Weinstein mflds w/out arboreal skeleta.

w/ Álvarez-Gavela + Large

§0. Preface.

X almost cx structure

TX cx v.b.

classified by map $X \to BU(n)$

$(X^{2n}, \omega = d\lambda)$ Weinstein mfld (+ conditions) $\rightsquigarrow \mathfrak{X}$ Liouville vf.

$\mathbb{L} = \bigcap_{t > 0} \varphi_{\mathfrak{X}}^{-t}(X)$ ← flow of $\mathfrak{X}$ — sing. isotropic skeleton. (strat. by isotropic ...)

e.g. (B^{2n}, ω_{std}) + Weinstein handle attachments.

$X_K := B^4$ + Weinstein 2 handle along Legendrian $K \subset S^3$

e.g. $(Y \setminus D, \lambda = -d^c \log \|s\|)$ for $s \in H^0(Y, \mathcal{O}(D))$, $D = s^{-1}(0)$.

(proj variety; ample divisor)

invariants e.g. $W(X)$ wrapped fuk. cat

$SH^*(X)$ symp. coh.

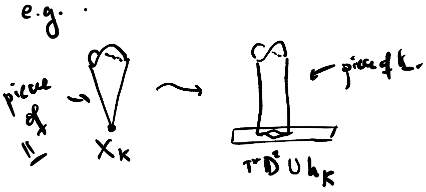
(X, λ_t) Weinstein $d\lambda_t = \omega$ $t \in [0,1]$

$(X_0, \lambda_0) \sim (X_1, \lambda_1)$ λ_t $t \in [0,1]$

Want: work up to Weinstein htpy: $\lambda \to \lambda_t$ (with $d\lambda_t = \omega$ $\forall t$) (X, λ_t) Weinstein.

n.b. get family $\mathfrak{X}_t$, $\mathbb{L}_t$ ← not all homeomorphic

e.g.:



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3 relevant properties of X.

(i) X has polarisation:

- $\exists$ global Lagn line field $\mathcal{L} \subset TX$

$\iff$ $TX = \mathcal{L} \otimes \mathbb{C}$

$\iff$ $\exists$ lift

$$\begin{tikzcd} & BO(n) \arrow[d] \\ X^{2n} \arrow[r, "(TX)"'] \arrow[ur, dashed] & BU(n) \end{tikzcd}$$

$\Longrightarrow$ $TX \simeq \overline{TX}$

$\Longrightarrow$ $c_i = -c_i \in H^*(X; \mathbb{Z})$ $\forall i$ odd

$\Longrightarrow$ $2c_1 = 0$

$\Longrightarrow$ $\mathcal{F}(X)$, $W(X)$ $\mathbb{Z}$ graded

alternatively: $R = \mathbb{Z}[\frac{1}{2}]$ $c_i \in \ker(H^*(BU; R) \to H^*(BO; R))$ $\forall i$ odd.

(ii) $(X,\lambda_0) \overset{\text{W. htpic}}{\sim} (X,\lambda_1)$ where

$\mathbb{L}_0 \subset X$, $\mathbb{L}_1 \subset X$

$\mathbb{L}_0 \sim \mathbb{L}_1$ ← with $\mathrm{sing}(\mathbb{L}_1)$ all <u>arboreal</u> (Nadler '13)

indexed by rooted trees

"X arborealisable."

in this case: $\mathcal{F}(X)$, $SH^*(X)$ determined combinatorially

e.g.: $\mathcal{F}(X)$ = htpy limit of $\operatorname{hocolim}_D \mathrm{Mod}(T_i)$

example of arborealisation: $X_K \to X_K'$ seen before.

(iii) • X has "Maslov data" (algebraic topology): null htpy of map

$$\begin{tikzcd}
X \arrow[r, "{[TX]}"] & BU \arrow[r, "\mathrm{Bott}"] \arrow[dr] & B^2(\mathbb{Z}\times BU) \arrow[r, "B^3 J"] & B^2 \mathrm{Pic}\, \mathbb{S} \\
 & & BU/O \arrow[ur] &
\end{tikzcd}$$

coherent orientations of moduli spaces.

$\overset{\text{expected}}{\Longrightarrow}$ $W(X)$ exists over $\mathbb{S}$

(following Large, Porcelli-Smith, ...)

$\Rightarrow$ $Sh(X,\lambda)/\mathbb{S}$ (Nadler-Shende '22)

[e.g. $\mathrm{Hom}_{W(X)}(L_1, L_2)$ upgraded to a <u>spectrum</u>]

expected: all Weinstein manifolds are arborealisable, Fuk. cat. det'd comb.

<u>evidence</u>:

(b) no local obstructions (Nadler '16)

(c) true for dim ≤ 4 (Starkston)

known: (i) $\Rightarrow$ (ii) (Álvarez-Gavela-Eliashberg-Nadler)

(i) $\Rightarrow$ (iii) (alg. top.)

theorem (Alvarez-Gavela - Laz. - W. '25):

- expectation is false: a.b $\Rightarrow$ $c_1 c_{2k} - c_{2k+1} = 0 \in H^*(X, \mathbb{Z}[\tfrac{1}{2}])$

e.g. $\mathbb{C}P^8 \setminus$ smooth divisor of degree 9

not arborealisable. ($\mathbb{C}P^n \setminus V_d$ generally not).

e.g. $\mathrm{Tot}((TS^6)^{23} \to S^6)$, structure comp. w/ acs on S^6.

not arborealisable, but admits Maslov data

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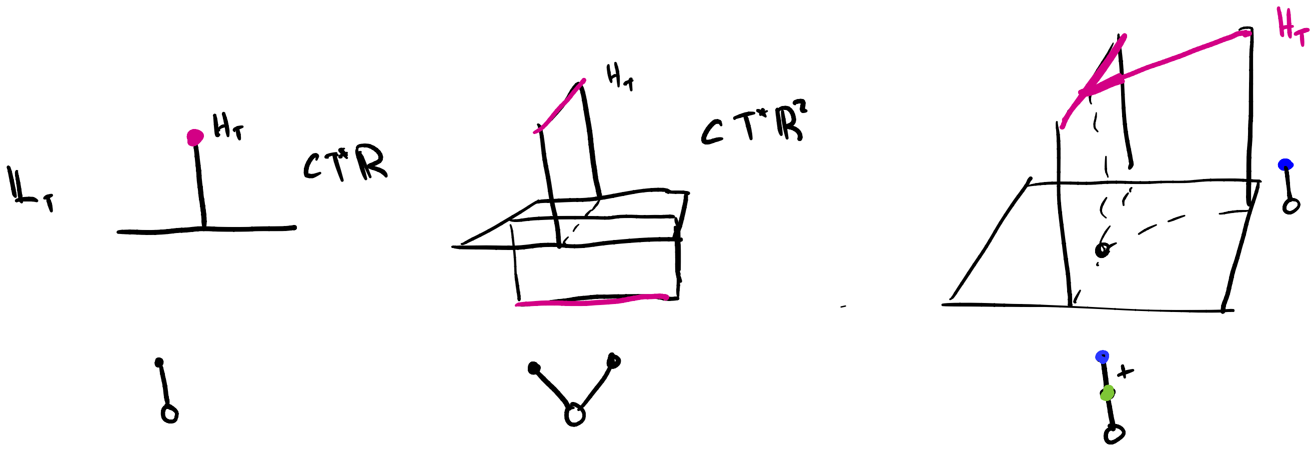
arboreal singularities (Nadler):

- described combinatorially from rooted trees w/ signs
- min'l class of sings closed under moves: · stabilisation: $L \mapsto L \times \mathbb{R} \subset T^*\mathbb{R}^{n+1}$
  · disjoint union of legendrian germs

germs $\mathbb{L}_T \subset T^* \mathbb{R}^{|v(T)|-1}$

$= \underline{\mathbb{R}}^n \cup C^+(H_T)$ ← legendrian



definition (a): symplectic vector bundle $E \xrightarrow{\mathrm{rk}\ n} M$ has $\underline{\text{I-structure}}_{I(n)}$ if

- $M = U \cup V$ open cover
- Lag'n $\mathcal{L}^n \subset E|_U$
- isotropic $G^{n-1} \subset E$
- $G \subset \mathcal{L}$ on $U \cap V$

(b) define I = homotopy pushout

$$\begin{tikzcd} BO(n-1) \times BO(1) \arrow[r] \arrow[d] & BO(n-1) \times BU(1) \arrow[d] \\ BO(n) \arrow[r] \arrow[ur, phantom, "h"] & I(n) \end{tikzcd}$$

iso on H^* over $\mathbb{R}$. $\longrightarrow$

E has I-structure if $\exists$ lift

$$\begin{tikzcd} & I(n) \arrow[d] \\ M \arrow[r] \arrow[ur, dashed] & BU(n) \end{tikzcd}$$

$\leftarrow$ from universal prop.

prop: $\forall$ I-structure $\Rightarrow$ $\overbrace{c_1 c_{2k}(X) - c_{2k+1}(X)}^{\alpha} = 0 \in H^*(X; \mathbb{R})$

proof: from calculating $\ker(H^*(BU(n)) \to H^*(I))$

MV sequence calculating $H^*(I)$ (pt of htpy pushout)

over $\mathbb{R}$, have

$$H^*(I) \longrightarrow H^*(BO \times BU(1)) \longrightarrow H^*(BO \times BO(1))$$

get:

$$\begin{tikzcd} H^*(BO \times BU(1)) \arrow[d, "\cong"', "\text{over } \mathbb{R}"] & H^*(BU) \arrow[l] \arrow[dl] \\ H^*(I) & \end{tikzcd}$$

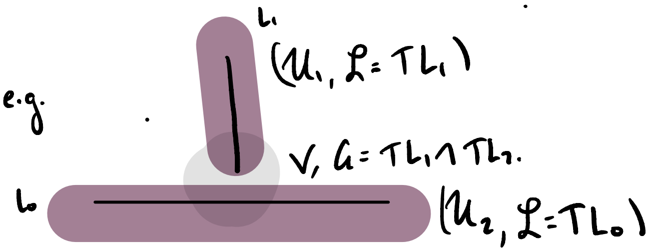
analyse: $\alpha \in \ker$.

(calc. chern $(G \otimes \mathbb{C}) \otimes \mathcal{L}$.)

theorem (Ah-L-W): if X admits arborealisable, X admits I structure with $V \overset{\text{def. retract.}}{\simeq} \operatorname{sing}(\mathbb{L})$.

$\mathcal{U} \simeq \bigcup_{\text{smth comp.}} \mathbb{L}_i$

e.g.



proof: replace I-structure with dual I structure: $((\mathcal{U}, \mathcal{L}), (V, K))$ ← coiso in $\mathcal{U}\cap V$

$K \pitchfork \mathcal{L}$ on $\mathcal{U} \cap V$

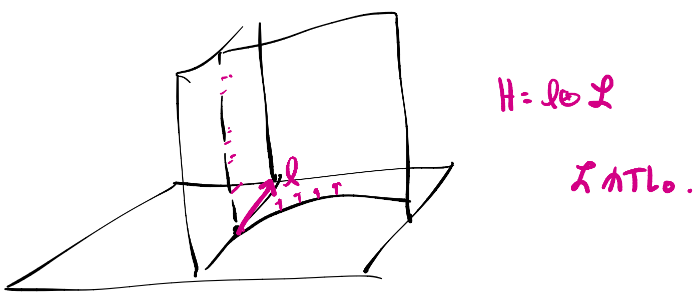
(go between: $K \longleftrightarrow \mathcal{G} \pitchfork K$, contr. choice.)

K iso on $\mathbb{L}$ collaring if $K_p \pitchfork TL_i$ $\forall L_i \ni p$.

[open condition!]

define: sheaf of collaring co isos on $\operatorname{sing} \mathbb{L}$.

e.g.



· global section. take that, and shrink $L_i \to (L_i \setminus \text{nbhd } \partial(L_i))$ *

$(\mathcal{U}_i = \text{nbhd}(*), TL_i)$

Questions:

(i) where is the obstruction??

(ii) class of sings beyond arb?

(iii) can I detect non-arb from $W(x)$?