum cohomology, where $w = \boldsymbol{\partial}_v \tau$ is given by

$$w = v + (v \cdot d)! \sum_{(c_1(X) - v)\cdot d = 1} \left\langle [\mathrm{pt}] \psi^{c_1(X)\cdot d - 2} \right\rangle_{0,1,d}^{X} (v \cdot d) Q^d.$$

Note that $w = v$ unless $c_1(X) - v$ is a primitive class. The polynomial $P(Q, v)$ is weighted homogeneous of degree N with respect to the standard degree of Q_a's and $\deg v = 1$. The new polynomial $P'(Q, w)$ is also weighted homogeneous but with different degree of Q_a; by the assumption that Y is Fano, the degrees of Q_a are still positive. Therefore we deduce that $P'(Q, w)$ is still a monic polynomial of degree N with respect to w.

Question 13. Is $P'(Q, w)$ divisible by w^m with

$$m = \sharp\{v\text{-Lefschetz blocks}\} = \dim \mathrm{Ker}(i^* \colon H^*(X) \to H^*(Y))?$$

This would follow if $v\star^X$ preserves each of Lefschetz blocks since then $P(Q, v)$ decomposes into the product of characteristic polynomials on each block.

Question 14. If the answer to the previous question is yes, is $P'(Q, w)/w^m$ the characteristic polynomial of $w\star^Y$ on the ambient part quantum cohomology $QH^*_{\mathrm{amb}}(Y)$? (Compare with Question 9.)

Question 15. What is the characteristic polynomial of $w\star^Y$ on the full quantum cohomology $QH^*(Y)$ including the primitive part? Is it just $P'(Q, w) w^{\dim H^*(Y) - \dim H^*(X)}$? [Or, if we are working with super-characteristic polynomials, we would ask if $P'(Q, w) w^{\chi(Y) - \chi(X)}$ is a super-characteristic polynomial of $w\star^Y$.] For hypersurfaces in $\mathbb{P}^n$, this seems to be true (including for index-one hypersurfaces) by the supertrace computation.

0.4. **Complete relations of quantum cohomology of hypersurfaces.** We describe a complete set of relations of $QH^*(Y)$ in terms of $QH^*(X)$ assuming that $v = c_1(L)$ satisfies the following:

$$(\dagger) \qquad v \cup \phi = 0 \implies i^*(v) = 0$$

for all $\phi \in H^*(X)$, where $i \colon Y \to X$ is the inclusion. Note that the converse $i^*\phi = 0 \Rightarrow v \cup \phi = 0$ is always true because $i_* i^* \phi = v \cup \phi$. The condition (†) holds if v is ample. A result of Mavlyutov shows that semiample hypersurfaces of toric orbifolds satisfy this condition (†).

Question 16. When does the condition (†) hold in general?

We rearrange the basis $\phi_0, \phi_1, \ldots, \phi_s$ of $H^*(X)$ in such a way that $\phi_{u+1}, \ldots, \phi_s$ form a basis of $\mathrm{Ker}(v\cup) = \mathrm{Ker}(i^* \colon H^*(X) \to H^*(Y))$. Let $A^k_{i,j}(Q, t)$, $C^j_i(Q, t)$ denote the structure constants of quantum cohomology of X:

$$\phi_i \star \phi_j = \sum_{k=0}^{s} A^k_{i,j}(Q, t) \phi_k, \quad v \star \phi_i = \sum_{j=0}^{s} C^j_i(Q, t) \phi_j.$$

In this section we work over the ring $\Lambda := \mathbb{C}[\![Q, t]\!]$ of formal power series. We use:

- $t = \sum_{j=0}^{s} t^j \phi_j$ for the parameter of the big quantum cohomology $QH^*(X)$;
- $\tau = \sum_{j=0}^{s} \tau^j \phi_j$ for the parameter of the twisted quantum cohomology $QH^*_{\mathrm{tw}}(X)$;
- $i^*\tau = \sum_{j=0}^{u} \tau^j (i^* \phi_j)$ for the parameter of the ambient part big quantum cohomology $QH^*_{\mathrm{amb}}(Y)$.

These parameters are related by the mirror map $\tau = \tau(t)$. By changing the base, we obtain the following $\Lambda := \mathbb{C}[\![Q, t]\!]$-algebras:

$$QH_{\rm tw}(X)_\Lambda := QH_{\rm tw}(X) \otimes_{\mathbb{C}[\![Q,\tau]\!]} \Lambda,$$
$$QH_{\rm amb}(Y)_\Lambda := QH_{\rm amb}(Y) \otimes_{\mathbb{C}[\![Q_Y,i^*\tau]\!]} \Lambda,$$

where we regard Λ as a $\mathbb{C}[\![Q,\tau]\!]$-algebra or as a $\mathbb{C}[\![Q_Y, i^*\tau]\!]$-algebra via the mirror map and the natural map of Novikov variables $Q_Y^d \mapsto Q^{i_* d}$ with $d \in H_2(Y,\mathbb{Z})$.

Proposition 17. *The twisted quantum cohomology $QH^*_{\rm tw}(X)_\Lambda$ is a Λ-algebra generated by*

$$\hat{\phi}_i = \partial_i \tau, \qquad \hat{v} = \boldsymbol{\partial}_v \tau.$$

and all relations among them are topologically[3] *generated by*

$$\hat{\phi}_i\hat{\phi}_j = \sum_{k=0}^{s} A_{i,j}^k(Q\hat{v}^v, t)\hat{\phi}_k, \quad \hat{v}\hat{\phi}_i = \sum_{j=0}^{s} C_i^j(Q\hat{v}^v, t)\hat{\phi}_j, \quad \hat{v} = \sum_{a=1}^{r} v^i\hat{\phi}_i$$

where we write $Q\hat{v}^v = (Q_1\hat{v}^{v^1}, \dots, Q_r\hat{v}^{v^r})$.

Proof. Corollary 5 and the divisor equation implies that these relations hold in the twisted quantum cohomology. On the other hand, these relations (Q-adically) define a free Λ-module of the expected rank $\dim H^*(X)$, because every appearance of $\hat{v}$ in the right-hand side associates non-zero powers of Q. Thus these relations are complete. □

Theorem 18. *Suppose that the class v satisfies the condition* (†). *The ambient quantum cohomology $QH^*_{\rm amb}(Y)_\Lambda$ is a Λ-algebra generated by*

$$\hat{\phi}_j = i^*(\partial_j\tau), \qquad \hat{v} = i^*(\boldsymbol{\partial}_v\tau)$$

and all relations among them are topologically generated by

$$\hat{\phi}_i\hat{\phi}_j = \sum_{k=0}^{s} A_{i,j}^k(Q\hat{v}^v, t)\hat{\phi}_k,$$
$$\hat{v}\hat{\phi}_i = \sum_{j=0}^{s} C_i^j(Q\hat{v}^v, t)\hat{\phi}_j \qquad \text{for } 0 \le i \le u$$
$$\hat{\phi}_i = \sum_{j=0}^{s} \frac{C_i^j(Q\hat{v}^v, t)}{\hat{v}}\hat{\phi}_j \qquad \text{for } u+1 \le i \le s$$
$$\hat{v} = \sum_{a=1}^{r} v^a\hat{\phi}_a$$

where we use the fact that $C_i^j(Q\hat{v}^v, t)$ is divisible by $\hat{v}$ for $u+1 \le i \le s$.

Proof. First we show that the third relation holds. (The other relations are obvious from Corollary 5). Suppose $u+1 \le i \le s$. We remark that $C_i^j(Q\hat{v}^v, t)$ is divisible by $\hat{v}$. To see this, it suffices to show that if Q^d appears in the series expansion of $C_i^j(Q,t)$, then $v \cdot d \neq 0$. Indeed, we have

$$C_i^j(Q,t) = \sum_{d\in \mathrm{Eff}(X)} \sum_{n=0}^{\infty} \frac{1}{n!}\left\langle v, \phi_i, \phi^j, t, \dots, t\right\rangle_{0,3+n,d} Q^d$$

[3]in the Q-adic topology

and the term with $(d,n) = (0,0)$ vanishes by the condition $v \cup \phi_i = 0$ and the terms with $(d,n) \neq (0,0)$, $v \cdot d = 0$ vanish by the divisor equation. Thus $C_i^j(Q\hat{v}^v, t)$ is divisible by $\hat{v}$. We have the differential relation

$$z\boldsymbol{\partial}_v z\partial_i J = \sum_j C_i^j(Q,t) z\partial_j J.$$

By Lemma 1, this yields

$$z\boldsymbol{\partial}_v z\partial_i \operatorname{modif}(J) = \sum_j C_i^j(Q,t)\Big|_{Q^d \mapsto (\prod_{l=0}^{v\cdot d-1}(z\boldsymbol{\partial}_v - lz))Q^d} z\partial_j \operatorname{modif}(J).$$

From what we have shown, we can factor out $z\boldsymbol{\partial}_v$ from the left, and obtain

$$z\boldsymbol{\partial}_v T = 0 \quad \text{with } T = \left(z\partial_i - \sum_j C_i^j(Q,t)\Big|_{Q^d \mapsto (\prod_{l=1}^{v\cdot d-1}(z\boldsymbol{\partial}_v - lz))Q^d} z\partial_j \right) \operatorname{modif}(J).$$

Setting $T = \sum_{d \in \mathrm{Eff}(X)} c_d Q^{d+p/z}$, we have $(v + (v \cdot d)z)c_d = 0$ for all d; this implies $c_d = 0$ for $v \cdot d \neq 0$ and $v \cup c_d = 0$ for $v \cdot d = 0$. The condition (†) implies that $i^* T = 0$. From this we obtain the relation in the hypersurface Y:

$$\hat{\phi}_i = \sum_{j=0}^{s} \frac{C_i^j(Q\hat{v}^v, t)}{\hat{v}} \hat{\phi}_j \qquad \text{for } u+1 \leq i \leq s.$$

The rest of the proof is similar to the previous proposition. The third relation determines $\hat{\phi}_i$, $u+1 \leq i \leq s$ in terms of the other $\hat{\phi}_j$'s and we can see that all these relations define a Λ-module of the expected rank $\dim H^*_{\mathrm{amb}}(Y) = u+1$. □

The same argument also shows the following:

Theorem 19. *Suppose that the class v satisfies the condition (†). Let $\hat{\phi}_j$, $\hat{v}$ be as in Theorem 18. As a $\Lambda[\hat{v}]$-module, the ambient quantum cohomology $QH^*_{\mathrm{amb}}(Y)_\Lambda$ is generated by $\hat{\phi}_j$, $j = 0, \ldots, s$ and all relations as a $\Lambda[\hat{v}]$-module are topologically generated by:*

$$\hat{v}\hat{\phi}_i = \sum_{j=0}^{s} C_i^j(Q\hat{v}^v, t)\hat{\phi}_j \qquad \text{for } 0 \leq i \leq u$$

$$\hat{\phi}_i = \sum_{j=0}^{s} \frac{C_i^j(Q\hat{v}^v, t)}{\hat{v}} \hat{\phi}_j \qquad \text{for } u+1 \leq i \leq s.$$

Remark 20. When X and Y are Fano, and if we are only interested in the *small* quantum cohomology (i.e. $t = 0$), everything is defined over the polynomial ring $\mathbb{C}[Q_1, \ldots, Q_r]$ for the degree reason, and the word "topologically" can be removed from the above statements.

Remark 21. Suppose we are only interested in the small quantum cohomology. Even in this case, in order to get the above presentation, we still need the mirror map $\tau(t)$ *up to the first order* in $t^0, \ldots, t^s$ (so that we get the generators $\hat{\phi}_i = \partial_i \tau|_{t=0}$).

Remark 22. The presentation here determines not only the abstract isomorphism class of the quantum cohomology rings, but also their structure constants.

0.5. **Characteristic polynomials, revisited.** We apply the above method to the case where Y is a *Fano* hypersurface in a *Fano* manifold X. We again assume the condition (†). This answers Questions 13, 14 affirmatively.

Theorem 23. *Suppose that Y is a Fano hypersurface in X and v satisfies (†). Let $P(\lambda, Q) = \det(\lambda - v \star^X)$ denote the characteristic polynomial of the small quantum product by v in $QH^*(X)$. Let $\hat{v} = \boldsymbol{\partial}_v \tau|_{t=0}$ be as in Theorem 18. Then the characteristic polynomial $\widehat{P}(\lambda, Q_Y)$ of the small quantum product by $\hat{v}$ in $QH^*_{\mathrm{amb}}(Y)$ satisfies:*

$$\widehat{P}(\lambda, Q_Y)\Big|_{Q_Y^d \to Q^{i_* d}} = \lambda^{-m} P(\lambda, Q\lambda^v)$$

with $m = s - u = \dim H^*(X) - \dim H^*_{\mathrm{amb}}(Y)$, where $Q\lambda^v = (Q_1 \lambda^{v^1}, \dots, Q_r \lambda^{v^r})$.

Remark 24. Actually the theorem holds separately for the even degree part and the odd degree part. In the following proof, we only consider the even degree part for simplicity of notation.

First we show the following:

Proposition 25. *With notation as above, the characteristic polynomial of the small twisted quantum product by $\hat{v}$ in $QH^*_{\mathrm{tw}}(X)$ is given by $P(\lambda, Q\lambda^v)$.*

Proof. The small quantum product by $\hat{v}$ in $QH^*_{\mathrm{tw}}(X)$ is determined by the formula:

$$\hat{v} \cdot \hat{\phi}_i = \sum_{j=0}^{s} C_i^j (Q\hat{v}^v) \hat{\phi}_j.$$

This gives a "self-referential" matrix representation of the multiplication by $\hat{v}$. What we want to show is that the characteristic polynomial of this self-referential matrix $(C_i^j(Q\hat{v}^v))$ becomes the characteristic polynomial of $\hat{v}$ after setting $\hat{v} = \lambda$. Note that, for the degree reason, the matrix $C_i^j(Q\hat{v}^v)$ is of the block-form:

$$\begin{bmatrix} * & \heartsuit & \heartsuit & \heartsuit & \heartsuit \\ * & * & \heartsuit & \heartsuit & \heartsuit \\ & * & * & \heartsuit & \heartsuit \\ & & * & * & \heartsuit \\ & & & * & * \end{bmatrix}$$

with respect to the decomposition $H^*(X) = H^0 \oplus H^2 \oplus \cdots \oplus H^{2n}$. Here $\hat{v}$ can only appear in the positions of $\heartsuit$ since both Q and $\hat{v}$ have positive degrees (here we use the fact that Y is Fano). We can obtain from $C_i^j(Q\hat{v}^v)$ the (usual) matrix presentation of $\hat{v}$ in the basis $\{\hat{\phi}_0, \dots, \hat{\phi}_s\}$ by the following procedure.

(1) Take the leftmost column that contains $\hat{v}$. Suppose it is the ith column $\sum_{j=0}^{s} C_i^j(Q\hat{v}^v)\hat{\phi}_j$.
(2) Take an entry $C_i^j(Q\hat{v})\phi_j$ of the ith column that contains $\hat{v}$. Then $|j| < |i|$. Replace an appearance of $\hat{v}\phi_j$ with $\sum_{k=0}^{s} C_j^k(Q\hat{v}^v)\phi_k$, which by assumption does not contain $\hat{v}$.
(3) If the matrix still contains $\hat{v}$, go back to (1) and repeat the same process.

This process terminates in finite steps because in each step the total number[4] of powers of $\hat{v}$ appearing in the matrix decreases. In the end we get a matrix presentation for $\hat{v}$.

[4]We count the "number of powers" as follows: each entry $c_0 + c_1\hat{v}^{e_1} + \cdots + c_k\hat{v}^{e_k}$ with $c_i \neq 0$ contributes $e_1 + \cdots + e_k$ to the number of powers.

Now it suffices to show that the characteristic polynomials with $\hat{v}$ set to be λ do not change under the replacement process in (2). Let C, C' denote the matrices, respectively, before or after the replacement. When we write

$$C = \begin{bmatrix} \cdots & c_{0j} & \cdots & c_{0i} & \cdots \\ & \vdots & & \vdots & \\ & c_{jj} & & a + b\hat{v} & \\ & \vdots & & \vdots & \\ \cdots & c_{sj} & \cdots & c_{si} & \cdots \end{bmatrix} \quad \text{with } c_{ji} = a + b\hat{v}$$

we have

$$C' = \begin{bmatrix} \cdots & c_{0j} & \cdots & c_{0i} + bc_{0j} & \cdots \\ & \vdots & & \vdots & \\ & c_{jj} & & a + bc_{jj} & \\ & \vdots & & \vdots & \\ \cdots & c_{sj} & \cdots & c_{si} + bc_{sj} & \cdots \end{bmatrix}.$$

Therefore

$$\det(\lambda I - C') = \begin{vmatrix} \cdots & -c_{0j} & \cdots & -c_{0i} - bc_{0j} & \cdots \\ & \vdots & & \vdots & \\ & \lambda - c_{jj} & & -a - bc_{jj} & \\ & \vdots & & \vdots & \\ \cdots & -c_{sj} & \cdots & -c_{si} - bc_{sj} & \cdots \end{vmatrix} = \begin{vmatrix} \cdots & -c_{0j} & \cdots & -c_{0i} & \cdots \\ & \vdots & & \vdots & \\ & \lambda - c_{jj} & & -a - b\lambda & \\ & \vdots & & \vdots & \\ \cdots & -c_{sj} & \cdots & -c_{si} & \cdots \end{vmatrix}$$

which equals $\det(\lambda I - C)$ after setting $\hat{v} = \lambda$. □

Proof of Theorem 23. Recall from (a Fano-adapted version of) Theorem 19 that $QH^*_{\mathrm{amb}}(Y)$ is a $\mathbb{C}[Q_1, \ldots, Q_r][\hat{v}]$-module generated by $\hat{\phi}_i$, $i = 0, \ldots, s$ with relations

$$\hat{v}\hat{\phi}_i = \sum_{j=0}^{s} C_i^j(Q\hat{v}^v, t)\hat{\phi}_j \qquad \text{for } 0 \le i \le u \tag{5}$$

$$\hat{\phi}_i = \sum_{j=0}^{s} \frac{C_i^j(Q\hat{v}^v, t)}{\hat{v}}\hat{\phi}_j \qquad \text{for } u + 1 \le i \le s. \tag{6}$$

We solve for $\hat{\phi}_i$, $u + 1 \le i \le s$ in terms of $\hat{\phi}_i$, $0 \le i \le u$ using the second equation (6), and replace ϕ_i, $u + 1 \le i \le s$ in the first equation (5) with those solutions. Then we again obtain a self-referential matrix of $\hat{v}$ in the basis $\hat{\phi}_0, \ldots, \hat{\phi}_u$:

$$\hat{v} \cdot \hat{\phi}_i = \sum_{j=0}^{u} D_i^j(Q, \hat{v})\hat{\phi}_j, \qquad 0 \le i \le u.$$

To be more precise, we write the matrix $C_i^j(Q\hat{v}^v)$ in the block form with respect to the partition $\{\hat{\phi}_0, \ldots, \hat{\phi}_u\} \cup \{\hat{\phi}_{u+1}, \ldots, \hat{\phi}_s\}$ of basis.

$$(C_i^j(Q\hat{v}^v))_{0 \le i,j \le s} = \begin{pmatrix} E & F \\ G & H \end{pmatrix}$$

Writing $\hat{\phi}(1) = (\hat{\phi}_0, \ldots, \hat{\phi}_u)$ and $\hat{\phi}(2) = (\hat{\phi}_{u+1}, \ldots, \hat{\phi}_s)$, we have from (6)

$$\hat{\phi}(2) = \hat{\phi}(1)F/\hat{v} + \hat{\phi}(2)H/\hat{v}.$$

Hence

$$\hat{\phi}(2) = \hat{\phi}(1)\hat{v}^{-1}F(I - \hat{v}^{-1}H)^{-1}.$$

By (5), we have

$$\hat{v}\hat{\phi}(1) = \hat{\phi}(1)E + \hat{\phi}(2)G = \hat{\phi}(1)\left(E + \hat{v}^{-1}F(I - \hat{v}^{-1}H)^{-1}G\right).$$

Therefore the matrix D above is given by:

$$D = E + \hat{v}^{-1}F(I - \hat{v}^{-1}H)^{-1}G.$$

By the same argument as in Proposition 17, we can show that the characteristic polynomial $\widehat{P}(\lambda, Q) := \widehat{P}(\lambda, Q_Y)|_{Q_Y^d \to Q^{i_* d}}$ of the small quantum product $\hat{v}\star^Y$ for Y (with the change of variables $Q_Y^d \mapsto Q^{i_* d}$) is given by:

$$\widehat{P}(\lambda, Q) = \det(\lambda - D)\Big|_{\hat{v}\to\lambda}.$$

On the other hand, we have for $C = (C_i^j(Q\hat{v}^v))_{0\le i,j\le s}$,

$$\begin{aligned}\det(\lambda I - C) &= \begin{vmatrix}\lambda - E & -F \\ -G & \lambda - H\end{vmatrix} = \begin{vmatrix}\lambda - E - F(\lambda - H)^{-1}G & -F \\ 0 & \lambda - H\end{vmatrix} \\ &= \det(\lambda - E - F(\lambda - H)^{-1}G)\det(\lambda - H) \\ &= \det(\lambda - E - \lambda^{-1}F(1 - \lambda^{-1}H)^{-1}G)\det(1 - \lambda^{-1}H)\lambda^{s-u}.\end{aligned}$$

Thus setting $\hat{v} = \lambda$, we obtain

$$\det(\lambda I - C)\Big|_{\hat{v}\to\lambda} = \det(\lambda - D)\Big|_{\hat{v}\to\lambda}\det(1 - \hat{v}^{-1}H)\Big|_{\hat{v}\to\lambda}\lambda^m.$$

Noting that H is divisible by $\hat{v}$, $\det(1 - \hat{v}^{-1}H)$ is of degree zero and that $\hat{v}^{-1}H = O(Q)$, we have that $\det(1 - \hat{v}^{-1}H) = 1$. Hence we obtain $P(\lambda, Q\lambda^v) = \lambda^m\widehat{P}(\lambda, Q)$ as required. □

Remark 26. It seems that we can generalize Theorem 23 to the case where $c_1(Y)$ is nef, with a little more effort. When $c_1(Y)$ is not even nef, $P(\lambda, Q\lambda^v)$ may not be a polynomial of degree $\dim H^*(X)$. A naive guess is that we can approximate (in the Q-adic topology) the actual characteristic polynomial by iteratively replacing higher powers of $\hat{v}$ appearing in $P(\lambda, Q\lambda^v)$, up to any given order.

0.6. **Subcanonical hypersurfaces, revisited.** Suppose that X is a Fano manifold of index ℓ, i.e. $-K_X = \ell h$ for a primitive ample class h, and that Y is a Fano hypersurface with respect to $v = kh$ with $0 < k < \ell$. Suppose that $P(\lambda, Q)$ is the characteristic polynomial of $h\star$ in the small quantum cohomology of X. Then the characteristic polynomial of $v\star$ is:

$$k^{s+1}P(\lambda/k, Q)$$

with $s + 1 = \dim H^*(X)$. By Theorem 23, the characteristic polynomial of $\hat{v} = kh'$ in the ambient quantum cohomology of Y is:

$$k^{s+1}\lambda^{-m}P(\lambda/k, Q\lambda^{kh})$$

with $m = s - u = \dim H^*(X) - \dim H^*_{\mathrm{amb}}(Y)$. Finally the characteristic polynomial of $h'\star$ in $QH^*_{\mathrm{amb}}(Y)$ is:

$$P'(\lambda, Q) = \lambda^{-m}P(\lambda, Q(k\lambda)^{kh}).$$

We write

$$P(\lambda, Q) = \lambda^{e_0}\prod_{i=1}^{c}\prod_{j=0}^{\ell-1}(\lambda - \zeta_\ell^j u_i)$$

with non-zero eigenvalues $u_1, \ldots, u_c$ of $h\star^X$, where $u_i = u_i(Q)$ satisfies the homogeneity

$$u_i(Q\epsilon^{\ell h}) = \epsilon u_i(Q).$$

Then

$$\begin{aligned} P'(\lambda, Q) &= \lambda^{-m}\lambda^{e_0}\prod_{i=1}^{c}\prod_{j=0}^{\ell-1}(\lambda - \zeta_\ell^j (k\lambda)^{k/\ell} u_i) \\ &= \lambda^{e_0-m}\prod_{i=1}^{c}(\lambda^\ell - k^k\lambda^k u_i^\ell) \\ &= \lambda^{e_0-m+ck}\prod_{i=1}^{c}(\lambda^{\ell-k} - k^k u_i^\ell). \end{aligned}$$

Corollary 27. *Let X, Y be as above. Let $h' \in H^*_{\mathrm{amb}}(Y)$ denote the class (3). Suppose that the small quantum product $h\star^X$ has $\{u_i\zeta_\ell^j : 1 \le i \le c, 0 \le j \le \ell - 1\}$ as the multi-set of non-zero eigenvalues. Then the quantum product $h'\star^Y$ on the ambient part has*

$$\{u_i'\zeta_k^j : 1 \le i \le c, 0 \le j \le \ell - k - 1\}$$

as the multi-set of non-zero eigenvalues, where u_i' is a solution to the equation:

$$(u_i')^{\ell-k} = k^k u_i^\ell.$$