Yuuki Shiraishi

Statistical patterns in the geography of Calabi-Yau 3-folds.

M. Miura

$/\mathbb{C}$

Def Y : normal proj var with at worst Gorenstein canonical sing

Y : Calabi-Yau $\iff \begin{cases} K_Y \simeq \mathcal{O}_Y \\ H^i(Y, \mathcal{O}_Y) = 0 \\ (0 < i < \dim Y) \end{cases}$

<u>Goal</u>

Geography of smooth

Calabi-Yau 3-folds

Y : sm CY3 $\rightsquigarrow$ $(h^{1,1}, h^{2,1})$ Hodge pair

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{11} & & 0 & \\ 1 & & h^{21} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \qquad \begin{array}{c} 1 \\ 0 \\ h^{1,1} \\ 2+2h^{2,1} \\ h^{1,1} \\ 0 \\ 1 \end{array} \qquad 2(h^{1,1}-h^{2,1}) = \chi$$

Hodge #　　　　Betti #　　　　Euler #

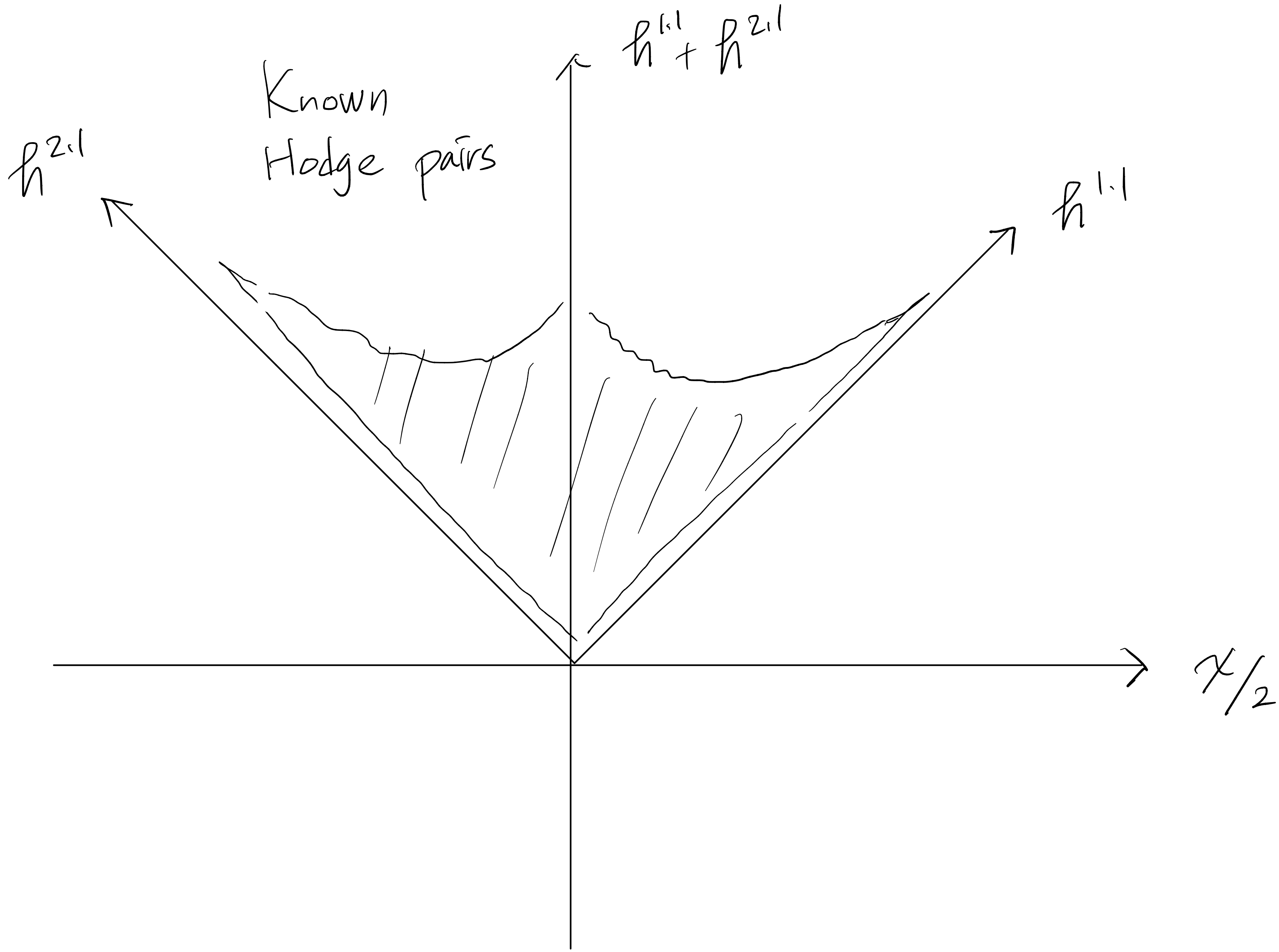
In physics

$|\chi/2|$ is regarded as

\# of generations of fermions.

Problem 0

Why three generations?



$h^{1,1}+h^{2,1}$
Known Hodge pairs
$h^{2,1}$
$h^{1,1}$
$\chi/2$

(11) Problem 1 (Boundedness)

$\#$ topological type of sm CY3 $< \infty$?

Families $\not\longrightarrow$

(up to birat / derived equiv) ?

$\#\{(h^{11}, h^{21}) \mid \exists$ sm CY$\} < \infty$?

$\underline{\{(h^{11}, h^{21}) \mid \exists \text{ sm CY}} = S_{all}$

$\# \quad \chi < \infty$?

$S \subset S_{all}$ finite

$$p(S) := \frac{\#\{(h^{11}, h^{21}) \in S \mid \chi/2 = h^{11} - h^{21} \text{ odd}\}}{\#S}$$

odd parity rate

Assumption

the sampling procedure of S is parity blind.

$\Rightarrow$ the standard error is

$$SE(\hat{S}) = \sqrt{\frac{p(S)(1-p(S))}{|S|}}.$$

95 % confidential interval (CI) is

$$p(S) \pm 1.96\, SE(S)$$

Ex 1

reflexive polytopes

Calabi-Yau hypersurfaces in toric 4-folds

$S_{KS} \subset S_{all}$ 30.108 Hodge pairs

---- the largest known database of Hodge pairs given by Krenzer-Skanke '00.

$$b(S_{KS}) = \frac{11106}{30108} \doteq 0.3689$$

$$SE(S_{KS}) \doteq 0.0028$$

$$\Longrightarrow \ 1/e \doteq 0.3679 \in 95\% \ CI.$$

Construction

$\nabla \subset N_{\mathbb{R}}$: 4-dim'l reflexive polytope

$\overset{\Longleftrightarrow}{\text{Batyrev}}$ $X_{\Sigma(\nabla)}$: Gorenstein toric Fano variety

$$\Sigma(\nabla) = \left\{ \mathrm{Cone}\,\theta \;\middle|\; \phi \leq \theta \underset{\text{face}}{\lneq} \nabla \right\}$$

$$\begin{array}{ccccl} X_{\hat{\Sigma}(\nabla)} & \supset & Y & : & \text{sm CY3} \underset{\text{explicit}}{\rightsquigarrow} (h^{11}, h^{21}) \\ \downarrow & & \downarrow & & \\ X_{\Sigma(\nabla)} & \supset & \bar{Y} & \in & |-K_{X_{\Sigma(\nabla)}}| \end{array}$$

S_{KS}

Ex 2

smoothing of CY hypersurfaces with ODPs

$S_{BK} \subset S_{all}$ 387 Hodge pairs

--- the largest known resource of

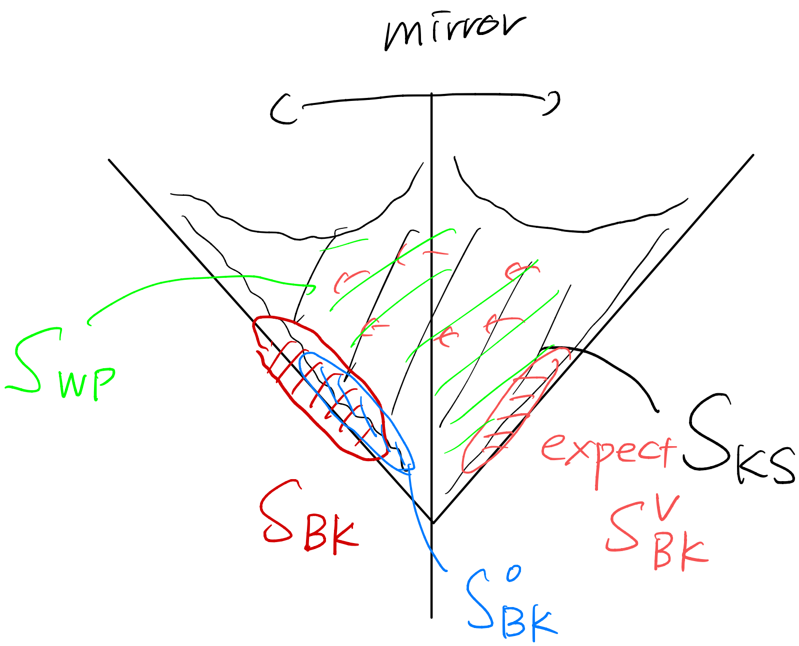
sm CY 3 w/ $h^{11} = 1$ given by

Batyrev - Kreuzer '08.

$$p(S_{BK}) = \frac{142}{387} \fallingdotseq 0.3669$$

$$SE(S_{BK}) \fallingdotseq 0.0245$$

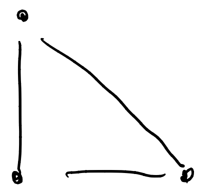
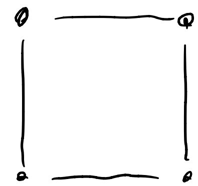
$$\Longrightarrow 1/e \in 95\%\ CI$$



Construction

$\nabla \subset N_{\mathbb{R}}$: 4-dim'l reflexive polytope

s.t. $\forall$ 2-face $\theta < \nabla$ is either

$\Longrightarrow$ $\overline{Y} \in |-K_{X_{\Sigma(\nabla)}}|$ has at worst ODPs.

Batyrev-Kreuzer

Sometimes $\quad \underset{\text{sm CY3}}{} \quad \rightsquigarrow (h^{11}, h^{21})$

$\exists$ smoothing $Y' \rightsquigarrow \overline{Y}$ $\qquad S_{BK}$

by flat deformation

$$\begin{array}{ccccc} X_{\hat{\Sigma}(\nabla)} & & X_{\Sigma(\nabla)} & & \\ \cup & & \cup & & \\ Y & \longrightarrow & \overline{Y} & \leftsquigarrow & Y' \\ S^{\circ}_{BK} & & \text{ODPs} & & S_{BK} \end{array}$$

Def

a geometric transition is an operation

$$\underset{\text{sm } CY_3}{Y} \xrightarrow[\substack{\text{birat} \\ \text{contr}}]{} \underset{\text{normal}}{\overline{Y}} \underset{\substack{\text{flat} \\ \text{def}}}{\rightsquigarrow} \underset{\text{sm } CY_3}{Y'}$$

birat contr $\Downarrow$ crep

normal $\Downarrow$ CY_3

Rem $Y \xrightarrow[g.t]{} Y'$

$\left\{ \begin{array}{l} \text{decreases } h^{11} \\ \text{increases } h^{21} \end{array} \right.$

(III) Problem 2 (Reid's fantasy)

- $\forall$ sm CY3 are connected via g.t ?
- S_{all} is a connected graph ?

(III) Problem 3 (Mirror Symmetry)

- $\underset{\text{sm CY3}}{Y}$ $\quad \exists\, \underset{\text{sm CY3}}{Y^\vee}$: mirror to Y
- $Y \xrightarrow[\text{g.t}]{} Y' \overset{\text{mirror}}{\longleftrightarrow} Y^\vee \xleftarrow[\text{g.t}]{} Y'^\vee$

Assumption fails in general.

$p(S)$ is highly construction-dependent.

Ex 3

(crep resolution of) Calabi-Yau hypersurfaces in 4 dim'l weighted projective spaces

$S_{WP} \subset S_{all}$ 10.237 Hodge pairs given by Sharke '96.

$$p(S_{wp}) = \frac{1886}{10237} \doteqdot 0.1842$$

$$SE(S_{wp}) \doteqdot 0.0038$$

$$\Longrightarrow 1/e \notin 95\% \text{ CI} \quad \text{but}$$

$$1/2e \doteqdot 0.1839 \in 95\% \text{ CI}.$$

(III) Problem 4 ("MMP" for g.t)

$$Y_0 \xrightarrow[g.t]{} Y_1 \longrightarrow \cdots \xrightarrow[g.t]{} Y_n$$

ends up with

"good" CY3.

[Gross '97] speculate implicitly

$$Y : \text{primitive} \Longrightarrow \text{either} \begin{cases} h^{11} = 1 \\ (Y \overset{}{\underset{\text{flop}}{\dashrightarrow}} Y') \\ Y \underset{\text{fibration}}{\longrightarrow} B \end{cases}$$

(birationally primitive)

Ex 4

$$\begin{tikzcd}
 & \mathbb{P}^2 \times \mathbb{P}^2 & \\
 & Y \arrow[u, phantom, "\cup" description] \arrow[dl] \arrow[dr] & \\
\mathbb{P}^2 & & \mathbb{P}^2
\end{tikzcd}$$

2,83

bicubic

birationally primitive & odd parity

$$\begin{tikzcd}
J \arrow[dr, "\text{small}"'] & & \\
 & \bar{J} \arrow[d, shift right] & J \arrow[l, squiggly] \arrow[dl] \\
 & \mathbb{P}^2 \arrow[u, shift right, "\exists s"'] &
\end{tikzcd}$$

relative Jacobian

189 ODPs

2,272

By [Căldăraru '02]

$D^b coh(Y) \simeq D^b coh(J, \alpha)$

$\alpha \in Br(J)$

Morrison - Taylor '12, Taylor '12

classifies generic Jacobian CY3 over sm toric base.

$S_{MT} \subset S_{all}$ 7524 Hodge pairs

$\rightsquigarrow$ all even parity $p(S_{MT}) = 0$

proposal (strategy)

$S_0 \subset S_1 \subset S_2 \cdots \subset S_{all}$

"1-step"

$g_i t$

$p(S_i) \longrightarrow ?$