A generalization of the Murnaghan-Nakayama rule for K-k-Schur and k-Schur functions

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1 Motivation

A function $f(x_1, x_2, \dots)$ is symmetric if $f(x_1, x_2, \dots) = f(x_{\sigma(1)}, x_{\sigma(2)}, \dots)$ for all $\sigma \in S_\infty$.

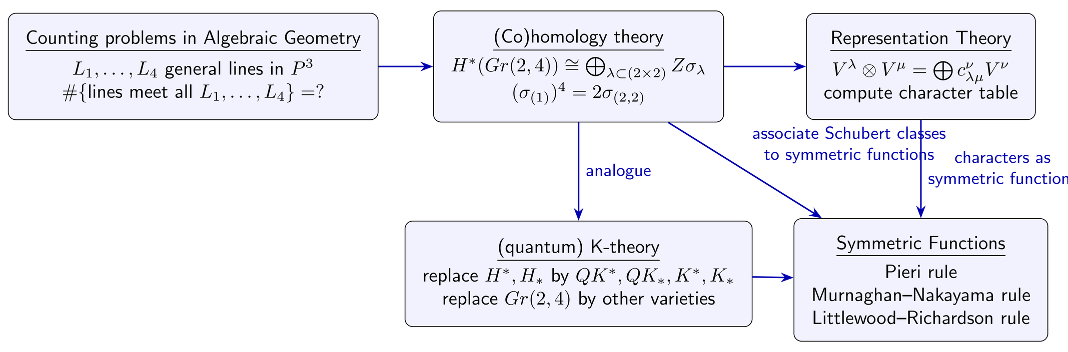
Example 1. *There are some fundamental polynomials in variables $x_1, \dots, x_n$:*

- $e_r = \sum_{1\le i_1<\dots<i_r\le n} x_{i_1}\dots x_{i_r}$ *elementary symmetric polynomial,*
- $h_r = \sum_{1\le i_1\le\dots\le i_r\le n} x_{i_1}\dots x_{i_r}$ *complete homogeneous symmetric polynomial,*
- $p_r = x_1^r + \dots + x_n^r$ *power-sum symmetric polynomial,*
- $s_\lambda = \frac{\det(x_i^{\lambda_j+n-j})_{n\times n}}{\prod_{1\le i<j\le n}(x_i-x_j)}$ *Schur function.*

$\Lambda = \bigoplus_{n\ge 0} \Lambda^n$: graded ring of symmetric functions in variables $x_1, x_2, \dots$ with coefficients in $\mathbb{Z}$. We know that Λ^n has a $\mathbb{Z}$-basis $\{s_\lambda \mid \lambda \text{ is a partition of } n\}$.

Fundamental rules: For $\lambda \vdash n$, $0 \le r \le n$,

- Pieri rule: $e_r.s_\lambda = \sum_{\mu\vdash n} *s_\mu$, $h_r.s_\lambda = \sum_{\mu\vdash n} *s_\mu$,
- Murnaghan-Nakayama rule: $p_r.s_\lambda = \sum_{\mu\vdash n} *s_\mu$,
- Littlewood-Richardson rule: $s_\lambda.s_\mu = \sum_{\nu\vdash n} *s_\nu$.



We consider $Gr = SL_{k+1}(\mathbb{C}((t)))/SL_{k+1}(\mathbb{C}[[t]])$.

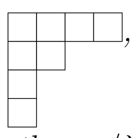
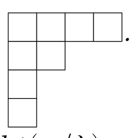
- $H_*(Gr) = \bigoplus_{\lambda_1\le k} \mathbb{Z}\sigma_\lambda^{(k)}$.
  $\sigma_\lambda^{(k)}$: Schubert class associated to λ.
  $s_\lambda^{(k)}$: k-Schur functions associated to λ.
  $\sigma_\lambda^{(k)}.\sigma_\mu^{(k)} = \sum_\nu *\sigma_\nu^{(k)}$ is computed by $s_\lambda^{(k)}.s_\mu^{(k)} = \sum_\nu *s_\nu^{(k)}$.
- $K_*(Gr) = \bigoplus_{\lambda_1\le k} \mathbb{Z}\theta_\lambda^{(k)}$.
  $\theta_\lambda^{(k)}$: Schubert class associated to λ.
  $g_\lambda^{(k)}$: K-k-Schur functions associated to λ.
  $\theta_\lambda^{(k)}.\theta_\mu^{(k)} = \sum_\nu *\theta_\nu^{(k)}$ is computed by $g_\lambda^{(k)}.g_\mu^{(k)} = \sum_\nu *g_\nu^{(k)}$.

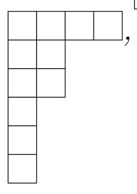
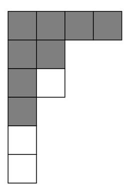
In [Ngu24], we introduce a new family of symmetric functions $\mathcal{F}_\lambda^{(k)}$, that generalizes the constructions via the Piere rule of K-k-Schur functions and k-Schur functions. Then we obtain the Murnaghan-Nakayama rule for the generalized functions. The rule are described explicitly in the cases of K-k-Schur functions and k-Schur functions, with concrete descriptions and algorithms for coefficients. Our work recovers the result of Bandlow, Schilling, and Zabrocki for k-Schur functions [BSZ11], and explains it as a degeneration of the rule for K-k-Schur functions. In particular, many other special cases and connections promise to be detailed in the future.

2 Definitions

$\lambda = (\lambda_1 \ge \lambda_2 \ge \dots)$, $\lambda_i \in \mathbb{Z}_{\ge 0}$: a partition $\sim$ Young diagram.
λ^t: conjugate partition of $\lambda \sim$ reflection of the Young diagram of λ through the main diagonal.
For $\lambda \le \mu$,

- μ/λ is the shape consisting of all boxes in μ but not in λ,
- we call μ/λ a ribbon if it does not contain $(2, 2)$,
- then the height of the skew shape $ht(\mu/\lambda)$ is defined by the number of vertical dominoes in μ/λ.

Example 2. $\lambda = (4, 2, 1, 1) \sim$ , $\lambda^t = (4, 2, 1, 1) \sim$ .

$\mu = (4, 2, 2, 1, 1, 1) \sim$ , *then* $\mu/\lambda =$  *and* $ht(\mu/\lambda) = 1$.

$\widetilde{S}_{k+1}$: affine symmetric group with generators $s_0, \dots, s_k$ satisfying relations

$$s_i^2 = 1 \text{ for all } i \qquad (1)$$
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \text{ for all } i, \qquad (2)$$
$$s_i s_j = s_j s_i \text{ for all } i - j \ne \pm 1, \qquad (3)$$

where the indices are taken from $\mathbb{Z}/(k+1)\mathbb{Z}$.
S_{k+1}: symmetric group with generators $s_1, \dots, s_k$.
$s_{i_1\dots i_r} := s_{i_1} \dots s_{i_r}$.
$\widetilde{S}^0_{k+1} := \{$minimum length coset representative of $\widetilde{S}_{k+1}/S_{k+1}\}$.
For $\lambda = (\lambda_1, \lambda_2, \dots)$ such that $\lambda_1 \le k$, we have bijections

$$\lambda \overset{\phi}{\mapsto} w_\lambda \in \widetilde{S}^0_{k+1} \overset{\mathfrak{s}}{\mapsto} \kappa_\lambda \overset{\mathfrak{p}}{\mapsto} \lambda.$$

Example 3. *Let $k = 4$, $\lambda = (4, 2, 1, 1)$. Then $w_\lambda = s_{23043210}$, and $\kappa_\lambda = (6, 2, 1, 1)$.*
For $\lambda \overset{\phi}{\mapsto} w_\lambda$, we use

	1	2	3	4	5	6	7	8	9
1	0	1	2	3	4	0	1	2	3
2	4	0	1	2	3	4	0	1	2
3	3	4	0	1	2	3	4	0	1
4	2	3	4	0	1	2	3	4	0
5	1	2	3	4	0	1	2	3	4
6	0	1	2	3	4	0	1	2	3
7	4	0	1	2	3	4	0	1	2

For $w_\lambda \overset{\mathfrak{s}}{\mapsto} \kappa_\lambda$, we use

	1	2	3	4	5	6	7	8	9
1	0	1	2	3	4	0	1	2	3
2	4	0	1	2	3	4	0	1	2
3	3	4	0	1	2	3	4	0	1
4	2	3	4	0	1	2	3	4	0
5	1	2	3	4	0	1	2	3	4
6	0	1	2	3	4	0	1	2	3
7	4	0	1	2	3	4	0	1	2

$\mathcal{A}_k$: associative algebra over $\mathbb{Z}$ with generators $A_0, \dots, A_k$ satisfying relations (2), (3).
$A_{i_1\dots i_r} := A_{i_1} \dots A_{i_r}$.
For $0 \le r \le k$, $A \subsetneq \mathbb{Z}/(k+1)\mathbb{Z}$, $|A| = r$, we set

$$d_A = A_{i_1\dots i_r}, \quad i_A = A_{i_r\dots i_1},$$

where $(i_1 \dots i_r)$ is a rearrangement of A such that if $i, i+1 \in A$, then $i+1$ occurs before i.

Example 4. *Let $k = 4$, $A = \{0, 2, 4\}$, then $d_A = A_{042} = A_{024}$, $i_A = A_{240} = A_{420}$.*

We define

- $h_r = \sum_{A\in\binom{[0,k]}{r}} d_A$: noncommutative homogeneous symmetric functions.
- $e_r = \sum_{A\in\binom{[0,k]}{r}} i_A$: noncommutative homogeneous symmetric functions.
- $s_{(r-i,1^i)} = \sum_{j=0}^{i} (-1)^j h_{r-i+j} e_{i-j}$: noncommutative hook Schur symmetric functions.
- $p_r = \sum_{i=0}^{r-1} (-1)^i s_{(r-i,1^i)}$: noncommutative power sum symmetric functions.

$\mathcal{A}_k \curvearrowright \mathbb{C}[\widetilde{S}_{k+1}]$ is defined by $\alpha *_\varphi w$.
$\psi : \mathcal{A}_k \times \widetilde{S}_{k+1} \to \mathbb{R}$ is said to be φ-compatiable if $\psi(\alpha\beta, w) = \psi(\alpha, \beta *_\varphi w)\psi(\beta, w)$.
Fix φ, ψ, let $\{\mathcal{F}_w^{(k)}\}_{w\in\widetilde{S}^0_{k+1}}$ be a family of symmetric functions such that

$$\mathcal{F}_{id}^{(k)} = 1,$$
$$h_r.\mathcal{F}_w^{(k)} = \sum_{\substack{A\in\binom{[0,k]}{r}\\ d_A *_\varphi w\in\widetilde{S}^0_{k+1}}} \psi(d_A, w)\mathcal{F}^{(k)}_{d_A *_\varphi w},$$
$$e_r.\mathcal{F}_w^{(k)} = \sum_{\substack{B\in\binom{[0,k]}{r}\\ i_B *_\varphi w\in\widetilde{S}^0_{k+1}}} \psi(i_B, w)\mathcal{F}^{(k)}_{i_B *_\varphi w},$$

for $w \in \widetilde{S}^0_{k+1}$, $0 \le r \le k$. We define $\mathcal{F}_\lambda^{(k)} = \mathcal{F}_{w_\lambda}^{(k)}$ for $w_\lambda \in \widetilde{S}^0_{k+1}$.

Example 5.

- *$\mathcal{A}_k \curvearrowright \mathbb{C}[\widetilde{S}_{k+1}]$ is defined by*

$$A_i * w = \begin{cases} s_i w & \text{if } l(s_i w) > l(w),\\ w & \text{if } l(s_i w) < l(w).\end{cases}$$

  $\psi(\alpha, w) = (-1)^{\widetilde{l}(\alpha)-l(\alpha*w)+l(w)}$ where $\widetilde{l}(\alpha)$ is the number of letters of α and $l(w)$ is the length of w in $\widetilde{S}_{k+1}$. Then $\mathcal{F}_w^{(k)} = g_w^{(k)}$ the K-k-Schur functions.

- *$\mathcal{A}_k \curvearrowright \mathbb{C}[\widetilde{S}_{k+1}]$ is defined by*

$$A_i.w = \begin{cases} s_i w & \text{if } l(s_i w) > l(w),\\ 0 & \text{otherwise.}\end{cases}$$

  $\psi(\alpha, w) = 1$. Then $\mathcal{F}_w^{(k)} = s_w^{(k)}$ the k-Schur functions.

For $u \in \mathcal{A}_k$,

- $S := supp(u)$,
- $I_S :=$ canonical cyclic interval of S:
  1. Let a be the minimum in $[0, k]$ such that $a \notin S$,
  2. then I_S is $a+1 < a+2 < \dots < a-1$.

Example 6. $u = A_{0424} \in \mathcal{A}_4$. *Then*

- $S := supp(u) = \{0, 2, 4\}$,
- $I_S = 2 < 3 < 4 < 0$.

u is called k-connected if S is an interval of I_S.
u is called a weak hook word if it has a reduced word of form $\bigvee$, say hook type V, or $\bigsqcup$ with $u_i = u_{i+1}$ for some i, say hook type U. In this case,

- $asc(u) := \#\{$ascents of hook forms $\bigvee, \bigsqcup$ with respect to the order $I_S\}$.
- $\mathcal{C}_u := \{$consecutive pairs $a < c$ s.t $a \ne c-1$, no $a < b < c$ in hook form of $u\}$.
- $c_{min} := min\{c \mid (a < c) \text{ in } \mathcal{C}_u\}$. Fact: $\#c_{min} \in \{0, 1, 2\}$.

Example 7. *$S = 2 < \not{3} < 4 < 0$, then u is not 4-connected.*
$u = A_{0424}$, then hook type of u is V, and $c_{min} = 4$ and it is in right and left sides of V.

$$u \begin{cases} \text{connected} \\ \text{disconnected} \begin{cases} \#c_{min} = 2 \\ \#c_{min} = 1 \begin{cases} c_{min} \text{is on the left side} \\ c_{min} \text{is on the right side} \end{cases} \end{cases} \end{cases}$$

We classify u into three types:

- wc: weak connected, for lines 1, 2.
- $\overline{wc}$, left: not weak connected, c_{min} is on the left side
- $\overline{wc}$, right: not weak connected, c_{min} is on the right side.

Notations: $u \in V^r_{i,wc}$ means

- the number of letters of u is r,
- weak hook type of u is V,
- $asc(u) = i$,
- weak connected.

Similar for $U^r_{i,wc}$, $U^r_{i,\overline{wc},\text{left}}$, $U^r_{i,\overline{wc},\text{right}}$.

Example 8. $A_{0424} \in V^4_{1,wc}$, $A_{4224} \in U^4_{1,wc}$, $A_{2240} \in U^4_{2,\overline{wc},right}$.

3 Main results

$f \doteq S$ means $f = \sum_{u\in S} u$.

Lemma 9. *For $1 \le r \le k$, we have*

$$p_r \doteq \sum_{i=0}^{r-1} (-1)^i V^r_{i,wc} + \sum_{i=1}^{r-1} (-1)^i (r-i) U^r_{i-1,wc} + \sum_{i=1}^{r-2} (-1)^i U_{i-1,\overline{wc},left}.$$

Theorem 10. *Suppose that $\psi(\alpha, w)$ depends only on $\widetilde{l}(\alpha)$, $\alpha *_\varphi w$, w, and we can write it as a function $\widetilde{\psi}$ on the three variables. Then for $1 \le r \le k$ and $w, w' \in \widetilde{S}^0_{k+1}$, the coefficient of $\mathcal{F}^{(k)}_{w'}$ in $p_r.\mathcal{F}^{(k)}_w$ is*

$$\widetilde{\psi}(r, w', w) \left(\sum_{i=0}^{r-1} (-1)^i |V^{r,w'}_{i,wc}| + \sum_{i=1}^{r-1} (-1)^i (r-i) |U^{r,w'}_{i-1,wc}| + \sum_{i=1}^{r-2} (-1)^i |U^{r,w'}_{i-1,\overline{wc},left}| \right),$$

where $X^{r,w'}_{i,con,side}$ is the subset of all weak hook words u in $X^r_{i,con,side}$ such that $u *_\varphi w = w'$.

$\mathcal{A}_k \curvearrowright \mathbb{C}[\widetilde{S}_{k+1}]$ is defined by

$$\eta(A_i)(w) = A_i * w = \begin{cases} s_i w & \text{if } l(s_i w) > l(w),\\ w & \text{if } l(s_i w) < l(w).\end{cases}$$

$\psi(\alpha, w) = (-1)^{\widetilde{l}(\alpha)-l(\alpha*w)+l(w)}$ where $\widetilde{l}(\alpha)$ is the number of letters of α and $l(w)$ is the length of w in $\widetilde{S}_{k+1}$. Then $\mathcal{F}_w^{(k)} = g_w^{(k)}$ the K-k-Schur functions.

Corollary 11. *For $1 \le r \le k$ and $w \in \widetilde{S}^0_{k+1}$, we have*

$$p_r.g_w^{(k)} = \sum_{w'\in\widetilde{S}^0_{k+1}} (-1)^{r-l(w')+l(w)} \left(\sum_{i=0}^{r-1} (-1)^i |V^{r,w'}_{i,wc}| + \sum_{i=1}^{r-1} (-1)^i (r-i) |U^{r,w'}_{i-1,wc}| + \sum_{i=1}^{r-2} (-1)^i |U^{r,w'}_{i-1,\overline{wc},left}| \right) g^{(k)}_{w'}.$$

Corollary 12. *For $1 \le r \le k$ and $\lambda \in \mathcal{P}_k$, we have*

$$p_r.g_\lambda^{(k)} = \sum_\mu (-1)^{r-|\mu|+|\lambda|} \left(\sum_{i=0}^{r-1} (-1)^i |V^{r,\mu}_{i,wc}| + \sum_{i=1}^{r-1} (-1)^i (r-i) |U^{r,\mu}_{i-1,wc}| + \sum_{i=1}^{r-2} (-1)^i |U^{r,\mu}_{i-1,\overline{wc},left}| \right) g^{(k)}_\mu,$$

where the sum runs over $\mu \in \mathcal{P}_k$ such that

(K0) $\lambda \subset \mu$, $\lambda^{(k)} \subset \mu^{(k)}$,
(K1) $|\mu/\lambda| \le r$,
(K2) $\mathfrak{p}^{-1}(\mu)/\mathfrak{p}^{-1}(\lambda)$ *is a ribbon,*
(K3) $supp(\mathfrak{p}^{-1}(\mu)/\mathfrak{p}^{-1}(\lambda))$ *is k-connected, or* $|supp(\mathfrak{p}^{-1}(\mu)/\mathfrak{p}^{-1}(\lambda))| < r$,
(K4) $ht(\mu/\lambda) + ht(\mu^{(k)}/\lambda^{(k)}) < r$.

$\mathcal{A}_k \curvearrowright \mathbb{C}[\widetilde{S}_{k+1}]$ is defined by

$$\zeta(A_i)(w) = A_i.w = \begin{cases} s_i w & \text{if } l(s_i w) > l(w),\\ 0 & \text{otherwise.}\end{cases}$$

$\psi(\alpha, w) = 1$. Then $\mathcal{F}_w^{(k)} = s_w^{(k)}$ the k-Schur functions.

Corollary 13 (Theorem 3.1, [BSZ11])**.** *For $1 \le r \le k$ and $w \in \widetilde{S}^0_{k+1}$, we have*

$$p_r.s_w^{(k)} = \sum_{w'\in\widetilde{S}^0_{k+1}} \left(\sum_{i=0}^{r-1} (-1)^i |V^{r,w'}_{i,c}| \right) s^{(k)}_{w'}.$$

Corollary 14 (Theorem 1.2, [BSZ11])**.** *For $1 \le r \le k$ and $\lambda \in \mathcal{P}_k$, we have*

$$p_r.s_\lambda^{(k)} = \sum_\mu \left(\sum_{i=0}^{r-1} (-1)^i |V^{r,\mu}_{i,c}| \right) s^{(k)}_\mu,$$

where the sum runs over $\mu \in \mathcal{P}_k$ such that

(k0) $\lambda \subset \mu$, $\lambda^{(k)} \subset \mu^{(k)}$,
(k1) $|\mu/\lambda| = r$,
(k2) $\mathfrak{p}^{-1}(\mu)/\mathfrak{p}^{-1}(\lambda)$ *is a ribbon,*
(k3) $supp(\mathfrak{p}^{-1}(\mu)/\mathfrak{p}^{-1}(\lambda))$ *is k-connected,*
(k4) $ht(\mu/\lambda) + ht(\mu^{(k)}/\lambda^{(k)}) = r - 1$.

Algorithm 15. *In both cases, we give an effective algorithm to find the sets which contribute to the decomposition. Hence, we can compute coefficients by hand easily.*

Example 16 (Murnaghan-Nakayama rule for K-k-Schur functions)**.** *Let $k = 4$, $r = 4$, $\lambda = (4, 2, 1, 1)$, $\mu = (4, 2, 2, 2, 1)$. We have $\kappa_\lambda = (6, 2, 1, 1)$, $\kappa_\mu = (7, 3, 2, 2, 1)$.*

	1	2	3	4	5	6	7	8	9
1	0	1	2	3	4	0	1	2	3
2	4	0	1	2	3	4	0	1	2
3	3	4	0	1	2	3	4	0	1
4	2	3	4	0	1	2	3	4	0
5	1	2	3	4	0	1	2	3	4
6	0	1	2	3	4	0	1	2	3
7	4	0	1	2	3	4	0	1	2

An algorithm on the skew tableau gives us weak hook words

$$A_{1034} \in V^{4,\mu}_{1,wc}, A_{3124} \in V^{4,\mu}_{2,wc}, A_{1340} \in V^{4,\mu}_{2,wc}, A_{3114} \in U^{4,\mu}_{1,\overline{wc},left}, A_{1134} \in U^{4,\mu}_{2,\overline{wc},right}, A_{3134} \in V^{4,\mu}_{2,wc}.$$

The coefficient of $g_\mu^{(4)}$ in $p_4.g_\lambda^{(4)}$ is

$$(-1)^{4+|\mu|-|\lambda|} \left(|V^{4,\mu}_{0,wc}| - |V^{4,\mu}_{1,wc}| + |V^{4,\mu}_{2,wc}| - |V^{4,\mu}_{3,wc}| - 3|U^{4,\mu}_{0,wc}| + 2|U^{4,\mu}_{1,wc}| - |U^{4,\mu}_{2,wc}| - |U^{4,\mu}_{0,\overline{wc},left}| + |U^{4,\mu}_{1,\overline{wc},left}| \right)$$
$$= -3.$$

4 Future directions

Our Murnaghan-Nakayama rule for $\mathcal{F}_w^{(k)}$ covers the rule for

- K-k-Schur functions $g_w^{(k)}$,
- k-Schur functons $s_w^{(k)}$,
- dual k-Schur functions,
- affine stable Grothendieck polynomials,
- closed K-k-Schur functions,
- etc.

Future directions:

- Give detail descriptions of the rule for the remaining cases (not $g_w^{(k)}$, $s_w^{(k)}$).
- Translate the rules to $QH^*(Fl_{k+1})$, $QK^*(Fl_{k+1})$ via Peterson isomorphism $H_*(Gr) \simeq QH^*(Fl_{k+1})$, K-Peterson isomorphism $K_*(Gr) \simeq QK^*(Fl_{k+1})$.

References

[BSZ11] Jason Bandlow, Anne Schilling, and Mike Zabrocki. The Murnaghan-Nakayama rule for k-Schur functions. *Journal of Combinatorial Theory, Series A*, 118(5):1588–1607, 2011.

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