Perverse Sheaf Gopakumar Vafa-Pandharipande Thomas Correspondence

for Local del Pezzo Surfaces (arXiv:2306.05547v2)

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What are GV Invariants? (Physical Origins)

For a Calabi-Yau 3-fold X and a curve class $\beta \in H_2(X, \mathbb{Z})$, string theory predicts the existence of integer invariants n_β^g.

Physical Meaning: These integers are counts of BPS states, arising from M2-branes wrapping curves of class β and genus g inside X. The integrality of n_β^g is a key physical prediction.

Mathematical Structure: The invariants are packaged in a generating series where the coefficients reflect the spin content of the BPS particles. The term $(q^{1/2} + q^{-1/2})^{2g}$ is the character of the $(2g+1)$-dimensional irreducible representation of $\mathfrak{su}(2)$.

$$\text{BPS Partition Function} \sim \sum_{g \geq 0} n_\beta^g (q^{1/2} + q^{-1/2})^{2g}$$

Relation to Geometry: These counts are expected to be equivalent to Gromov-Witten (GW), Donaldson-Thomas (DT), and Pandharipande-Thomas (PT) invariants after specific transformations. The challenge is to find a direct geometric definition of n_β^g.

The Modern Viewpoint

The integers n_β^g can be rigorously defined and extracted from the topology of moduli spaces of 1-dimensional sheaves on X, using the tools of perverse sheaves.

A Landscape of Curve-Counting Theories

There are several major theories for counting curves in a Calabi-Yau 3-fold X.

Gromov-Witten (GW): Counts stable maps $f: C \to X$ from curves C into the target space. It is defined using integration over a virtual fundamental class on the moduli space of maps.

Donaldson-Thomas (DT): Counts ideal sheaves $\mathcal{I}_Z \subset \mathcal{O}_X$ of 0-dimensional subschemes $Z \subset X$.

Pandharipande-Thomas (PT): Counts "stable pairs" (F, s), where F is a 1-dimensional coherent sheaf on X and $s : \mathcal{O}_X \to F$ is a section with 0-dimensional cokernel.

The GV/PT Correspondence (Numerical)

Heuristics from physics (Katz-Klemm-Vafa) predict a precise relationship between the generating series of PT invariants and the GV invariants:

$$\sum_{n,\beta} P_{n,\beta}\, q^n Q^\beta = \prod_{\beta \in H_2(X,\mathbb{Z})} \left(\prod_{j=1}^{\infty} (1 + (-1)^{j+1} q^j Q^\beta)^{j n^0_\beta} \cdot \prod_{g=1}^{\infty} \prod_{k=0}^{2g-2} \left(1 + (-1)^{g-k} q^{g-1-k} Q^\beta\right)^{(-1)^{k+g} n^g_\beta \binom{2g-2}{k}} \right),$$

This formula suggests Gopakumar-Vafa invariants are the "primitive" counts from which other enumerative invariants are built. Our goal is to lift this numerical correspondence to the level of sheaves.

The HST Definition (for Smooth Moduli Spaces)

When the moduli space is smooth, Hosono-Saito-Takahashi gave a definition using the BBDG decomposition theorem.

Let $\mathrm{Sh}_\beta(X)$ be the moduli space of Gieseker-stable 1-dimensional sheaves on X. Assume it is smooth.

Consider the support map to the Chow variety, $\pi^\beta : \mathrm{Sh}_\beta(X) \to \mathrm{Chow}_\beta(X)$.

The pushforward of the constant sheaf (shifted to be a perverse sheaf), $\mathrm{R}\pi^\beta_* \mathbb{Q}_{\mathrm{Sh}_\beta}[\dim \mathrm{Sh}_\beta]$, decomposes into a direct sum of shifted perverse sheaves:

$$\mathrm{R}\pi^\beta_* \mathcal{IC}_{\mathrm{Sh}_\beta} \cong \bigoplus_{i \in \mathbb{Z}} {}^pR^i \pi^\beta_* \mathcal{IC}_{\mathrm{Sh}_\beta}[-i]$$

Defining GV Invariants via Perverse Cohomology

The GV integers n^g_β are defined as the unique integers satisfying the identity:

$$\sum_{i \in \mathbb{Z}} \chi(\mathrm{Chow}_\beta(X), {}^pR^i \pi^\beta_* \mathcal{IC}_{\mathrm{Sh}_\beta})\, q^i = \sum_{g \geq 0} n^g_\beta (q^{1/2} + q^{-1/2})^{2g}$$

The polynomial on the left is palindromic due to Poincaré duality, which makes this decomposition possible. This provides a solid definition, but it fails when $\mathrm{Sh}_\beta(X)$ is singular.

Orientation Data

In general, $\mathrm{Sh}_\beta(X)$ is singular. It has the structure of a **d-critical locus**, locally modeled by the critical locus $Z(df)$ of a function f on a smooth manifold U.

- **Local Object:** On each chart (U, f), we have the perverse sheaf of vanishing cycles ϕ_f, which captures the change in topology near the singular locus.
- **The Gluing Problem:** To construct a global perverse sheaf ϕ_{Sh_β} on the moduli space, one must glue the local sheaves ϕ_f on chart overlaps. This requires a consistent choice of isomorphisms, which is controlled by a square root of the canonical bundle of the d-critical locus, $K^{1/2}_{\text{d-crit}}$. This choice is called an **orientation data**.

The Central Obstacle

For a long time, it was not known if a canonical, deformation-invariant choice of orientation data existed for moduli of sheaves. This motivated approaches to define GV invariants that bypass this problem.

(The work of Joyce-Upmeier has constructed such data, but its application remains an active area of research.)

Two Approaches to the Orientation Problem

Kiem-Li's Approach (c. 2012)

Instead of the full sheaf of vanishing cycles ϕ, use its **associated graded**, $\mathrm{gr}\,\phi$. This object is independent of the orientation data, so a global object can be defined without making a choice.
The Flaw: As shown by Maulik-Toda, $\mathrm{gr}\,\phi$ is **not** deformation invariant. The resulting invariants would depend on the complex structure of X.

Maulik-Toda's Solution (c. 2018)

Do not construct a global perverse sheaf, instead via **weighted Euler characteristics**. The count is defined as an integral over the Chow variety of local contributions from the vanishing cycles:

$$n_\beta^g \text{ defined via } \int_{\mathrm{Chow}_\beta} \chi_{vir}(\pi^{-1}(c), \phi_\pi) d\chi(c)$$

where χ_{vir} uses the Behrend function.

Toy Model GV/PT

The structure of the GV/PT correspondence is beautifully illustrated by the case of a single smooth curve C of genus g.

GV Side (Sheaves on C): The moduli space of rank 1, degree 0 sheaves on C is the Jacobian, $\mathrm{Jac}(C)$. Its Poincaré polynomial is the "GV partition function":

$$Z_{GV}(q,x) = P(\mathrm{Jac}(C), qx) = (1+qx)^{2g}$$

PT Side (Stable Pairs on X): Stable pairs on X are essentially subschemes of C. The moduli space is the Hilbert scheme of points, $C^{[k]}$. The PT partition function is the generating series of their Poincaré polynomials:

$$Z_{PT}(q,x) = \sum_{k\geq 0} P(C^{[k]}, x) q^k$$

Macdonald's Formula is the Correspondence

A classical formula of Macdonald relates these two generating series, which gives the "cohomological shadow" of the full GV/PT correspondence:

$$Z_{PT}(q,x) = \sum_{k\geq 0} P(C^{[k]}, x) q^k = \frac{(1+qx)^{2g}}{(1-q)(1-qx^2)} = \frac{Z_{GV}(q,x)}{(1-q)(1-qx^2)}$$

Crash Course on *perverse sheaf*

Ambient. Fix a complex algebraic variety X and the constructible derived category $D_c^b(X)$.

Definition (middle perversity). A complex $K \in D_c^b(X)$ is *perverse* if it lies in the heart of the middle perverse t–structure, equivalently (dimension conditions)

$$\dim \operatorname{Supp} \mathcal{H}^{-i}(K) \le i \quad \text{and} \quad \dim \operatorname{Supp} \mathcal{H}^{-i}(\mathbb{D}K) \le i \quad (\forall\, i),$$

where $\mathbb{D}$ is Verdier duality. Intuitively: cohomology is concentrated "near the middle dimension."

Basic examples.

- If U is smooth of (complex) dimension d, then $\mathbb{Q}_U[d]$ is perverse.
- For a pure Z with smooth open $j\colon U \hookrightarrow Z$ and local system L on U, the intersection complex $\mathcal{IC}_Z(L)$ is perverse.
- (*Vanishing cycles*) If $f : M \to \mathbb{C}$ with M smooth, then $\phi_f(\mathbb{Q}_M[\dim M])$ is perverse on $\mathrm{Crit}(f)$.

Key properties used here.

Stability: $\mathbb{D}$ preserves perversity; perverse sheaves form an abelian category.

Decomposition Theorem: For proper $\pi : Y \to X$, $\mathrm{R}\pi_*\mathcal{IC}_Y$ splits into shifts of $\mathcal{IC}(\text{stratum}, \mathcal{L})$; its perverse cohomology ${}^pR^i\pi_*\mathcal{IC}_Y$ is semisimple.

Perverse Hard Lefschetz: ${}^pR^i \to {}^pR^{-i}$ is an isomorphism when f is projective.

Goresky–MacPherson inequality (codimension bound). Let L be a local system defined on an open subset of X. Let $f : X \to Y$ be proper with X of relative dimension d. Assuming that a subvariety S appears in the decomposition theorem for $\mathrm{R}f_*\mathcal{IC}_X(L)$, then $\mathrm{codim}(S) \leq d$.

The Perverse Filtration on Cohomology

The decomposition of $\mathrm{R}\pi_*\mathcal{IC}_{M_\beta}$ induces a fundamental structure on the cohomology of the moduli space itself.

Definition (Perverse Filtration)

The perverse Leray spectral sequence for π is E_2-degenerate and gives an isomorphism

$$H^k(M_\beta, \mathbb{Q}) \cong \bigoplus_{i \in \mathbb{Z}} H^{k-i}(B, {}^pR^i\pi_*\mathcal{IC}_{M_\beta})$$

This decomposition defines a filtration $P^\bullet$ on $H^*(M_\beta, \mathbb{Q})$, where

$$P^j H^k(M_\beta, \mathbb{Q}) := \bigoplus_{i \geq j} H^{k-i}(B, {}^pR^i\pi_*\mathcal{IC}_{M_\beta})$$

The associated graded pieces are the cohomology of the summands:

$$\mathrm{gr}_j^P H^k(M_\beta, \mathbb{Q}) \cong H^{k-j}(B, {}^pR^j\pi_*\mathcal{IC}_{M_\beta})$$

Main Idea

To understand the structure of $H^*(M_\beta, \mathbb{Q})$, we must understand the perverse sheaves ${}^pR^j\pi_*\mathcal{IC}_{M_\beta}$. This is the key input for Gopakumar-Vafa theory and related invariants.

The Gopakumar–Vafa Definition

Let S be a smooth projective surface and $\beta \in H_2(S, \mathbb{Z})$ a curve class.

- $M_\beta(S)$: moduli of Gieseker-stable 1-dimensional sheaves on S with support class β.
- B: a parameter for supports (e.g. $B = \mathrm{Chow}_\beta(S)$).
- $\pi : M_\beta(S) \to B$: the support map.

Central IC Package (HST viewpoint)

Study the pushforward $\mathrm{R}\pi_* \mathcal{IC}_{M_\beta} \in D^b_c(B)$. For proper π,

$$\mathrm{R}\pi_* \mathcal{IC}_{M_\beta} \;\cong\; \bigoplus_{i \in \mathbb{Z}} {}^pR^i \pi_* \mathcal{IC}_{M_\beta}[-i].$$

In the local Calabi–Yau $X = \mathrm{Tot}(K_S)$, this IC pushforward is the *GV block* over the base B.

Gopakumar–Vafa integers (HST packaging)

Define the Gopakumar-Vafa number $\{n^g_\beta\}_{g \geq 0}$ by the palindromic identity

$$\boxed{\sum_{i \in \mathbb{Z}} \chi\big(B,\ {}^pR^i \pi_* \mathcal{IC}_{M_\beta}\big)\, q^i \;=\; \sum_{g \geq 0} n^g_\beta \left(q^{1/2} + q^{-1/2}\right)^{2g}}$$

The GV Side for Local $\mathbb{P}^2$ is "Clean"

Let $X = \mathrm{Tot}(K_{\mathbb{P}^2})$ and $\beta = dH$.

Theorem

The pushforward of the intersection complex from the moduli of sheaves Sh_{dH} has no supports on reducible or non-reduced loci. It is completely determined by the geometry over the open stratum of smooth, integral curves $\mathrm{Chow}^{\mathrm{sm}}_{dH}$.

$$\mathrm{R}\pi^{dH}_{*}\mathcal{IC}_{\mathrm{Sh}_{dH}} \cong \bigoplus_{i=-g_d}^{g_d} \mathcal{IC}(\wedge^{i+g_d}\mathbb{V})[-i]$$

where $\mathbb{V} = R^1\pi^{\mathrm{sm}}_{*}\mathbb{Q}$ is the local system from the universal compactified Jacobian over $\mathrm{Chow}^{\mathrm{sm}}_{dH}$.

Proof Idea: The key is Ngô's support theorem, which gives a necessary condition $\mathrm{codim}(Z) \leq \delta_Z$ for a stratum Z to be a support. I prove that for any boundary stratum in $\mathrm{Chow}_{dH}(\mathbb{P}^2)$, the opposite inequality $\mathrm{codim}(Z) > \delta_Z$ holds, ruling them out.

The PT Side is More Complex

The PT side is studied via the Hilbert-Chow map for relative Hilbert schemes, $\pi_d^{[n]} : \mathcal{C}_d^{[n]} \to \mathrm{Chow}_{dH}$, since $P_{1-g_d+n}(X, dH) \cong \mathcal{C}_d^{[n]}$ for $n \leq d+2$.

The "Clean Range" ($0 \leq n \leq d-2$): Here, the PT pushforward is also supported only on the smooth locus. The decomposition is analogous to the GV side, but with symmetric powers $\mathbb{S}^k = \wedge^k \oplus \wedge^{k-2} \oplus \ldots$ replacing exterior powers.

$$\mathrm{R}\pi_{d*}^{[n]}\mathbb{Q}_{\mathcal{C}_d^{[n]}}[\mathrm{dim}] \cong \bigoplus_{i=-n}^{n} \mathcal{IC}(\mathbb{S}^{i+n}\mathbb{V})[-i]$$

The Boundary Range ($n \geq d-1$): The decomposition is no longer clean! New perverse sheaves appear, supported on the loci of reducible curves. For example, at $n = d-1$, a new term appears, supported on the locus of curves splitting as (degree $d-1$) $\cup$ (line).

The Puzzle

How can the "clean" GV side be related to the more complex PT side, which sees these boundary contributions?

Main Conjecture: A Perverse GV/PT Correspondence

I propose that the structure of the PT side is universally generated by the simple structure of the GV side via a symmetric product formula.

Conjecture (Z.)

Let $Z^c_{PT} = \sum_{n,\beta}(q[-2])^{1-g_\beta}\,\mathrm{R}\pi_*^{n,\beta}\Phi_{n,\beta}\, q^n Q^\beta$ be the generating series of PT pushforwards. Then:

$$\boxed{Z^c_{PT} = \prod_\beta \mathrm{Sym}\left(\frac{\text{GV-BuildingBlock}(\beta)}{(1-q)(1-q[-2])}Q^\beta\right)}$$

where the ingredients are:

GV-BuildingBlock(β): $\bigoplus_{i=-g_\beta}^{g_\beta} q^{i+1}\, {}^pR^i\pi_*^\beta\mathbb{Q}[-i-1]$. This is constructed purely from the "clean" GV data over the smooth locus.

Denominators: $(1-q)(1-q[-2])$ are the Macdonald factors, lifted to the derived category (where multiplication is tensor product and q carries a shift).

Sym **Operator:** A generalization of the symmetric power to formal power series of complexes. This algebraic operation generates all the boundary terms on the PT side.

Thm: This formula implies the numerical GV/PT correspondence

Refined Invariants and the Motivic Story

The Euler characteristic is a shadow of a richer structure, the Hodge structure on cohomology.

Refined GV Invariants: By replacing the Euler characteristic with the Poincaré polynomial in the HST definition, we get a bigraded polynomial:

$$\sum_{i,j} \dim \mathbb{H}^j(\mathrm{Chow}_\beta, {}^pR^i\pi_*^\beta\mathcal{IC})\, r^j q^i = \sum_{j_L, j_R \in \frac{1}{2}\mathbb{Z}} N^\beta_{j_L, j_R} \frac{r^{2j_L+1} - r^{-2j_L-1}}{r - r^{-1}} \cdot \frac{q^{2j_R+1} - q^{-2j_R-1}}{q - q^{-1}}$$

The Hard Lefschetz theorem implies an $\mathfrak{sl}_2 \times \mathfrak{sl}_2$ action on this bigraded cohomology. The refined GV numbers $N^\beta_{j_L, j_R}$ are the dimensions of the irreducible representations in this decomposition.

Refined PT Invariants: Conjecturally, there exists a perverse sheaf $\phi_{n,\beta}$ on the PT moduli space $P_n(X, \beta)$. Its hypercohomology should give rise to motivic invariants.

Corollary (Z.)

The conjectural perverse sheaf correspondence will imply:

$$\sum_{n,\beta} [P_n(X,\beta)]_{\mathrm{vir}} q^n Q^\beta = \prod_{j_L, j_R \in \frac{1}{2}\mathbb{Z}} \prod_{m_{L/R} = -j_{L/R}}^{j_{L/R}} \prod_{m=1}^{\infty} \prod_{j=0}^{m-1} \left(1 - (-q)^{m-2m_L} \mathbb{L}^{-m/2+1/2+j-m_R}\right)^{(-1)^{2j_L+2j_R} N^\beta_{j_L, j_R}}.$$

Here $\mathbb{L} = [\mathbb{A}^1]$ is the class of affine line.

Main Theorem: Perverse GV/PT on "reducible-only" components

Fix a class β on a del Pezzo surface S and work on a connected component $B_0 \subset \mathrm{Chow}_\beta(S)$ whose discriminant $\Delta_0 := B_0 \setminus B_0^{\mathrm{sm}}$ consists *only* of loci of **reducible** curves (no nonreduced multiples or worse singularities). Let $X = \mathrm{Tot}(K_S)$, and let

$$\pi^\beta : \mathrm{Sh}_\beta(X) \to B_0, \qquad \pi_\beta^{[n]} : P_n(X, \beta) \to B_0.$$

Theorem

On B_0 one has the perverse identity in $D_c^b(B_0)$:

$$Z_{PT}^c\big|_{B_0} = \mathrm{Sym}\left(\frac{\bigoplus_{i=-g_\beta}^{g_\beta} q^{i+1}\, {}^pR^i\pi_*^\beta \mathcal{IC}_{\mathrm{Sh}_\beta}[-i-1]}{(1-q)(1-q[-2])} \right)\Bigg|_{B_0}, \quad g_\beta = 1 + \tfrac{1}{2}\beta\cdot(\beta + K_S).$$

Equivalently, the coefficient of q^n on the PT side is a direct sum of $\mathcal{IC}$-extensions supported on B_0^{sm} and on *reducible* boundary strata $\Sigma \subset \Delta_0$, with local systems and shifts exactly given by the symmetric-product expansion of the GV block.

Example for $\mathbb{P}^2$

Let $S = \mathbb{P}^2$, $\beta = dH$, and $M_\beta = \mathrm{Sh}_{dH}(\mathbb{P}^2)$. Let $B = \mathrm{Chow}_{dH}(\mathbb{P}^2)$.

Theorem

The pushforward complex $\mathrm{R}\pi_*\mathcal{IC}_{\mathrm{Sh}_{dH}}$ *is supported entirely on the open locus of smooth integral curves* $\mathrm{Chow}^{\mathrm{sm}}_{dH}$. *The decomposition is given by:*

$$\boxed{\mathrm{R}\pi_*\mathcal{IC}_{\mathrm{Sh}_{dH}} \cong \bigoplus_{i=-g_d}^{g_d} \mathcal{IC}(\mathrm{Chow}^{\mathrm{sm}}_{dH}, \wedge^{i+g_d}\mathbb{V})[-i]}$$

where $\mathbb{V} = R^1\pi^{\mathrm{sm}}_*\mathbb{Q}$ *is the local system from the universal compactified Jacobian, and* $g_d = \frac{(d-1)(d-2)}{2}$.

Interpretation: For $\mathbb{P}^2$, the perverse cohomology sheaves ${}^pR^i\pi_*\mathcal{IC}$ are simple. They are the intersection cohomology extension of a local system from the smooth locus. There are **no** contributions from reducible or non-reduced curve loci.

Proof Sketch: Why are there no boundary supports?

The proof relies on Ngô's support theorem for abelian fibrations.

Let $B_\lambda \subset \mathrm{Chow}_{dH}$ be a stratum of reducible or non-reduced curves, e.g., cycles of the form $C = \sum m_i \Gamma_i$.

Ngô's theorem states that if B_λ is a support of $\mathrm{R}\pi_*\mathcal{IC}$, its codimension must be bounded by a geometric invariant δ_λ.

$$\text{If } B_\lambda \text{ is a support } \implies \text{codim } B_\lambda \leq \delta_\lambda := g_d - \sum g(\widetilde{\Gamma}_i)$$

where δ_λ measures the "excess genus" lost upon specialization.

For any such stratum B_λ in $\mathrm{Chow}_{dH}(\mathbb{P}^2)$, we prove the opposite inequality via a direct calculation:

$$\boxed{\text{codim } B_\lambda > \delta_\lambda}$$

Example: For $C = C_{d_1} \cup C_{d-d_1}$,

$$\text{codim} = d_1(d - d_1) + 1, \qquad \delta = d_1(d - d_1) - 1,$$

so indeed codim $> \delta$.

This violation of Ngô's condition forbids these strata from being supports, proving the "clean" decomposition on the previous slide.

Cohomological χ–independence

Set-up. Let S be a del Pezzo surface, $\beta \in H_2(S, \mathbb{Z})$ effective, and $M_{\beta,\chi}(S)$ the moduli of Gieseker–stable 1-dimensional sheaves on S with class β and Euler characteristic χ. Let $B = \mathrm{Chow}_\beta(S)$ and $\pi_{\beta,\chi} : M_{\beta,\chi}(S) \to B$ be the support map, with smooth-curve locus $B^{\mathrm{sm}} \subset B$.

Theorem (Maulik–Shen — *cohomological χ–independence*)

For toric del Pezzo S and ample curve class β any χ_1, χ_2 in the same stability chamber, the perverse pushforwards along the support map are canonically

$$ {}^pR^i\pi_{\beta,\chi_1 *}\mathcal{IC}_{M_{\beta,\chi_1}} \cong {}^pR^i\pi_{\beta,\chi_2 *}\mathcal{IC}_{M_{\beta,\chi_2}} \quad \text{in } D^b_c(B) \quad (\forall\, i), $$

Proof Idea Roadmap (PT Side)

Goal: To understand the decomposition of the PT pushforward $\mathrm{R}(\pi_d^{[n]})_*\mathbb{Q}$ and relate it to the GV side. Our main tool is a cohomological version of the exponential formula of Migliorini-Shende-Viviani.

Start with the Main Conjecture. It claims the complex PT generating series is a symmetric product of a simple building block from the GV side:

$$Z_{PT}^c = \mathrm{Sym}\left(\frac{\text{GV-Block}}{(1-q)(1-q[-2])}\right)$$

Unpack the Sym Operator. The symmetric product operator Sym expands this into a sum over partitions $\lambda = (d_1, \dots, d_\ell)$ of the curve degree d. Each partition corresponds to a boundary stratum in Chow_d where curves degenerate.

Boundary Contributions. The term for a partition λ is governed by the union map $\times_\lambda : \prod \mathrm{Chow}_{d_j} \to \mathrm{Chow}_d$. The pushforward is given by an intermediate extension:

$$(\times_\lambda)_{!*}\left(\boxtimes_{j=1}^{\ell}\left[\text{Contribution from degree } d_j\right]\right)$$

Monodromy and Topology. If a partition has repeated parts (e.g., $d = 2 = 1 + 1$), the symmetric group $\mathfrak{S}_\ell$ acts. The pushforward to the boundary stratum decomposes according to representations of $\mathfrak{S}_\ell$, producing non-trivial local systems (e.g., the sign representation L).

Shifts and Denominators. The Macdonald-type denominators $(1 - q[-2])$ in the formula correspond geometrically to contributions from punctual Hilbert schemes.

Worked Example $d = 2$: Pairs of Lines

Let's test the conjecture on the boundary stratum $\Sigma \subset \mathrm{Chow}_{2H}$ parameterizing pairs of lines $L_1 \cup L_2$. The fundamental group $\pi_1(\Sigma^{\text{gen}}) \cong \mathbb{Z}_2$ supports two local systems: trivial $\mathbb{Q}$ and sign L.

- We compute the PT pushforward $\mathrm{R}(\pi_2^{[1]})_*\mathbb{Q}$ directly. The PT space is the universal curve $\mathcal{C}_2$.
- Over Σ, the fiber is a nodal conic $C = \mathbb{P}^1 \cup_p \mathbb{P}^1$.
- Monodromy around a loop in Σ swaps the two $\mathbb{P}^1$ components.
- This induces an action on the fiber's cohomology $H_c^2(C) \cong \mathbb{Q} \oplus \mathbb{Q}$. The local system $R^2(\pi_2^{[1]})_*\mathbb{Q}$ decomposes into the ± 1 eigenspaces of this action.
- This gives $R^2(\pi_2^{[1]})_*\mathbb{Q} \cong \mathbb{Q} \oplus L$. The full complex is:

$$\mathrm{R}(\pi_2^{[1]})_*\mathbb{Q}\big|_\Sigma \cong (\text{trivial sheaves}) \oplus L[-2]$$

Example $d = 2$: General n via Node Convolution

For higher n, the geometry on the PT side becomes richer, but the correspondence holds. The pushforward over the nodal locus Σ can be computed via a stratification of the Hilbert scheme.

Geometric Structure of $\mathcal{C}_d^{[n]}|_\Sigma$

A point in $\mathcal{C}_d^{[n]}|_\Sigma$ corresponds to choosing n points on a nodal conic $C = L_1 \cup_p L_2$. These points can be distributed between the smooth components ($L_1 \setminus \{p\}$, $L_2 \setminus \{p\}$) and the node p. This leads to a convolution formula for the pushforward:

$$\mathrm{R}(\pi_2^{[n]})_*\mathbb{Q}\big|_\Sigma \cong \bigoplus_{k=0}^{n} \underbrace{M_k}_{\text{from node}} \otimes \underbrace{L_k}_{\text{from smooth parts}}$$

M_k: The contribution from placing k points at the node. This is the pushforward from the **punctual Hilbert scheme** $(C, p)^{[k]}$. Its complex cohomology provides the powers of the shift operator '[-2]', corresponding to the Macdonald denominator $(1 - q[-2])$.

L_k: The contribution from placing $n - k$ points on the smooth locus $C_{sm} = \mathbb{A}^1 \sqcup \mathbb{A}^1$. The monodromy action swaps the two $\mathbb{A}^1$'s, so the cohomology of $((\mathbb{A}^1 \sqcup \mathbb{A}^1)^{[n-k]})$ carries the local system information ($\mathbb{Q}$ and L).

Local model for higher d on $\mathbb{P}^2$: partitions, nodes, and punctual factors

Fix $d \geq 3$ and a boundary stratum $\Sigma_\lambda \subset \mathrm{Chow}_{dH}$ corresponding to a partition $\lambda : d = d_1 + \cdots + d_\ell$ with a *generic* union $C_\lambda = C_1 \cup \cdots \cup C_\ell$ of smooth plane curves ($\deg C_i = d_i$) meeting transversely. Then the node set has size

$$\#\mathrm{Nodes}(C_\lambda) = \sum_{i<j} d_i d_j.$$

Local structure of $\mathrm{R}(\pi_d^{[n]})_*\mathbb{Q}$ along Σ_λ (schematic)

Distribute n points among the smooth parts of the components and the nodes:

$$\mathrm{R}(\pi_d^{[n]})_*\mathbb{Q}\Big|_{\Sigma_\lambda} \cong (\times_\lambda)_{!*} \bigoplus_{\substack{n_1,\dots,n_\ell \geq 0 \\ \{k_p\} \geq 0 \\ \sum_i n_i + \sum_p k_p = n}} \left(\boxtimes_{i=1}^{\ell} L_{n_i}^{(i)} \right) \otimes \left(\bigotimes_{p \in \mathrm{Nodes}} M_{k_p}^{(p)} \right)$$

Hilbert scheme of Nodal planar Curves

Local set-up. Let $R = \Bbbk[x, y]/(xy)$ be the local ring of a node (and $\widehat{R}$ its completion).

Theorem (Z. Ran). For each $m \geq 0$, the Hilbert scheme $\mathrm{Hilb}^m(R)$ (resp. $\mathrm{Hilb}^m(\widehat{R})$) is a *chain* of $m+1$ smooth m–dimensional germs (resp. formal schemes)

$$\mathrm{Hilb}^m(R) \;=\; D_0^m \;\cup\; D_1^m \;\cup\; \cdots \;\cup\; D_{m-1}^m \;\cup\; D_m^m,$$

where

- D_i^m meets only its neighbors $D_{i\pm1}^m$ *transversely* in dimension $m-1$;
- a generic point of D_i^m corresponds to a subscheme consisting of $(m-i)$ points on the x–axis and i points on the y–axis.

(Ran, *Geometry of Hilbert schemes of a node*, Thm. 2)

Local model for higher d on $\mathbb{P}^2$: partitions, nodes, and punctual factors

$L_m^{(i)}$: contribution from placing m points on the smooth locus of C_i; fiberwise this is the *length-m slice* of $\mathrm{Sym}(\text{GV-block}(d_i))$, i.e. built from $\mathrm{Sym}^\bullet(\mathbb{V}_i)$ with $\mathbb{V}_i := R^1\pi_*^{\mathrm{sm},(i)}\mathbb{Q}$.
$M_k^{(p)}$: contribution from placing k points at a node p; this is the (complex) cohomology of the **punctual Hilbert scheme** $(C_\lambda, p)^{[k]}$. Its q–generating series yields the *Macdonald denominator* factor $1/\big((1-q)(1-q[-2])\big)$ per node.

Monodromy. The pushforward $(\times_\lambda)_{!*}$ records the $\mathfrak{S}_\ell$-action permuting components of equal degree, producing nontrivial local systems (Specht isotypical pieces) on Σ_λ.

Generating-series form.

$$\sum_{n\geq 0} \mathrm{R}(\pi_d^{[n]})_*\mathbb{Q}\, q^n\Big|_{\Sigma_\lambda} \cong (\times_\lambda)_{!*}\Big(\boxtimes_{i=1}^{\ell}\mathrm{Sym}(\text{GV-block}(d_i)) \Big)\cdot \prod_{p\in\mathrm{Nodes}} \frac{1}{(1-q)(1-q[-2])}.$$

Local model for general β on $X = \mathrm{Tot}(K_S)$ (del Pezzo S)

Let $\beta > 0$ and fix a boundary stratum $\Sigma_\lambda \subset \mathrm{Chow}_\beta(S)$ for a partition $\lambda : \; \beta = \beta_1 + \cdots + \beta_\ell$ where a generic member $C_\lambda = \bigcup_{i=1}^{\ell} C_i$ has C_i smooth and the C_i meet transversely. Then the node count is

$$N(\lambda) = \#\mathrm{Nodes}(C_\lambda) = \sum_{i<j} \beta_i \cdot \beta_j \quad \text{(intersection product on } S\text{)}.$$

Local structure of $\mathrm{R}(\pi_\beta^{[n]})_*\mathbb{Q}$ along Σ_λ (schematic)

With $n = \sum_i n_i + \sum_{p=1}^{N(\lambda)} k_p$,

$$\mathrm{R}(\pi_\beta^{[n]})_*\mathbb{Q}\Big|_{\Sigma_\lambda} \cong (\times_\lambda)_{!*} \bigoplus_{\substack{n_1,\ldots,n_\ell \geq 0 \\ k_1,\ldots,k_{N(\lambda)} \geq 0}} \left(\boxtimes_{i=1}^{\ell} L_{n_i}^{(i)} \right) \otimes \left(\bigotimes_{p=1}^{N(\lambda)} M_{k_p}^{(p)} \right)$$

Local model for higher d on local del Pezzo: partitions, nodes, and punctual factors

$L_m^{(i)}$ comes from the smooth part of C_i and equals the length-m slice of $\mathrm{Sym}(\text{GV-block}(\beta_i))$, i.e. built from the Jacobian local system $\mathbb{V}_i := R^1\pi_*^{\mathrm{sm},(i)}\mathbb{Q}$ on the family of smooth curves in class β_i.
Each node contributes a punctual factor $M_k^{(p)}$ with generating series $1/\big((1-q)(1-q[-2])\big)$; altogether these yield $((1-q)(1-q[-2]))^{-N(\lambda)}$.

Monodromy & supports. The union map $(\times_\lambda)_{!*} : \prod_i \mathrm{Chow}_{\beta_i} \to \mathrm{Chow}_\beta$ encodes the $\mathfrak{S}_\ell$-symmetry among equal classes, producing boundary $\mathcal{IC}$-summands with (nontrivial) local systems on Σ_λ.

Generating-series form.

$$\sum_{n\geq 0} \mathrm{R}(\pi_\beta^{[n]})_*\mathbb{Q}\, q^n\Big|_{\Sigma_\lambda} \;\cong\; (\times_\lambda)_{!*}\Big(\,\boxtimes_{i=1}^{\ell}\mathrm{Sym}(\text{GV-block}(\beta_i))\,\Big)\cdot\frac{1}{\big((1-q)(1-q[-2])\big)^{N(\lambda)}}.$$

This is the local incarnation of the symmetric-product generation of PT from the GV blocks on del Pezzo surfaces.

Main Theorem on "Reducible-Only" Components

The full Chow variety Chow_d contains strata of "badly singular" curves (e.g., cuspidal, non-reduced). Let's restrict to a large open set $B_0 \subset \mathrm{Chow}_d$ that excludes these, containing only smooth curves and their nodal degenerations.

Definition

Let $Z_{\mathrm{nonred}} \subset \mathrm{Chow}_d$ denote the locus of *totally nonreduced* cycles (i.e. some component appears with multiplicity ≥ 2, including thickened/embedded structures). Define the open *reduced* locus

$$B_0 \ := \ \mathrm{Chow}_d \setminus Z_{\mathrm{nonred}}.$$

Then B_0 contains the smooth locus and all *reduced* degenerations (irreducible nodal, unions of smooth components meeting transversely, etc.); only *nonreduced* cycles are removed.

Theorem (Zhao)

On this open set B_0, the perverse GV/PT correspondence holds exactly. In the derived category $D^b_c(B_0)$, we have the equality:

$$\boxed{Z^c_{PT}\Big|_{B_0} = \mathrm{Sym}\left(\frac{\bigoplus_{i=-g}^{g} q^{i+1}\ {}^pR^i\pi_*\mathcal{IC}_{\mathrm{Sh}}[-i-1]}{(1-q)(1-q[-2])} \right)\Bigg|_{B_0}}$$

Interpretation: This theorem states that for the most well-behaved types of degenerations (nodal unions of smooth curves) perfectly generates all boundary contributions on the PT side from the clean GV input.

Concluding remark: why the *nonreduced* strata are hard

What are they? Loci where the support acquires multiplicity (e.g. mC, double lines, ribbons) and/or embedded points. Scheme-theoretically, local equations are *thickenings*, not simple normal crossings.

PT-side difficulty (local): To describe $R\pi_*^{[n]}\mathbb{Q}$ perverse–sheaf theoretically, one needs the *vanishing-cycle* sheaf on d-critical charts of relative Hilbert/Pair moduli near fat points. Unlike the nodal case:

- local models are higher-order thickenings (ribbons, A_k-type and beyond) with more parameters;
- monodromy can be more complicated (higher unipotent index), not captured by the simple $\{\mathbb{Q}, L\}$ split;
- there is no universal "punctual package" replacing $\frac{1}{(1-q)(1-q[-2])}$ for thickenings;
- Behrend weights give *numbers* (MT), but the *perverse-level* structure of ϕ_f remains subtle.

Consequence. Our theorem covers components with *only reducible* boundary; extending to nonreduced strata requires a finer local analysis of vanishing cycles on punctual/nested Hilbert schemes of thickenings.

Outlook. Develop canonical "punctual factors" for thickenings (analogs of Macdonald denominators), via explicit critical charts, nested Hilbert schemes of ribbons, and monodromy/weight filtrations—then integrate them into the symmetric-product framework.