P-adic Gamma classes and overconvergent Frobenius structures for quantum connections

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Quantum connection

Let M be a closed symplectic manifold which is monotone,

$$[\omega_M] = c_1(M) \in H^2(M; \mathbb{R}).$$

The quantum connection on $H^*(M; \mathbb{C})[q^{\pm 1}]$ differentiates with respect to the variable q:

$$\nabla_q x = \partial_q x + q^{-1}([\omega_M] *_q x),$$

where $*_q$ is the small quantum product,

$$x *_q y = x *^{(0)} y + q\, x *^{(1)} y + q^2\, x *^{(2)} y + \cdots$$

Local monodromy theorem

Let $\bar{S}$ be a complex, smooth projective curve and let

$$S = \bar{S} - \{p_1, \cdots, p_r\}$$

be the complement of a finite number of points. Suppose we have a proper smooth algebraic family $W : Y \longrightarrow S$ with fibers of complex dimension n.

Theorem (Griffiths, Grothendieck, Katz)

- *The local monodromy about each missing point p_i has eigenvalues which are roots of unity and Jordan blocks of size at most $n+1$.*
- *The underlying algebraic vector bundle equipped with its Gauss-Manin connection has regular singularities.*

Monodromies of the quantum connection

Theorem (P.-Seidel, Chen)

Suppose M is a monotone symplectic manifold.

- *The Fourier transform $FT(QH^*(M))$ has only regular singularities at precisely the eigenvalues of $c_1(M)*_{q=1}$ (and ∞).*
- *The monodromies about each singular point are quasi-unipotent (have eigenvalues which are roots of unity).*

Theorem (P.-Seidel)

Suppose M is a monotone symplectic manifold which carries a smooth "anti-canonical divisor" D. Any Jordan block for the monodromy about any singular point of $FT(QH^*(M))$ is size $\leq \dim_{\mathbb{C}}(M)$.

Questions about the monodromy

From a symplectic geometry context, a natural question is:

Question

Is there a monotone symplectic manifold for which $QH^*(M)$ is "undeformed"?

Question

Is the representation

$$\pi_1(\mathbb{A}^1 \setminus \{\lambda_i\}) \longrightarrow GL_n(\mathbb{V}_x) \quad (1)$$

coming from $FT(QH^*(M))$ semi-simple?

Remark

The last question should be compared to Deligne's well-known semi-simplicity result.

Setup

To study arithmetic properties, we switch to coefficients in a p-adic field. Let p be an odd prime. The field is $K = \mathbb{Q}_p(\mu)$, where μ is a primitive p-th root of unity. The p-adic valuation on K, normalized so that $\mathrm{val}(p) = 1$, takes values in $\frac{1}{p-1}\mathbb{Z}$. The p-adic norm is

$$|x| = p^{-\mathrm{val}(x)}. \tag{2}$$

Consider formal power series connections in one variable t, of rank r, with a nilpotent simple pole, and having coefficients in the field K. This means

$$\nabla_{t\partial_t} = t\partial_t + A(t), \quad A(t) = A_0 + tA_1 + t^2A_2 + \cdots, \quad A_0 \text{ nilpotent}, \tag{3}$$

where the A_m are $r \times r$ matrices over K.

Classification

Lemma

There is a unique isomorphism $\Theta(t) = \Theta_0 + t\Theta_1 + \cdots, \quad \Theta_0 = I$, *between* $\nabla^{\mathrm{pole}}_{t\partial_t} = t\partial_t + A_0$ *and the original connection. Concretely, this means that*

$$t\partial_t\Theta + A(t)\Theta(t) - \Theta(t)A_0 = 0. \tag{4}$$

Write out (4) order by order:

$$m\Theta_m + [A_0, \Theta_m] = -\sum_{0\le i<m} A_{m-i}\Theta_i, \quad m \ge 1. \tag{5}$$

Let R_m be the right hand side of (5). Since A_0 is nilpotent, we can solve for Θ_m by the formula

$$\Theta_m = R_m/m - [A_0, R_m]/m^2 + [A_0, [A_0, R_m]]/m^3 + \cdots, \tag{6}$$

where the last term is divided by m^{2N}.

Frobenius

A Frobenius structure is an isomorphism between the original equation and its pullback under the coordinate change $t \mapsto t^p$. Explicitly, the Frobenius pullback of a connection (3) is

$$\nabla^{(p)}_{t\partial_t} = t\partial_t + pA(t^p). \tag{7}$$

Definition

A Frobenius structure for $\nabla_{t\partial_t}$ is an isomorphism

$$\Phi(t) = \Phi_0 + t\Phi_1 + \cdots, \quad \Phi_0 \text{ invertible}, \tag{8}$$

intertwining (7) and the original connection. Explicitly, this satisfies

$$t\partial_t\Phi + A(t)\Phi(t) - p\Phi(t)A(t^p) = 0. \tag{9}$$

Lemma

(i) The constant term Φ_0 *of any Frobenius structure satisfies*

$$A_0 \Phi_0 = p \Phi_0 A_0. \tag{10}$$

Conversely, every invertible Φ_0 *with* (10) *is the constant term of a unique Frobenius structure.*

(ii) Suppose that the map $\Theta(t)$ *from the Lemma converges on an open disc of radius* ρ*. Then any Frobenius structure converges on the open disc of radius* $\min(\rho, \rho^{1/p})$*.*

(iii) Suppose that our equation admits a Frobenius structure Φ *which converges on the open disc of radius* 1*. Then, if the map* Θ *from the previous Lemma converges on some open disc around* 0*, it must converge on the open disc of radius* 1 *(and by (ii), the same will then hold for all Frobenius structures).*

Example

Example

Consider the equation

$$df/dt = f, \tag{11}$$

The function $\exp(t)$ has radius of conv. $p^{-1/(p-1)}$ (Legendre's formula). The Frobenius pullback is $df^{(p)}/dt = pt^{p-1}f^{(p)}$. The relation between the two equations is that

$$f(t) = \Phi(t)f^{(p)}(t), \;\; \Phi(t) = e^{(t-t^p)}, \tag{12}$$

so multiplication by $\Phi(t)$ is a Frobenius structure (and the unique one, up to multiplying by a nonzero constant). The function $\Phi(t)$, defined by the usual power series, has p-adic convergence radius < 1.

Overconvergent Frobenius

Definition

A Frobenius structure Φ is called overconvergent if it converges on a disc of radius > 1.

Example

Our field K contains a unique element π (known as Dwork's constant) with

$$\pi^{p-1} = -p, \quad |(1+\pi) - \mu| \leq p^{-2/(p-1)}. \tag{13}$$

Consider the equation

$$df/dt = \pi f, \tag{14}$$

$$D(\pi t) = \exp(\pi(t - t^p)) \text{ has radius of convergence } p^{(p-1)/p^2} > 1.$$

Thus, (14) has an overconvergent Frobenius structure.

Key example: Bessel equation

Example

Dwork constructed an overconvergent Frobenius $\Phi(t)$ on the Bessel equation. This is the following 2x2 connection

$$t\partial_t + 2\begin{pmatrix} 0 & \pi^2 t^2 \\ 1 & 0 \end{pmatrix}. \tag{15}$$

Over zero, the Frobenius $\Phi(0)$ is given by

$$\Phi(0) = \begin{pmatrix} 1 & 0 \\ 2\Gamma_p'(0) & 1/p \end{pmatrix}, \tag{16}$$

where Γ_p is the p-adic Γ function recalled below.

Rescaling QH^*

Returning to the quantum connection, now with K-coefficients, we rescale the variable to

$$t = q/\pi \tag{17}$$

For simplicity, we also restrict to the even degree part of cohomology (one could carry out a parallel discussion for the odd degree part). The outcome is

$$\nabla_{t\partial_t} : H^{\mathrm{even}}(M;K)[t] \longrightarrow H^{\mathrm{even}}(M;K)[t],$$
$$\nabla_{t\partial_t} x = t\partial_t x + c_1(M) *_{\pi t} x = t\partial_t x + c_1(M) \smile x + \pi t c_1(M) *^{(1)} x + \cdots \tag{18}$$

p-adic Gamma function

The Morita p-adic Gamma function $\Gamma_p(z) \in \mathbb{Z}_p^\times$ is a continuous function of $z \in \mathbb{Z}_p$. One way to define it is by the collection of Mahler expansions:

$$\Gamma_p(-j-pz) = \sum_{m\geq 0} (-p)^m d_{mp+j} z(z-1)\cdots(z-m+1), \quad j \in \{0,\ldots,p-1\},\ z \in \tag{19}$$

where the coefficients are Taylor coefficients of $D(z) = e^{z+z^p/p}$. Note that the formulae (19) for different j compute Γ_p on disjoint subsets of $\mathbb{Z}_p$. Nevertheless, they are related by:

$$\Gamma_p(-j-pz) = \frac{\Gamma_p(-pz)}{(pz+1)\cdots(pz+j)}, \quad j, z \text{ as before.} \tag{20}$$

p-adic Gamma class

Let

$$\Gamma_p(E) \in H^{even}(M; \mathbb{Q}_p) \tag{21}$$

be the multiplicative characteristic class associated to the p-adic Gamma function, applied to a complex vector bundle $E \to M$. In terms of Chern roots $c(E) = 1 + c_1(E) + c_2(E) + \cdots = \prod_{i=1}^{n}(1 + r_i)$,

$$\begin{aligned} \Gamma_p(E) = \prod_{i=1}^{n} \Gamma_p(r_i) &= 1 + \Gamma_p'(0)c_1(E) + \Gamma_p'(0)^2(c_1(E)^2/2) + \\ &+ \Gamma_p'''(0)\big(c_3(E)/2 - c_1(E)c_2(E)/2 + c_1(E)^3/6\big) \\ &+ \Gamma_p'(0)^3\big(-c_3(E)/2 + c_1(E)c_2(E)/2\big) + \cdots \end{aligned} \tag{22}$$

Candidate Frobenius

We now return to the quantum connection. The map Θ from the Lemma above is essentially Givental's fundamental solution:

$$\int_M x \smile \Theta(y) = \int_M x \smile y + \sum_{m>0} q^{\dagger}(-1)^{m+1}\langle x, \psi^m(y)\rangle, \tag{23}$$

Conjecture (Conjecture A)

For the quantum connection on any monotone symplectic manifold, the Frobenius structure with constant term

$$\Phi_0(x) = p^{-\mathrm{deg}(x)/2}\Gamma_p(TM) \smile x \tag{24}$$

is overconvergent.

Theorem (Theorem A)

Conjecture A holds for all Fano toric varieties and Grassmannians.

Convergence of fundamental solution

- When overconvergence holds, we get that:

Claim

The fundamental solution Θ converges on an open p-adic disc of radius one.

- This would follow if the denominators of $< x, \psi^m(y) >$ divide $m!$ (which they seem to do in examples).
- For $< 1, \psi^m(pt) >$ and when M contains a monotone torus, this follows from Tonkonog's formula.
- General case can be studied by Ionel-Parker type arguments? On the other hand, if you want to disprove overconvergence (or even convergence along a disc of radius one) that would be one way.

Eigenvalues of Frobenius

Dwork's interest in overconvergent Frobenius structures on the Bessel equation was to relate their eigenvalues to arithmetic quantities.

Conjecture (Conjecture B)

Take the Frobenius structure from Conjecture A and consider $\Phi(\theta)$ for some $(p-1)$-st root of unity θ. Then, the valuations of the eigenvalues of $\Phi(\theta)$ are nonnpositive integers; and among them (listed with multiplicities), the number of times that each k appears equals the Betti number $b_{-2k}(M)$. Note that for elementary reasons, the valuations under consideration are independent of θ.

Theorem (Theorem B)

Conjecture B holds for all Grassmannians.

Toric mirror symmetry

Take a fan in $N \otimes \mathbb{R}$ giving rise to a smooth projective Fano toric variety M, on which T acts. Let $e_1, \ldots, e_r \in N$ be the primitive generators of the rays of the fan. Our mirror superpotential is:

$$W_t = \sum_{i=1}^{r} \pi t z^{e_i} \in K[t^{\pm}, z_j^{\pm}]. \tag{25}$$

We consider differential forms $\Omega^* := \Omega^*_{K[t^{\pm}, z_i^{\pm}]/K[t^{\pm}]}$. These carry operations:

$$\begin{aligned} d + dW_t &: \Omega^* \longrightarrow \Omega^{*+1}, \\ t\nabla_{\partial_t} = t\partial_t + (W_t) &: \Omega^* \longrightarrow \Omega^*. \end{aligned} \tag{26}$$

Theorem

There is an isomorphism compatible with connections:

$$QH^*(M; K[t^{\pm}]) \cong H^*(\Omega^*, d + dW_t), \tag{27}$$

Lattice in differential forms

We are concerned with the behavior at $t = 0$, so we need a refinement of the above mirror symmetry. We consider differential forms with coefficients in the "positive action" ring

$$\theta = \sum_{i \geq w(\mathbf{n})} a_{i,\mathbf{n}} t^i z^{\mathbf{n}}, \tag{28}$$

where $w(\mathbf{n})$ is the piecewise linear function on the toric fan defined by $-\mathcal{K}_M$.

Theorem

There is an isomorphism of D-modules:

$$H^*(\Omega^*_{\geq 0}, d + dW_t) \cong QH^*(M) \tag{29}$$

Dwork inverse Frobenius

The definition of Dwork's inverse Frobenius comes in two parts.

1. The first one is given by multiplication with an invertible power series constructed from the Dwork exponential:
$$\theta \longmapsto \prod D(\pi t z^{e_i})\theta. \tag{30}$$
This relates the differential $d + d(W_t)$ to a modified $d + d(W_t^{(p)})$.
2. The second step is to make the substitution $z_i \mapsto z_i^{1/p}$ and throw away non-integer powers of z_i. Explicitly, in one variable
$$z^k \longmapsto \begin{cases} z^{k/p} & p|k, \\ 0 & \text{otherwise} \end{cases} \qquad z^k \frac{dz}{z} \longmapsto \begin{cases} (1/p) z^{k/p} \frac{dz}{z} & p|k, \\ 0 & \text{otherwise.} \end{cases} \tag{31}$$
This relates modified differential $d + d(W_t^{(p)})$ back to $d + d(W_{t^p})$.

P-adic Banach space

The main task is to find a p-adic analytic space of functions where the Dwork exponential converges, but the usual exponential does not.

Definition

Fix a constant $b > 0$. Let

$$R_{L(b)} := \{\sum a_{i,\mathbf{n}} t^i z^{\mathbf{n}}, w(\mathbf{n}) \leq i, \quad \exists c \quad \mathrm{val}(a_{i,\mathbf{n}}) \geq b \cdot i + c.\}, \tag{32}$$

Let $L(b)$ denote the functions with $\mathbf{n} = 0$ (convergent on the disc where $\mathrm{val}(t) > -b$):

Key Theorem

Suppose $b > \frac{1}{p-1}$. The inclusion $\Omega^*_{\geq 0} \subset \Omega^*_{L(b)}$ induces an isomorphism

$$H^n(\Omega^*_{\geq 0}, d + dW_t) \otimes_{K[t]} L(b) \cong H^n(\Omega^*_{R_{L(b)}}, d + dW_t). \tag{33}$$

One ingredient in proof

Let $R_{K\langle t\rangle}$ be the space of series

$$\sum_{\substack{k,\,\mathbf{n}\\ k\geq w(\mathbf{n})}} a_{k,\mathbf{n}}\pi^{w(\mathbf{n})}t^k z^{\mathbf{n}} \tag{34}$$

which satisfy the condition that for any $C \in \mathbb{R}$, there are only finitely many $(k, \mathbf{n})$ with $\mathrm{val}(a_{k,\mathbf{n}}) \leq C$.

There is also an integral version $R_{\mathcal{O}\langle t\rangle}$ which satisfies:

$$R_{\mathcal{O}\langle t\rangle}/(\pi) \cong SR_{\mathbb{F}_p}(\Delta), \tag{35}$$

where $SR_{\mathbb{F}_p}(\Delta)$ is the Stanley-Reisner ring with product structure:

$$t^v z^{\mathbf{m}} \cdot t^w z^{\mathbf{n}} = \begin{cases} t^{v+w} z^{\mathbf{m}+\mathbf{n}} & \text{if } \mathbf{m}, \mathbf{n} \text{ lie in the same cone of the fan,} \\ 0 & \text{otherwise.} \end{cases} \tag{36}$$

Leading order term

$$\Psi : \frac{dz}{z} \to p^{-1} \sum_{\substack{k,l\geq 0 \\ k\equiv l \bmod p}} d_k d_l \pi^{k+l} t^{k+l} z^{(k-l)/p} \frac{dz}{z}. \tag{37}$$

- For the right hand side of (37), all terms with both $k > 0$ and $l > 0$ represent classes in $t^p \bar{L}$. Abbreviate $\bar{x}_0 = dz/z, \bar{x}_1 = (\pi t)zdz$.

$$[(\pi \cdot t)^{pk} z^k (dz/z)] \equiv [(\pi \cdot t)^{pk} z^{-k} (dz/z)] \equiv -(k-1)! p^k \bar{x}_1 \bmod t^p \bar{L}. \tag{38}$$

$$\begin{aligned} \Psi(x_0) &= p^{-1}\Big[\frac{dz}{z}\Big] + \sum_{k>0} p^{-1} d_{pk}\Big[(\pi t)^{pk} z^k \frac{dz}{z}\Big] + \sum_{l>0} p^{-1} d_{pl}\Big[(\pi t)^{pl} z^{-l} \frac{dz}{z}\Big] \\ &= p^{-1}\bar{x}_0 - 2\sum_{l>0} (l-1)!\, p^{l-1} d_{pl}\, \bar{x}_1 \ = p^{-1}\bar{x}_0 - 2\Gamma_p'(0)\bar{x}_1 \mod t^p \bar{L}. \end{aligned} \tag{39}$$

Grassmannians

Let $M = Gr(k, N)$. There is a Satake isomorphism:

$$S : \Lambda^k H^*(\mathbb{C}P^{N-1}; \mathbb{Q}) \cong H^{*-k(k-1)}(M; \mathbb{Q}). \tag{40}$$

In terms of Schur polynomials:

$$S(x^{d_1} \wedge \cdots \wedge x^{d_k}) = \frac{\det\big((r_i^{d_j})_{ij}\big)}{\det\big((r_i^{k-j})_{ij}\big)}. \tag{41}$$

Here, $x \in H^2(\mathbb{C}P^{N-1})$ is the hyperplane class. On the right hand side, both numerator and denominator are determinants of $k \times k$ matrices with coefficients in $\mathbb{Q}[r_1, \ldots, r_k]$; the quotient is a symmetric polynomial.

Lemma

The map (40) *induces an isomorphism of quantum D-modules.*

Grassmannians (cont)

We apply this to the p-adic Gamma classes:

Lemma

Let $\Phi_{\mathbb{C}P^{N-1},0}$ and $\Phi_{M,0}$ be the endomorphisms (24) for those two manifolds. They fit into a commutative diagram

$$\begin{array}{ccc} \Lambda^k H^*(\mathbb{C}P^{N-1};\mathbb{Q}_p) & \xrightarrow[\cong]{S} & H^{*-k(k-1)}(M;\mathbb{Q}_p) \\ \Big\downarrow{\scriptstyle \Lambda^k_{\mathrm{Group}}(\Phi_{\mathbb{C}P^{N-1},0})} & & \Big\downarrow{\scriptstyle p^{-k(k+1)/2}\Phi_{M,0}} \\ \Lambda^k H^*(\mathbb{C}P^{N-1};\mathbb{Q}_p) & \xrightarrow[\cong]{S} & H^{*-k(k-1)}(M;\mathbb{Q}_p) \end{array} \tag{42}$$

Here $\Lambda^k_{\mathrm{Group}}(\Phi_{\mathbb{C}P^{N-1},0})$ is the restriction of $\Phi_{\mathbb{C}P^{N-1},0} \otimes \cdots \otimes \Phi_{\mathbb{C}P^{N-1},0}$ to the antisymmetric part.

Symplectic cohomology

Suppose for simplicity that M carries a snc anti-canonical divisor D.

- Then using a standard model for a tubular neighborhood of D, the complement $X := M \setminus D$ is a convex symplectic manifold. One can therefore associate to X its symplectic cohomology $SC^*(X)$.
- $SC^*(X)$ carries the structure of an L_∞-algebra (in fact E_2 algebra).
- Roughly speaking, in the case D is smooth, the generators of this complex look like

$$C^*(X) \bigoplus_{w \geq 1} C^*(SD) z^w$$

- We can arrange that $SC^*(X)$ is concentrated in non-negative degrees and carries an "approximate action" filtration by the winding number $w(x_0)$ of an orbit around the divisor D.

Deformed symplectic cohomology

- The symplectic cohomology carries a certain class $\theta \in SH^0(X)$ which measures how compactifying $X \subset M$ *deforms* its Floer theory. More precisely, θ gives a Maurer-Cartan element $q\theta \in MC^\bullet(SC^*(X)[q])$ (just for degree reasons).
- The S^1-equivariant symplectic cohomology $SC^*_{S^1}(X)$ is an L_∞ module over $SC^*(X)$.
- One can use the class θ to deform the S^1-equivariant symplectic cohomology, $SC^*_{S^1}(M, D)$. For general reasons, this acquires a connection $u\nabla_q$, where u is the distinguished generator of $H^2(BS^1)$.

Deformed SH^* versus QH^*

Theorem (P-Seidel)

Suppose for simplicity D is smooth. There is a canonical isomorphism

$$\mathsf{G} : H^*(M)[q,u] \oplus \bigoplus_{w \geq 1} H^*(D)[u]z^w \cong H^*(SC^*_{S^1}(M,D)),$$

The map is q-linear on the $H^*(M)[q,u]$ component.

Corollary

After inverting q, we obtain an isomorphism

$$QH^*(M)[u,q^{\pm}] \cong H^*(SC^*_{S^1}(M,D)) \otimes_{\mathbb{C}[q]} \mathbb{C}[q^{\pm}],$$

taking the quantum connection to $u\nabla_q$.

Dwork symplectic cohomology

Fix a constant $b > 0$. We can consider a version of symplectic cohomology, $SH^*_{S^1, L(b)}(M, D)$ which roughly speaking includes cochains of the form:

$$\{\sum a_{i,\mathbf{n}} t^i x_0, \quad i - w(x_0) \geq 0, \quad \exists c \quad \mathrm{val}(a_{i,\mathbf{n}}) \geq b \cdot i + c.\}, \tag{43}$$

Theorem

Suppose $b > \frac{1}{p-1}$. The above map gives an isomorphism compatible with connections:

$$G : H^*(M; L(b)) \longrightarrow SH^*_{S^1, L(b)}(M, D) \tag{44}$$

Dwork exponential on symplectic cohomology?

There is an isomorphism:

$$SC^*_{S^1}(X)((u)) \cong H^*(X)((u)). \tag{45}$$

We can integrate the Gauss-Manin connection to give an isomorphism:

$$(SH^*_{S^1}(X)((u))[[q]], d_q) \longrightarrow (SH^*_{S^1}(X)((u))[[q]], d_q^{(p)}). \tag{46}$$

Question

Does this "converge" to an isomorphism:

$$SH^*_{S^1, L(p-1/p)}(M, D)((u)) \longrightarrow SH^*_{S^1, L(\frac{p-1}{p^2})}(M, D)((u))^{(p)}, \tag{47}$$

compatible with connections on both sides?