Asymptotically Mittag-Leffler: A Discrete Version of the Gamma Conjecture I

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Background (J-function and Gamma Conjecture I)

Let X be a smooth Fano manifold. Consider the Givental J-function in the form:

$$J(c_1(X)\log t, 1) = \sum_{m=0}^{\infty} J_m t^{rm+c_1(X)}$$

where $r = \max\{n \in \mathbb{N} \mid \frac{1}{n}c_1(X) \in H^2(X,\mathbb{Z})\}$ is the Fano index of X.
Assume that $c_1(X)\star_0 : H^*(X) \to H^*(X)$ has a simple rightmost eigenvalue. The asymptotic equivalence of $J(t)$ as $t \to +\infty$ is known [1]:

$$J(t) = t^{-\frac{1}{2}\dim X}\, e^{Tt} \cdot \left(\hat{A}_X + o(1)\right),$$

where T is the rightmost eigenvalue of $c_1(X)\star_0$, and $\hat{A}_X$ is the principal asymptotic class of X.

It was predicted in [1] as the Gamma Conjecture I that the Gamma class of X gives the principal asymptotic class of X, i.e.,

$$\hat{\Gamma}_X \in \mathbb{C} \cdot \hat{A}_X.$$

Recently counterexamples to Gamma Conjecture I, $\mathbb{P}_{\mathbb{P}^n}(\mathcal{O} \oplus \mathcal{O}(n))$ for even $n \geq 4$, have been found [2]. In that case, $\hat{\Gamma}_X \operatorname{Ch}(E) \in \mathbb{C} \cdot \hat{A}_X$ for some K-class E. The problem becomes: Is it true that for all Fano manifolds, we have $\hat{\Gamma}_X \operatorname{Ch}(E) \in \mathbb{C} \cdot \hat{A}_X$ for some K-class E? How can we find such E?

Definitions

We say that $J(t)$ is (λ, A_λ)-*scaled asymptotically Mittag-Leffler* ((λ, A_λ)-aML) for $\lambda \in \mathbb{C}$ and $A_\lambda \in H^*(X)$ if the coefficients J_m satisfy the asymptotic equivalence as $m \to \infty$:

$$J_m = \frac{e^{(rm+\frac{1}{2}\dim X + c_1(X))\lambda}}{\Gamma\left(1 + rm + \frac{1}{2}\dim X + c_1(X)\right)} \cdot (A_\lambda + o(1)).$$

Properties

The aML condition satisfies some simple properties:

Let $J(t)$ be (λ, A_λ)-aML with $A_\lambda \neq 0$. Then:
1. The scale is unique up to $(\lambda, A_\lambda) \longleftrightarrow (\lambda + 2\pi\mathtt{i}\frac{k}{r}, e^{-2\pi\mathtt{i}\frac{k}{r}\beta}A)$ for $k \in \mathbb{Z}$,
2. $\frac{d}{dt}J(t)$ is $(\lambda, \exp(\lambda)A)$-aML (but generally $t \cdot J(t)$ is not aML for any scaling),
3. the tail $\sum_{m=M}^{\infty} J_{rm}\, t^{rm+c_1(X)} = \sum_{m=0}^{\infty} J_{r(m+M)}\, t^{rm+(rM+c_1(X))}$ is also (λ, A_λ)-aML.

Specifically, when λ is real, it is stable under taking product and Fano hypersurface:

If $J(t)$ is (λ_X, A_X)-aML for some real λ_X, then:
1. if another Fano manifold Y also has an (λ_Y, A_Y)-aML J-function for some real λ_Y, then the J-function of $X \times Y$ is also aML, with scaling $(log(\exp\lambda_X + \exp\lambda_Y), r_{X\times Y}\frac{A_X}{r_X}\frac{A_Y}{r_Y})$,
2. if Z is a degree d Fano hypersurface of X in the linear system $|-\frac{d}{r}K_X|$ with $d = 1, \ldots, r-1$ and r is the Fano index of X, then the J-function of Z is also asymptotically Mittag-Leffler.

Main Theorem

Assume $J(t)$ is (λ, A_λ)-aML for some $\lambda \in \mathbb{C}$ and $A_\lambda \in H^*(X)$. Then $J(t)$ exhibits the following asymptotic equivalence as $t \to +\infty$,

$$J(t) = \begin{cases} \dfrac{1}{r}\, e^{\mathtt{i}(\operatorname{Im}\lambda)(\frac{1}{2}\dim X + c_1(X))}\, t^{-\frac{1}{2}\dim X}\, e^{\exp(\operatorname{Re}\lambda)\, t} \cdot (A_\lambda + o(1)) & \text{when } \operatorname{Im}\lambda \in \frac{2\pi}{r}\mathbb{Z}, \\ t^{-\frac{1}{2}\dim X}\, e^{\exp(\operatorname{Re}\lambda)\, t} \cdot o(1) & \text{otherwise.} \end{cases}$$

Comparing with the continuous asymptotic of the J-function, when $c_1(X)\star_0$ has a simple rightmost eigenvalue T and the J-function being (λ, A_λ)-aML with a real λ, the scale (λ, A_λ) can be realized by:

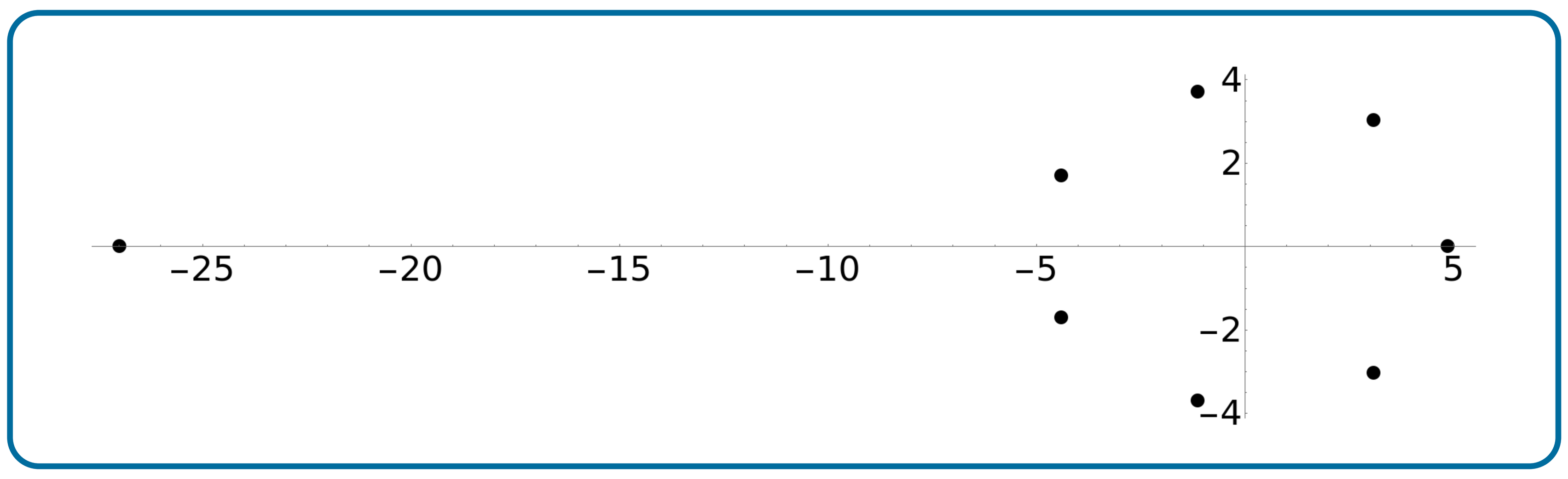
$$\begin{aligned} \exp\lambda &= T \quad \text{(the rightmost eigenvalue of } c_1\star_0\text{)}, \\ A_\lambda &\in \mathbb{C} \cdot \hat{A}_X \quad \text{(the principal asymptotic class)}. \end{aligned}$$

So one may compute the principal asymptotic class by finding the (discrete) asymptotic of the coefficients of the J-function. This approach is more computable when the J-function is already explicitly known as a series.

Notice that under the above conditions, the rightmost eigenvalue must be positive. Do there exist Fano manifolds such that the rightmost eigenvalue is negative?

Examples

It is easy to check that $\mathbb{P}^n$ is aML. By the stability under taking product and hypersurface, all Fano hypersurfaces of projective space and its products are also aML. We have also computed another example: $\mathbb{P}_{\mathbb{P}^3}(\mathcal{O} \oplus \mathcal{O}(3))$. This is an interesting example because the eigenvalues of $c_1(X)\star_0$ look like this:



In this case, the outmost eigenvalue is negative. For this example, we have computed the following numbers by putting $\lambda = \log(\rho(c_1\star_0)) + \pi\mathtt{i}$:

m	$J_m\, m!\, m^{2+c_1}\, e^{-\lambda(m+2+c_1)}$
15	$(0., -0.249072, 0.747216, 0.184877 + 0.782483\mathtt{i}, -0.554631 - 2.34745\mathtt{i}, 2.19172 - 0.580808\mathtt{i}, -6.57515 + 1.74242\mathtt{i}, -3.95630 - 4.31122\mathtt{i}) \times 10^{-3}$
16	$(0., -0.249367, 0.748102, 0.182541 + 0.783411\mathtt{i}, -0.547422 - 2.35023\mathtt{i}, 2.19584 - 0.573469\mathtt{i}, -6.58753 + 1.72041\mathtt{i}, -3.93822 - 4.32113\mathtt{i}) \times 10^{-3}$
17	$(0., -0.249635, 0.748904, 0.180477 + 0.784251\mathtt{i}, -0.541431 - 2.35275\mathtt{i}, 2.19952 - 0.566985\mathtt{i}, -6.59856 + 1.70096\mathtt{i}, -3.92230 - 4.32991\mathtt{i}) \times 10^{-3}$
30	$(0., -0.251845, 0.755534, 0.166211 + 0.791193\mathtt{i}, -0.498632 - 2.37358\mathtt{i}, 2.22748 - 0.522166\mathtt{i}, -6.68243 + 1.56650\mathtt{i}, -3.81519 - 4.39490\mathtt{i}) \times 10^{-3}$
A_λ	$(0., -0.252094, 0.756281, 0.145512 + 0.791976\mathtt{i}, -0.436537 - 2.37593\mathtt{i}, 2.23873 - 0.457141\mathtt{i}, -6.71619 + 1.37142\mathtt{i}, -3.63163 - 4.42768\mathtt{i}) \times 10^{-3}$

The A_λ above is our expected asymptotic class, given by $\hat{\Gamma}_X \operatorname{Ch}(\mathcal{O}_E \otimes \mathcal{O}(-2H))$, where $E = \mathbb{P}_{\mathbb{P}^3}(\mathcal{O}(3))$ and H is the pullback of the hyperplane class. The above computations give evidence that $\mathbb{P}_{\mathbb{P}^3}(\mathcal{O} \oplus \mathcal{O}(3))$ is aML, but with a non-real λ and $A_\lambda \neq \hat{\Gamma}_X$.

Questions

- When does a Fano manifold admit an aML J-function?
- To what level can we relate the scaling (λ, A) with eigenvalues of $c_1(X)\star_0$?

To investigate those questions, we have formulated three levels of correlation:

The J-function of X is (λ, A_λ)-aML for some $\lambda \in \mathbb{C}$ and non-zero $A \in H^*(X)$.

Level 0 is just the space of aML phase being non-empty. The scaling (λ, A_λ) may not be related to other geometric properties.

The J-function of X is (λ, A_λ)-aML for some non-zero $A \in H^*(X)$. (λ, A_λ) can be realized by:
1. $\exp\lambda$ is an eigenvalue of the operator $c_1(X)\star_0 : H^*(X) \to H^*(X)$,
2. $|\exp\lambda| = \rho(c_1(X)\star_0)$.
3. $\hat{\Gamma}_X \operatorname{Ch}(E) \propto A_\lambda$ for some K-class E.

Level 1 corresponds to the aML phase being a subset of spectral phase (eigenvalues with maximum modulus). There may be eigenvalues with maximum modulus that cannot be read from the coefficients of the J-function.

For all eigenvalues of $\exp\lambda$ of $c_1(X)\star_0 : H^*(X) \to H^*(X)$ with maximum modulus, there exist non-zero A_λ such that the J-function is (λ, A_λ)-aML.

There exist some K-class E_λ such that $\hat{\Gamma}_X \operatorname{Ch}(E_\lambda) \propto A_\lambda$.

Level 2 is the full correspondence between aML phase and spectral phase. This results in a specific distribution of eigenvalues of $c_1(X)\star_0$ that resembles positive matrices.

Reference

[1] Sergey Galkin, Vasily Golyshev, and Hiroshi Iritani. "Gamma classes and quantum cohomology of Fano manifolds: gamma conjectures". In: *Duke Math. J.* 165.11 (2016), pp. 20052077. ISSN: 0012-7094,1547-7398.

[2] Sergey Galkin, Jianxun Hu, Hiroshi Iritani, Huazhong Ke, Changzheng Li, and Zhitong Su. *Revisiting Gamma conjecture I: counterexamples and modifications*. 2025. arXiv: 2405.16979 [math.AG].

[3] ChunYin Hau. *Continuous and Discrete Asymptotic Behaviours of the J-function of a Fano Manifold*. 2025. arXiv: 2503.06570 [math.DG].